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CHARACTERISTIC CURVES OF HOLOMORPHIC FOLIATIONS

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Abstract. Let $\mathcal{F}$ be a germ of holomorphic foliation with an isolated singularity at $0 \in \mathbb{C}^2$. A characteristic curve of $\mathcal{F}$ is a continuous one-dimensional curve tending to $0 \in \mathbb{C}^2$, tangent to $\mathcal{F}$ and having some "tame" oscillating behavior, which is a kind of generalization of a separatrix. We define a notion of resolution of the set of characteristic curves of $\mathcal{F}$ and show that this process gives another way of obtaining the resolution of singularities of the foliation.

1. Introduction

We consider a germ of one-dimensional holomorphic foliation $\mathcal{F}$ on a complex smooth surface V, with a unique singularity at $p \in V$. In local coordinates $(\mathbb{C}^2, 0) \simeq (V, p)$ the foliation $\mathcal{F}$ is generated by a holomorphic vector field Z with an isolated singularity at $0 \in \mathbb{C}^2$. The singularity at $p \in V$ is called *reduced* if the linear part of Z has eigenvalues $\lambda_1, \lambda_2 \in \mathbb{C}$ with $\lambda_1 \neq 0$ and such that $\lambda = \frac{\lambda_2}{\lambda_1}$ is not a rational positive number. This last number will be called the eigenvalue ratio of the singularity $p \in V$ and allows us to classify reduced singularities in the following way:

(1) The singularity at p is *hyperbolic* if λ is a non-real complex number. In this case there are linearizing coordinates where the foliation is generated by the vector field $x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$. This singularity has exactly two formal separatrices, both convergent and given by the coordinate axes.

(2) The singularity at p is a *node* if λ is a positive irrational number. As in previous case, in suitable holomorphic coordinates the foliation is generated by the vector field $x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}$. This singularity also has exactly two formal separatrices, both convergent and given by the coordinate axes.

(3) The singularity at p is a *saddle* if λ is a negative real number. In this case the linearization is not always possible, but the singularity still has exactly two formal separatrices which are convergent, smooth and transverse.

(4) The singularity at p is a *saddle-node* if $\lambda = 0$. In this case, according to Dulac, there are holomorphic coordinates such that the foliation is generated by the vector field

$$x^{p+1}\frac{\partial}{\partial x} + \big(y + xA\big)\frac{\partial}{\partial y},$$

where $p \in \mathbb{N}$ and A is a holomorphic function in a neighborhood of the origin. This singularity has exactly two formal separatrices. The first one is convergent and given by the y-axis —this separatrix corresponds to the nonzero eigenvalue and is called *strong separatrix*. The second one, called *weak separatrix*, is tangent to the x-axis and can be non-convergent. We remark that this is the only reduced singularity that can have a purely formal separatrix.

If $\mathcal{F}$ is not reduced, there exists a well known reduction process which we briefly describe in what follows. Let $\pi\colon M \to V$ be a finite composition of successive complex blow-ups at p. The exceptional divisor of π is the set $E = \pi^{-1}(p)$, which is a finite union of smooth rational curves with normal crossings. Outside the exceptional divisor E, the map π is a biholomorphism onto $V\backslash\{p\}$, so $\mathcal{F}$ induces a holomorphic foliation on $M\backslash E$. This foliation has a unique extension to

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E as a holomorphic foliation $\tilde{\mathcal{F}}$ with finitely many singularities which are all contained in E. The foliation $\tilde{\mathcal{F}}$ is called the strict transform of $\mathcal{F}$ by π. We say that π is a resolution —also a reduction or desingularization— of $\mathcal{F}$ if for any point $q \in E$ the foliation $\tilde{\mathcal{F}}$ has following local description:

(1) if $\tilde{\mathcal{F}}$ is regular at q, there are local holomorphic coordinates (x, y) at q such that E is contained in $\{xy = 0\}$ and $\tilde{\mathcal{F}}$ is defined by $dy = 0$;

(2) if $\tilde{\mathcal{F}}$ is singular at q, then this singularity is reduced and the irreducible branches of (E, q) are separatrices of $(\tilde{\mathcal{F}}, q)$.

The map π is a minimal resolution if any other resolution is obtained from π by doing some additional blow-ups. It is well known that a minimal resolution for $\mathcal{F}$ always exists [8, 1] —of course this minimal resolution is unique. We describe now the set of separatrices of $\mathcal{F}$ in relation with the resolution process. Recall first the definition of separatrix: if ω is a holomorphic 1-form defining the foliation $\mathcal{F}$ at p, a formal separatrix of $\mathcal{F}$ is an irreducible formal series $f \in \mathbb{C}[[x, y]]$ such that

$$\omega \wedge df = fh dx \wedge dy$$

for some formal series $h \in \mathbb{C}[[x, y]]$. When f is convergent the equation above means that the curve $f = 0$ is invariant by $\mathcal{F}$. We denote by $\hat{\text{Sep}}(\mathcal{F})$ the set of formal separatrices of $\mathcal{F}$ and by $\text{Sep}(\mathcal{F})$ the set of convergent ones. After the resolution process, a separatrix S of $\mathcal{F}$ becomes a curve in M transverse to some irreducible component D of the exceptional divisor E at a non-corner point $q \in D$. We have the following two possibilities:

(1) The component D is *dicritical*: this means that the foliation $\tilde{\mathcal{F}}$ is totally transverse to D. Then the separatrix S is contained in a leaf of $\tilde{\mathcal{F}}$, so S is always convergent. Clearly, for each non-corner point of D passes a leaf of $\tilde{\mathcal{F}}$ which gives a convergent separatrix of $\mathcal{F}$. This kind of separatrices are called *dicritical*.

(2) The component D is invariant by $\tilde{\mathcal{F}}$. Then q is a reduced singularity of $\tilde{\mathcal{F}}$ having S as a separatrix —the separatrix S could be non-convergent and this can only happen if the singularity $(\tilde{\mathcal{F}}, q)$ is a saddle-node whose strong separatrix is contained in D. These separatrices are called *isolated*. Clearly there are only finitely many isolated separatrices and, in particular, only finitely many purely formal ones.

We can define a notion of resolution —or desingularization— of the set $\hat{\text{Sep}}(\mathcal{F})$ of formal separatrices of $\mathcal{F}$ as follows. If $\pi\colon M \to V$ is any finite composition of successive complex blow-ups at p, we say that π is a resolution of $\hat{\text{Sep}}(\mathcal{F})$ if the strict transforms of the branches in $\hat{\text{Sep}}(\mathcal{F})$ are pairwise disjoint and transverse to the exceptional divisor of π. Clearly we can define the minimal resolution of $\hat{\text{Sep}}(\mathcal{F})$ in the sense that it generates any other resolution by doing additional blow-ups. Obviously, a resolution of $\mathcal{F}$ is also a resolution of $\hat{\text{Sep}}(\mathcal{F})$. On the other hand, in [1] we find the following converse result valid for a generic class of foliations.

Theorem 1. *If $\mathcal{F}$ is a generalized curve foliation at $(\mathbb{C}^2, 0)$, then the minimal resolution of $\hat{\text{Sep}}(\mathcal{F})$ is also the minimal resolution of $\mathcal{F}$.*

Recall that a generalized curve is a foliation having no saddle-node in its resolution. Then, in this case we have no purely formal separatrices and therefore $\hat{\text{Sep}}(\mathcal{F}) = \text{Sep}(\mathcal{F})$. Theorem 1 can be extended to the greater class of *second type* foliations [5, 6], which are the foliations that, after resolution, present no saddle-node whose weak separatrix is contained in the exceptional divisor —of course a second type foliation can have purely formal separatrices. In general, the resolution of $\hat{\text{Sep}}(\mathcal{F})$ can be viewed as a first step in the resolution of the foliation $\mathcal{F}$ —the separatrices of a foliation are concrete objects and their resolution is in some sense a simpler process. However, in general the resolution of $\hat{\text{Sep}}(\mathcal{F})$ does not complete the resolution of the foliation. In this work, we extend the notion of a separatrix to a more general object called *characteristic curve*. Roughly speaking, a characteristic curve of $\mathcal{F}$ is a class of real curves tangent to $\mathcal{F}$, tending to the singularity and having some "tame" oscillating behavior. We denote by $C(\mathcal{F})$ the set of characteristic curves of $\mathcal{F}$. The formal separatrices of $\mathcal{F}$ define a particular kind of characteristic curves of $\mathcal{F}$, so we will have $\hat{\mathrm{Sep}}(\mathcal{F}) \subset C(\mathcal{F})$. The equality $\hat{\mathrm{Sep}}(\mathcal{F}) = C(\mathcal{F})$ holds if and only if $\mathcal{F}$ is a second type foliation. We define a notion of resolution of the set of characteristic curves of $\mathcal{F}$ extending the notion of resolution of $\hat{\mathrm{Sep}}(\mathcal{F})$ and prove the following theorem generalizing Theorem 1.

Theorem 2. *Let $\mathcal{F}$ be a non-reduced foliation at (V, p). Let $\pi\colon M \to V$ be a finite composition of successive complex blow-ups at p. Then, π is a resolution of $\mathcal{F}$ if and only if π is a resolution of $C(\mathcal{F})$.*

It is worth mentioning that the term "characteristic curve" is already defined in [6] in a more restrictive way, as we explain in Section 6.

2. Iterated tangents and oscillation

In this section we establish the notions of iterated tangents and weak oscillation for sets — similar notions are introduced in [3] in the case of curves that are orbits of analytic vector fields in $\mathbb{R}^3$. Consider a point p in a complex manifold M of complex dimension $m \in \mathbb{N}$. Let X be a set in $M\backslash\{p\}$ having p as a limit point and let $\pi\colon \tilde{M} \to M$ be the complex punctual blow-up at p. We say that X has a complex tangent at p if $\pi^{-1}(X)$ has a unique limit point p_1 in the exceptional divisor of π. If this is the case, we say that X has a second complex tangent at p if the set $\pi^{-1}(X)$ has a complex tangent at p_1. This means that, if π_1 is the complex blow-up at p_1, then the set $\pi_1^{-1}\pi^{-1}(X)$ has a unique limit point p_2 in the exceptional divisor of π_1. If this process can be indefinitely iterated we say that X has *complex iterated tangents* at p. The successive limit points $p_1, p_2, \ldots$ obtained in this process is the sequence of iterated tangents of X, which is in fact the sequence of infinitely near points to p naturally associated to X. In other words, we can say that a set having complex iterated tangents at p is a set that is stable by successive punctual complex blow-ups at p.

In an analogous way, we say that the set X has a real tangent at p if X is stable by a real blow-up at p, that is, after a real blow-up at p the strict transform of X has a unique limit point in the exceptional divisor. The set X has *real iterated tangents* at p if X is stable by successive real blow-ups at p. We recall that a real blow-up at p is the standard punctual blow-up at p considering M as a real analytic manifold of dimension $2m$.

Finally, we say that the set X is *weakly oscillating* at p if we have the following properties:

(1) X has complex iterated tangents at p, thus, denoting $p = p_0$ we have a well-defined sequence $p_0, p_1, p_2, \ldots$ of points equal or infinitely near to p associated to X;
(2) for infinitely many $n \geq 0$, the corresponding strict transform of X has a real tangent at p_n.

Remark 3. It is worth mentioning that X can effectively have a real tangent at p_n without having real tangents at the points $p_0, p_1, \ldots, p_{n-1}$. For example, it can be verified that the image of the curve $\gamma\colon [1, +\infty) \to \mathbb{C}^2$ defined by $\gamma(t) = (e^{-\frac{1}{t}}\cos\frac{1}{t} + ie^{-\frac{1}{t}}\sin\frac{1}{t}, te^{-\frac{1}{t}}\cos\frac{1}{t} + ite^{-\frac{1}{t}}\sin\frac{1}{t})$ has complex iterated tangents at $0 \in \mathbb{C}^2$, has no real tangent $p_0 = 0$ and has a real tangent at p_1.

We are specially interested in sets given by continuous curves. Thus, given a continuous curve $\gamma\colon (0, \epsilon] \to M\backslash\{p\}$ with $\lim\limits_{t\to 0}\gamma(t) = p$, we say that γ has complex iterated tangents, has real iterated tangents or is weakly oscillating if the corresponding property holds for the image of γ.

Flat contact. Let X and Y be sets having complex (resp. real) iterated tangents at $p \in M$. We say that X has a flat complex (resp. real) contact with Y at p if both X and Y have the same complex (resp. real) iterated tangents at p. This happens, for example, if one of the sets contains the other. If C is a formal complex irreducible curve at p, we say that X has flat contact with C at p if both X and C define the same sequence of infinitely near points at p.

Real tangents vs complex tangents. If X has a real tangent at p, then it is easy to see that X has also a complex tangent at p. Nevertheless, this implication is not true for more than one blow-up as the following example shows. Consider the function $h = e^{-\frac{1}{t}}$, $t > 0$ and define the curve

$$\gamma(t) = \left(h + ith, \cos\Big(\frac{1}{t}\Big)h^2 + i\sin\Big(\frac{1}{t}\Big)h^2\right).$$

After a real blow-up at the origin we obtain the curve

$$t \mapsto \left(h + it, \cos\Big(\frac{1}{t}\Big)h + i\sin\Big(\frac{1}{t}\Big)h\right).$$

This curve has real iterated tangents, since it has flat contact with the real analytic curve $t \mapsto (it, 0)$. Then γ has real iterated tangents at the origin. On the other hand, if we perform a complex blow-up at the origin the curve γ becomes

$$t \mapsto \left(h + ith, \frac{\cos(\frac{1}{t})h + i\sin(\frac{1}{t})h}{1 + it}\right).$$

This curve tends to the origin of coordinates, so we can do an additional complex blow-up and we obtain the curve

$$t \mapsto \left(h + ith, \frac{\cos(\frac{1}{t}) + i\sin(\frac{1}{t})}{(1 + it)^2}\right),$$

which accumulates to a circle in the exceptional divisor, so γ does not have a second complex tangent at the origin.

Branches of curves. The most simple examples of sets with complex iterated tangents are given by the branches of analytic curves: if C is an irreducible analytic curve at p, then C has complex iterated tangents at p. Moreover, if $\gamma\colon (0, \epsilon] \to C\backslash\{p\}$ is any continuous curve tending to p, then γ has the same complex iterated tangents as C. Clearly γ does not have necessarily a real tangent at p. On the other hand, if γ does have a real tangent at p, it turns out that γ has a real tangent at each of its infinitely near points, as we show next — so γ will be weakly oscillating. Suppose that $\gamma(t) = (x(t), y(t))$ has a real tangent and is contained in the irreducible analytic curve C at $0 \in \mathbb{C}^2$. We can assume that C is tangent to $\{y = 0\}$, so C is given by the Puiseux equation $y = \sum_{j\geq\nu} a_j x^{\frac{j}{n}}$, $a_\nu \neq 0$, $\nu > n$. The existence of a real tangent for $\gamma(t)$ at $0 \in \mathbb{C}^2$ means that $x(t)$ has a real tangent at $0 \in \mathbb{C}$. After a blow-up at $0 \in \mathbb{C}^2$, in the coordinates $(x, \tilde{y}) = (x, y/x)$, the curve C becomes $\tilde{y} = \sum_{j\geq\nu} a_j x^{\frac{j-n}{n}}$ and can be expressed in the form $y = \big(\varphi(x^{1/n})\big)^m$, where $m = \nu - n$ and φ is a germ of biholomorphism in $\mathrm{Diff}(\mathbb{C}, 0)$. Since $x(t)$ has a real tangent at $0 \in \mathbb{C}$, it is easy to see that any determination of $(x(t))^{1/n}$ has a real tangent at $0 \in \mathbb{C}$. Thus, since φ is a biholomorphism, $\varphi\big(x^{1/n}(t)\big)$ has a real tangent at $0 \in \mathbb{C}$ and the same holds for $\tilde{y}(t) = \varphi^m\big(x^{1/n}(t)\big)$. Thus, after the blow-up γ becomes a curve $\tilde{\gamma}(t) = (x(t), \tilde{y}(t))$ with a complex tangent at $0 \in \mathbb{C}^2$ and such that $x(t)$ and $\tilde{y}(t)$ have real tangents at $0 \in \mathbb{C}$. This implies that $\tilde{\gamma}$ has a real tangent at $0 \in \mathbb{C}^2$. By induction we conclude that $\tilde{\gamma}$ has a real tangent at each of its infinitely near points.

Nodal separators. Suppose that M is a complex surface and consider holomorphic coordinates $(M, p) \simeq (\mathbb{C}^2, 0)$. Fix an irrational positive number λ and consider the set

$$S = \{|y| = |x|^\lambda\}.$$

The set S is a real three-dimensional manifold with an isolated singularity at p called *nodal separator* at p. The set S is invariant by the foliation

$$\lambda y dx - x dy = 0,$$

which has the multi-valued first integral $yx^{-\lambda}$, so we obtain —because λ is irrational— that any leaf in S is dense in S. Let us do a complex blow-up at p. We assume $\lambda > 1$ — the other case is similar. If we replace $y = tx$ the set S becomes

$$\{|t| = |x|^{\lambda-1}\},$$

which is again a nodal separator at the origin p_1 of the plane (t, x). Since this set is closed and intersects the coordinate axes only at p_1, we deduce that p_1 defines the complex tangent of S at p. Thus, a nodal separator always has a complex tangent and its strict transform is again a nodal separator. This proves that a nodal separator has complex iterated tangents. More generally, we say that a set S is a nodal separator at p if there exists a finite composition π of complex blow-ups at p and holomorphic coordinates (x, y) at some point $\tilde{p}$ in the exceptional divisor E of π such that

(1) the exceptional divisor E is contained in $\{xy = 0\}$, and
(2) the strict transform of S by π is given by the set

$$\{|y| = |x|^{\lambda}\},$$

where λ is an irrational positive number.

Nodal separators naturally arise [4, 2, 7] as invariant sets of holomorphic foliations near a singularity: let $\mathcal{F}$ be a holomorphic foliation at $(\mathbb{C}^2, 0)$ and suppose that after the resolution process $\mathcal{F}$ has a node at some $\tilde{p}$ in the exceptional divisor. Then we can find holomorphic coordinates (x, y) at $\tilde{p}$ such that

(1) the reduced foliation is defined by the form $\lambda y dx - x dy$, where λ is the eigenvalue of the nodal singularity at $\tilde{p}$, and
(2) the exceptional divisor of the resolution is contained in $\{xy = 0\}$.

The closure of any leaf outside $\{xy = 0\}$ is a set of the form $\{|y| = c|x|^{\lambda}\}$, $c > 0$. Such a set can be expressed in the form $\{|y| = |x|^{\lambda}\}$ by a linear change of coordinates, so the projections of these sets define nodal separators at $(\mathbb{C}^2, 0)$.

Remark 4. It is not difficult to see that, after a sufficient number of blow-ups, a nodal separator always attaches to a corner point in the exceptional divisor. Thus, a nodal separator cannot have flat contact with any formal curve.

3. Tame ends of foliations

Let p be a regular point in the regular complex surface V and let

$$\gamma \colon (0, \epsilon] \to V \backslash \{p\}$$

be a continuous curve at p such that $\lim_{t \to 0} \gamma(t) = p$.

Definition 5. We say that γ is an *end* of $\mathcal{F}$ at $p \in V$ if the image of γ is contained in a leaf of $\mathcal{F}$. Furthermore, γ is a *valuative end* if γ has complex iterated tangents and γ is a *tame end* if γ is weakly oscillating. We say that two valuative ends of $\mathcal{F}$ are equivalent if they have a flat complex contact.

Example 6. *Convergent separatrices.* Consider a convergent separatrix S of a holomorphic foliation $\mathcal{F}$ at (V, p). Then any continuous curve $\gamma \colon (0, \epsilon] \to S \backslash \{p\}$ with $\lim \gamma = p$ and having a real tangent at p is a tame end of $\mathcal{F}$. All these ends are equivalent because they share the sequence of infinitely near points defined by the separatrix S. Thus, a convergent separatrix defines a class of tame ends.

Example 7. *A non-valuative end.* Consider the foliation $\mathcal{F}$ defined by

$$(x + y)dx + (y - x)dy = 0,$$

which is a real focus. This complex equation has the following family of integral curves:

$$\gamma(t) = \big(ce^t\cos(t), ce^t\sin(t)\big),\ c \in \mathbb{R}.$$

If we fix $c \neq 0$ and restrict t to real values, γ becomes a real curve tending to $0 \in \mathbb{C}^2$ as $t \to -\infty$. Thus, up to reparametrization the curve γ is an end of the foliation $\mathcal{F}$. Nevertheless, γ is not a valuative end because γ does not have a complex tangent at $0 \in \mathbb{C}^2$.

Example 8. *The saddle-node.* Consider a saddle-node foliation $\mathcal{F}$. There exist holomorphic coordinates such that $\mathcal{F}$ is defined by the 1-form

$$x^{p+1}dy - \big(y + xA\big)dx,$$

where $p \in \mathbb{N}$ and A is a holomorphic function in a neighborhood of the origin. Thus, the weak separatrix S is transverse to the y-axis and then there is a formal series $\hat{s}(x) = \sum_{j=1}^{\infty} c_j x^j$ such that S is given by $y = \hat{s}(x)$. It is known that there exists a small sector

$$V = \{x \in \mathbb{C} \colon 0 < |x| < \epsilon,\, \alpha < \arg x < \beta\},$$

with $\epsilon > 0, \beta > \alpha > 0$ and a family $\mathcal{H}$ of holomorphic functions on V such that

(1) each $f \in \mathcal{H}$ has the series $\sum\limits_{j=1}^{\infty} c_j x^j$ as asymptotic expansion at $0 \in \mathbb{C}$;
(2) for each $f \in \mathcal{H}$, the graph $V_f = \{(x, f(x)) : x \in V\}$ is contained in a leaf of the foliation $\mathcal{F}$;
(3) the union $\bigcup\limits_{f \in \mathcal{H}} V_f$ is an open set;
(4) if $\hat{s}$ is convergent, then $\hat{s} \in \mathcal{H}$.

Fix $f \in \mathcal{H}$ and $\eta \in V$ and define

$$\gamma(t) = (\eta t, f(\eta t)),\ t \in (0, 1].$$

Then the curve γ is contained in a leaf of $\mathcal{F}$ and has flat complex contact with the formal separatrix S —so γ has complex iterated tangents. Let us show that γ is weakly oscillating. It is easy to see that γ has a real tangent at the origin. Let $p_1, p_2, \ldots$ be the infinitely near point associated to γ and fix $k \in \mathbb{N}$. By changing coordinates if necessary, we can assume that

$$c_1 = \cdots = c_{k+1} = 0.$$

Then we have $f(x) = o(|z|^{k+1})$ and therefore $\gamma(t) = \big(\eta t, o(t^{k+1})\big)$. Thus, after doing k successive blow-ups we have coordinates (u, v) at p_k such that the canonical projection is given by

$$(u, v) \mapsto (x, y) = (u, vu^k).$$

Then, in the coordinates (u, v) the curve γ is given by

$$\left(\eta t, \frac{o(t^{k+1})}{(\eta t)^k}\right) = (\eta t, o(t))$$

and it follows that γ has a real tangent at p_k. Thus, γ has a real tangent at each of its infinitely near points and therefore γ is a tame end of $\mathcal{F}$ having flat contact with the formal separatrix S. We notice that if S is convergent, clearly we can take $f \neq \hat{s}$, so almost all these tame ends are not contained in the weak separatrix. This example is generalized in Example 9.

Example 9. *Purely formal separatrices.* Consider a foliation $\mathcal{F}$ at $0 \in \mathbb{C}^2$ with a purely formal separatrix S. We know that, after reduction of singularities, the strict transform $\tilde{S}$ of S is the weak separatrix of a saddle-node singularity $\tilde{p}$ of the strict transform $\tilde{\mathcal{F}}$ of $\mathcal{F}$. Then, by Example 8 we can find tame ends of the strict transform foliation at $\tilde{p}$ having a flat contact with $\tilde{S}$. Clearly this tame ends induce tame ends of $\mathcal{F}$ at $0 \in \mathbb{C}^2$ having a flat contact with S. Thus, a purely formal separatrix —like in the convergent case— define a class of equivalent tame ends.

Example 10. *Nodal singularities.* Let $\mathcal{F}$ be a foliation with a node at $0 \in \mathbb{C}^2$. Then, in suitable holomorphic coordinates $\mathcal{F}$ is defined by the form

$$\lambda y dx - x dy,\ \lambda \in \mathbb{R}^+ \backslash \mathbb{Q},$$

hence $\mathcal{F}$ has the multi-valued first integral $yx^{-\lambda}$. Then the closure of any leaf other than the separatrices is a set of the type

$$S_c = \{|y| = c|x|^\lambda\},$$

with $c > 0$, which is called a nodal separator [4, 7]. The nodal separator S_c has a well defined sequence of infinitely near points depending only on the continuous fraction of λ —it does not depend on the constant $c > 0$. Then any curve γ contained in a leaf in S_c and tending to the singularity will be a valuative end of $\mathcal{F}$ having a flat complex contact with the nodal separator S_c. Thus, the nodal separator S_c defines a class of valuative ends. In this class we can find some ends that are tame and others that are not: for example, it can be proved that $\gamma(t) = (t, ct^\lambda)$, $t \in (0, \epsilon]$ is a tame end in S_c, whereas $\tilde{\gamma}(t) = (te^{i\log t}, ct^\lambda e^{i\lambda \log t})$ is not.

Example 11. *Non-nodal singularities with nodal ends.* Let $\mathcal{F}$ be a complex hyperbolic singularity at $0 \in \mathbb{C}^2$. In suitable holomorphic coordinates $\mathcal{F}$ is defined by the form

$$\lambda y dx - x dy,\ \lambda \notin \mathbb{R}.$$

It is not difficult to see that, given $\alpha \in \mathbb{R}^+ \backslash \mathbb{Q}$, we can find $\theta \in \mathbb{C}$ such that

(1) θ and $\theta\lambda$ have negative real parts;
(2) $\alpha = \frac{\mathrm{Re}(\theta\lambda)}{\mathrm{Re}(\theta)}$.

Take $x_0, y_0 \in \mathbb{C}^*$ and define the curve

$$\gamma(t) = (x_0 e^{\frac{\theta}{t}}, y_0 e^{\frac{\theta\lambda}{t}}),\ t > 0.$$

Then it is easy to see that the following properties hold:

(1) γ is contained in a leaf of $\mathcal{F}$;
(2) γ tends to $0 \in \mathbb{C}^2$ as $t \to 0$;
(3) if $c = \frac{|y_0|}{|x_0|^\alpha}$, then γ is contained in the nodal separator

$$S_\alpha = \{|y| = c|x|^\alpha\}.$$

Thus, γ is a valuative end of $\mathcal{F}$ having a flat complex contact with the nodal separator S_α. This example shows that a hyperbolic singularity has an uncountable set of non-equivalent valuative ends. Nevertheless, it is worth mentioning that $\mathcal{F}$ may not have any tame end outside the separatrices. For example, in Lemma 18 will be proved that if $\mathrm{Re}\,\lambda \leq 0$ the foliation $\mathcal{F}$ has no tame end outside the separatrices.

Non-reduced examples. Let $\pi\colon M \to V$ be a finite composition of complex blow-ups at p and let $\tilde{\mathcal{F}}$ be the strict transform of $\mathcal{F}$ by π. Let $\tilde{\gamma}$ be a tame end of $\tilde{\mathcal{F}}$ at a point $\tilde{p}$ in the exceptional divisor of π. Then, if γ does not intersect the exceptional divisor, it is easy to see that $\pi\tilde{\gamma}$ is a tame end of $\mathcal{F}$ at p. This allows us to construct tame ends of non-reduced foliations from the examples above.

4. Characteristic curves

As we have seen in the previous section, there exist tame ends having a flat contact with nodal separators. The nodal separators are objects for which we have notions of equisingularity and topological equivalence —as in the case of curves— which are in fact equivalent notions [7]. Nevertheless, unlike the case of curves, the information contained in the sequence of infinitely near points of a nodal separator is essentially infinite, meaning that there is no resolution process for a nodal separator. In this sense, the infinite information contained in the sequence of infinitely near points of a nodal separator is not relevant in the goal of describing a resolution process of a foliation singularity. Thus, we arrive to our main definition:

Definition 12. Let $\mathcal{F}$ be a holomorphic foliation at (V, p), where p is a regular point of the complex surface V. A *characteristic curve* of $\mathcal{F}$ is an end equivalent to a tame end, whose sequence of iterated tangents is not the sequence of iteradted tangents of a nodal separator.

As we can see, among the examples of Section 3 only examples 6, 8 and 9 are examples of characteristic curves. In all those examples the characteristic curves have flat contact with a formal separatrix, so the following example will be important.

4.1. **A characteristic curve which does not come from a separatrix.** Consider the foliation $\mathcal{F}$ defined by the form

$$(y^2 + yx)dx - x^2dy.$$

By a straightforward computation we can verify that

$$\gamma(t) = (e^{-\frac{1}{t}}, te^{-\frac{1}{t}}),\ t > 0$$

is an end of $\mathcal{F}$. Le π be the blow-up at $0 \in \mathbb{C}^2$. In the coordinates

$$(t, x) = (\frac{y}{x}, x)$$

the strict transforms $\tilde{\gamma}$ and $\tilde{\mathcal{F}}$ of γ and $\mathcal{F}$ are given by

$$t^2dx - xdt = 0 \quad \text{and} \quad \tilde{\gamma}(t) = (t, e^{-\frac{1}{t}}).$$

Thus, $\tilde{\mathcal{F}}$ has saddle-node singularity at $\tilde{p} = (0, 0)$ with a convergent weak separatrix S contained in the exceptional divisor $\{x = 0\}$. On the other hand, the curve $\tilde{\gamma}$ clearly has flat contact with S at $\tilde{p}$. From this we obtain the following conclusions:

(1) The curve γ has complex iterated tangents: its infinitely near points are $\tilde{p}$ and the points infinitely near to $\tilde{p}$ that lie on S.
(2) Since $\tilde{\gamma}$ has a flat contact with S at $\tilde{p}$ and $\tilde{\gamma}$ has a real tangent at $\tilde{p}$, it is not difficult to prove that γ is weakly oscillating —this was essentially done in Example 8. Then γ is a tame end of $\mathcal{F}$.
(3) Since $\tilde{\gamma}$ has a flat contact with a complex curve at $\tilde{p}$, then $\tilde{\gamma}$ cannot have a flat contact with a nodal separator (Remark 4) and γ cannot either.
(4) Since $\tilde{\gamma}$ has a flat contact with the exceptional divisor, by successive blow-ups this curve remains tangent to the exceptional divisor, so γ has no flat contact with any formal curve at $0 \in \mathbb{C}^2$.

The previous example give rise to the following definition.

Definition 13. Let $\mathcal{F}$ be a holomorphic foliation at (V, p) and let γ be a characteristic curve of $\mathcal{F}$. We say that γ is a tangent characteristic curve of $\mathcal{F}$ if γ has not flat contact with any formal curve at p.

Proposition 14. *Let $\mathcal{F}$ be a holomorphic foliation at (V, p) and let γ be a characteristic curve of $\mathcal{F}$. Then one of the following properties holds:*

(1) γ has a flat contact with a formal curve S at p. In this case S is a formal separatrix of $\mathcal{F}$.
(2) γ is a tangent characteristic curve. In this case, in any resolution of $\mathcal{F}$ we find a saddle-node $\tilde{p}$ such that:
   (a) the weak separatrix $\tilde{S}$ of $\tilde{p}$ is contained in the exceptional divisor;
   (b) γ has a flat contact with $\tilde{S}$ at $\tilde{p}$.

   In particular, there exist finitely many tangent characteristic curves.

As a first step we prove the following proposition.

Proposition 15. *Let $\mathcal{F}$ have a reduced singularity at (V,p) and let γ be a characteristic curve of $\mathcal{F}$. Then γ has a flat contact with a separatrix S of $\mathcal{F}$. Moreover at least one of the following possibilities holds:*

(1) the separatrix S is convergent and contains γ, or
(2) the singularity at p is a saddle-node having S as its weak separatrix.

Remark 16. According to [5], a saddle-node whose weak separatrix is contained in the exceptional divisor of a resolution is called *tangent* saddle-node. Thus, the tangent characteristic curves of $\mathcal{F}$ are associated to tangent saddle-nodes in the resolution of $\mathcal{F}$.

We need some lemmas.

Lemma 17. *Let γ be a valuative end of a holomorphic foliation $\mathcal{F}$ at (V,p). Let π be a resolution of $\mathcal{F}$ and denote by $\tilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ by π. Then we have the following alternative:*

(1) either γ is contained in a dicritical separatrix, or
(2) γ tends to a singularity of $\tilde{\mathcal{F}}$.

In particular, the second possibility always holds if the foliation is non-dicritical.

Proof. Let E be the exceptional divisor of π and let q be the limit point of γ in E. Suppose that the second possibility does not hold. Then q is a regular point of $\tilde{\mathcal{F}}$. Thus, in a neighborhood of q the leaves are closed and the only one accumulating to q is the leaf L passing through q. Then, since $\gamma \to q$, we deduce that $\gamma \subset L$ and consequently L is not contained in the exceptional divisor E. Thus, $L \nsubseteq E$ is a leaf of $\tilde{\mathcal{F}}$ passing through a regular point $q \in E$. Therefore, since π is a resolution, (L,q) defines a dicritical separatrix. □

We say that a singularity is *non-degenerate* if its eigenvalue ratio is nonzero.

Lemma 18. *Suppose that $\mathcal{F}$ has a reduced non-degenerate singularity at (V,p). Let γ be a tame end of $\mathcal{F}$ which is not contained in any separatrix. Then*

(1) $\mathcal{F}$ has an eigenvalue ratio with positive real part,
(2) γ is tangent to a separatrix S of $\mathcal{F}$ at p, and
(3) γ has not a flat contact with S at p.

Proof. Let $p_1, p_2, \ldots$ be the points infinitely near to p associated to γ and denote $p = p_0$. Since γ is weakly oscillating, then γ has a real tangent at its n-th infinitely near point p_n for some $n \geq 0$. We will do the proof by induction on n: suppose first that γ has a real tangent at $p = p_0$. If we do a blow-up at p we have that γ tends to p_1 and, by Lemma 17, this point p_1 is a singularity of the strict transform of $\mathcal{F}$. Since this singularity is necessarily non-degenerate, there exists a separatrix S through p_1 which is not contained in the exceptional divisor. Then S defines a separatrix of $(\mathcal{F},p)$ and clearly γ is tangent to S at p —property (2) is proved. Suppose that $\mathcal{F}$ can be linearized —we will prove later that the other case is not possible. Then we can assume that $\mathcal{F}$ is generated by the vector field

$$x\frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}.$$

Set

$$\gamma(t) = \big(x(t), y(t)\big),\ t \in (0,\epsilon].$$

Without loss of generality we can suppose that $\gamma(t)$ has the vector $(1,0) \in \mathbb{C}^2$ as its real tangent at $0 \in \mathbb{C}^2$. Thus, given $\delta > 0$, by reducing ϵ if necessary we can assume that

$$|\arg x(t)| < \delta \text{ for all } t \in (0,\epsilon]. \tag{4.1}$$

Then we have $x = e^f$, where

$$f\colon (0,\epsilon] \to \{z \in \mathbb{C}\colon |\mathrm{Im}(z)| < \delta\}$$

is a continuous function. Since $\mathcal{F}$ has the liouvillian first integral $yx^{-\lambda}$ and γ is contained in a leaf, we can express $y = ce^{\lambda f}$, for some constant $c \in \mathbb{C}^*$. Thus, if $\lambda = a + ib$ and $f = u + iv$, we have

$$|y| = |c|e^{\mathrm{Re}(\lambda f)} = |c|e^{au-bv} = |c|e^{-bv}|x|^a.$$

Then, since $|v| = |\arg(x)| \leq \delta$ we obtain the inequalities

$$|c|e^{-\delta|b|}|x|^a \leq |y| \leq |c|e^{\delta|b|}|x|^a. \tag{4.2}$$

Since x and y tend to zero as $t \to 0$, from the first inequality in (4.2) we deduce that $a = \mathrm{Re}\,\lambda > 0$ —property (1) is proved. Now, again from the inequalities (4.2) property (3) is easily obtained. It remains to prove that the non-linearizable case does not happen: suppose that $\mathcal{F}$ is not linearizable —so $\mathcal{F}$ has a real negative eigenvalue ratio λ. It is easy to find holomorphic coordinates at p such that $\mathcal{F}$ is generated by the vector field

$$Z = x\frac{\partial}{\partial x} + \lambda y(1+H)\frac{\partial}{\partial y},$$

where H is a holomorphic function vanishing at $0 \in \mathbb{C}^2$. Clearly the separatrices are given by the coordinate axes. As before we can assume that inequality (4.1) holds. Given $t \in (0, \epsilon]$, let $\phi(s)$ be the real orbit of the vector field Z such that $\phi(0) = \gamma(t)$. If $\phi(s) = (x(s), y(s))$ and $\phi(0) = (a_t, b_t)$, we obtain $x(s) = a_t e^s$ and

$$y'(s) = \lambda y\big(1 + H(x,y)\big).$$

Then, if we set $f(s) = |y(s)|^2$ we have

$$f' = 2\,\mathrm{Re}(\bar{y}y') = 2\lambda|y|^2\,\mathrm{Re}(1+H). \tag{4.3}$$

Since $H(0,0) = 0$, we have that $\mathrm{Re}(1+H) > 0$ on some bidisc

$$D = \{|x| \leq r, |y| \leq r\},\ r > 0$$

and without loss of generality we can assume $\gamma \subset D$. Then from (4.3) we obtain $f' < 0$ and therefore $|y(s)|$ is decreasing. Thus, the orbit $\phi(s)$ is contained in D until we have $|x(s_t)| = r$ for some $s_t \geq 0$. So we set

$$\tilde{\gamma}(t) = \phi(s_t) = (x(s_t), y(s_t)).$$

Since $|y(s_t)| \leq |y(0)| = |b_t|$ and $\gamma(t) = (a_t, b_t) \to 0$ as $t \to 0$, we have that $\tilde{\gamma}(t)$ tends to the axis $\{y = 0\}$ as $t \to 0$. Let us show that this is not possible. The curve $\tilde{\gamma}$ is contained in a leaf of the foliation and is also contained in the set $\{|x| = r\}$, hence $\tilde{\gamma}$ is contained in a leaf of the real one-dimensional foliation $\mathcal{G}$ induced by $\mathcal{F}$ on $\{|x| = r\}$. Since $\mathcal{G}$ is the suspension of the holonomy map associated to the loop $\{|x| = 1, y = 0\}$, the curve $\tilde{\gamma}$ tends to $\{y = 0\}$ only if $\tilde{\gamma}$ winds around $\{x = 0\}$. But this is not possible because $\tilde{\gamma}$, like γ, is contained in $\{|\arg x| < \delta\}$, because the x-coordinates of γ and $\tilde{\gamma}$ are respectively a_t and $a_t e^{s_t}$. Therefore the first step of induction is done. Suppose that Lemma 18 holds when γ has a real tangent at its n-th infinitely near point. Let γ be as in Lemma 18 and having a real tangent at its $(n+1)^{\text{th}}$ infinitely near point. If we do a blow-up at p, then γ becomes a tame end $\tilde{\gamma}$ of the strict transform $\tilde{\mathcal{F}}$ of $\mathcal{F}$ at the point p_1. Clearly the singularity $(\tilde{\mathcal{F}}, p_1)$ is non-degenerate and $\tilde{\gamma}$ has a real tangent at p_{n+1}, which is its n^{th} infinitely near point. Then, by the inductive hypothesis we have that

(a) the eigenvalue ratio of $(\tilde{\mathcal{F}}, p_1)$ has a positive real part,
(b) $\tilde{\gamma}$ is tangent to a separatrix S_1 of $(\tilde{\mathcal{F}}, p_1)$, and
(c) $\tilde{\gamma}$ has not a flat contact with S_1 at p_1.

Let S be the separatrix of $(\tilde{\mathcal{F}}, p_1)$ that is not contained in the exceptional divisor. Then S defines a separatrix of $(\mathcal{F}, p)$ and γ is obviously tangent to S at p. If $S_1 \neq S$, by item (b) above γ has not a flat contact with S at p; if $S_1 = S$, it follows from item (c) above that γ has not a flat contact with S at p. Finally, from item (a) we have that the Camacho-Sad index $\mathrm{CS}(\tilde{\mathcal{F}}, S, p_1)$ has a positive real part, so we have that

$$\mathrm{CS}(\mathcal{F}, S, p) = \mathrm{CS}(\tilde{\mathcal{F}}, S, p_1) + 1$$

has also a positive real part, which finishes the proof. □

Lemma 19. *Suppose that $\mathcal{F}$ has a reduced non-degenerate singularity at (V, p). Let γ be a characteristic curve of $\mathcal{F}$. Then γ is contained in a separatrix of $\mathcal{F}$.*

Proof. Suppose that γ is not contained in any separatrix of $\mathcal{F}$. Let $p_1, p_2, \ldots$ be the points infinitely near to $p \in V$ that lie on γ, that is:

(1) p_1 is the only point in the exceptional divisor E_1 of the blow-up at $p \in V$ that lies in γ;
(2) p_j is the only point in the exceptional divisor E_j of the blow-up at p_{j-1} that lies in γ $(j \geq 2)$.

All the strict transforms of E_j by subsequent blow-ups are also denoted by E_j. By Lemma 18 the curve γ is tangent to a separatrix of $\mathcal{F}$, which we denote by E_0. This means that after the first blow-up we have $p_1 \in E_0$. If γ is tangent to E_0 at p_1, then we have $p_2 \in E_0$. By Lemma 18 we cannot have $p_k \in E_0$ for all $k \in \mathbb{N}$, otherwise γ would have a flat contact with E_0. Then there exists $n_0 \in \mathbb{N}$ such that $p_1, \ldots, p_{n_0} \in E_0$ and $p_{n_0+1} \notin E_0$. Then γ is not tangent to E_0 at p_{n_0}. Observe that p_{n_0} is a non-degenerate singularity having E_0 and E_{n_0} as separatrices. Thus, by Lemma 18 the curve γ is tangent to E_{n_0} at p_{n_0} and therefore $p_{n_0+1} \in E_{n_0}$. Again by Lemma 18, since γ has not a flat contact with E_{n_0}, there exists $n_1 \in \mathbb{N}$ such that $p_{n_0+1}, \ldots, p_{n_0+n_1} \in E_{n_0}$ and $p_{n_0+n_1+1} \notin E_{n_0}$. Now, γ is not tangent to E_{n_0} at $p_{n_0+n_1}$ and this point is a non-degenerate singularity having E_{n_0} and $E_{n_0+n_1}$ as separatrices. Thus, again by Lemma 18 the curve γ is tangent but has not a flat contact with $E_{n_0+n_1}$ and therefore we find $n_2 \in \mathbb{N}$ such that $p_{n_0+n_1+1}, \ldots, p_{n_0+n_1+n_2} \in E_{n_0+n_1}$ and $p_{n_0+n_1+n_2+1} \notin E_{n_0+n_1}$. Proceeding in this way we define a sequence $n_0, n_1, \ldots$ of natural numbers which determines the sequence p_j of infinitely near points of γ. Consider the irrational number defined by the continuous fraction $\lambda = [n_0, n_1, \ldots]$. Let (x, y) be holomorphic coordinates at (V, p) such that the separatrices of $\mathcal{F}$ are given by the coordinate axes and $E_0 = \{y = 0\}$.

Consider the nodal separator $S = \{|y| = |x|^\lambda\}$ at p and let $\tilde{p}_1, \tilde{p}_2, \ldots$ be the sequence of infinitely near points associated to S. Since $n_0 < \lambda < n_0 + 1$, it is easy that $\tilde{p}_1, \ldots, \tilde{p}_{n_0} \in E_0$ and $\tilde{p}_{n_0+1} \notin E_0$. Then $\tilde{p}_j = p_j$ for $j = 1, \ldots, n_0$. Then, by replacing $y = y_1^{n_0} x$ we have that S is given by

$$\left\{ |x| = |y_1|^{\frac{1}{\lambda - n_0}} \right\}$$

and

$$E_{n_0} = \{x = 0\}$$

at p_{n_0}. Since

$$n_1 < \frac{1}{\lambda - n_0} < n_1 + 1,$$

we have that $\tilde{p}_{n_0+1}, \ldots, \tilde{p}_{n_0+n_1} \in E_{n_0}$ and $\tilde{p}_{n_0+n_1+1} \notin E_{n_0}$. Then $\tilde{p}_j = p_j$ for $j = n_0 + 1, \ldots, n_1$. Proceeding in this way we easily prove that $\tilde{p}_j = p_j$ for all $j \in \mathbb{N}$. Then γ has a flat contact with a nodal separator, which is a contradiction. □

Proof of Proposition 15. In view of Lemma 19 we can assume that p is a saddle-node. Let S_s and S_w be the strong and weak separatrices of p, respectively. We recall first some basic facts about the blow-up of a saddle node. If we do a blow-up at p, then we obtain an invariant divisor D and exactly two singular points $s_1 = S_s \cap D$ and $w_1 = S_w \cap D$. The point w_1 is again a saddle-node having S_w as weak separatrix and s_1 is a non-degenerate singularity. If we do a second blow-up at w_1, we obtain again a saddle-node $w_2 \in S_w$ and a new non-degenerate singularity in the corner of the exceptional divisor. We do a new blow-up at w_2 and continue with this process. Thus, after k blow-ups we have the following properties:

(1) the exceptional divisor E is invariant by the strict transform $\tilde{\mathcal{F}}$ of $\mathcal{F}$,
(2) the point $w_k = S_w \cap E$ is a saddle-node having S_w as weak separatrix,
(3) all the other singularities are non-degenerate, and
(4) besides w_k, the only non-corner singularity is $s_1 = S_s \cap E$.

Returning to the proof, if we suppose that S_w has not a flat contact with γ, for k large enough we will have that γ attaches to a point $q \in E$ which is different from w_k. Then, since by Lemma 17 the point q is a singularity, we have by item (3) that q is a non-degenerate singularity of $\tilde{\mathcal{F}}$. By Lemma 19 the singularity q cannot be a corner, otherwise γ will be contained in the exceptional divisor E. Then by item (4) above we have $q = s_1$. Finally, again by Lemma 19 we conclude that γ is contained in S_s. □

Proof of Proposition 14. Suppose that γ has a flat contact with a formal curve S at p. Let $\pi \colon M \to V$ be the desingularization of S and let $\tilde{\mathcal{F}}$ be the strict transform of $\mathcal{F}$ by π. Then the curve S becomes a non-singular formal curve transverse to the exceptional divisor E at a non-corner point q —of course γ tends to q. By doing some additional blow-ups at q if necessary we can assume that the foliation at q is either regular or reduced. Suppose first that q is a regular point of the foliation. Then the only leaf accumulating to q is the leaf L passing through it, so we necessarily have $\gamma \subset L$. Then L cannot be contained in the exceptional divisor and therefore (L, q) defines a separatrix of $\mathcal{F}$. Since γ obviously has flat contact with L at q, then L has a flat contact with S and consequently $S = L$, which finishes the proof in this case. Suppose now that q is a reduced singularity of $\tilde{\mathcal{F}}$. Also assume that by additional blow-ups the curve S always pass through a singular point of the strict transfor foliation — ortherwise we would return to the first case. By doing additional blow-ups if necessary, we can assume that one of the separatrices of q, say S_e, is contained in the exceptional divisor E. Let S' be the formal separatrix of q that is transverse to E. Suppose that $S \neq S'$. By doing some additional blow-ups if necessary we can assume that S is transverse to S'. Let us do an additional blow-up at q. Since q is reduced, we obtain an exceptional divisor D invariant by the strict transform foliation. Moreover, the separatrices S_e and S' become transverse to D at points q_e and q', which are the only singularities in D of the strict transform foliation. Nevertheless, since S was transverse to S_e and S', then S becomes transverse to D at some point $\tilde{q}$ different from q_e and q', which is a contradiction because $\tilde{q}$ has to be singular.

Suppose now that γ is a tangent characteristic curve. Let π be a resolution of $\mathcal{F}$ and denote by $\tilde{\mathcal{F}}$ the corresponding strict transform foliation. By Lemma 17 the curve γ tends to a singularity q of $\tilde{\mathcal{F}}$ in the exceptional divisor E of π. Since γ is a characteristic curve of $\tilde{\mathcal{F}}$ at q, by Proposition 15 we have that γ has a flat contact with a separatrix S of $\tilde{\mathcal{F}}$ at q. Then S is necessarily contained in the exceptional divisor, otherwise S would define a separatrix of $\mathcal{F}$ and γ would not be a tangent characteristic curve of $\mathcal{F}$. Thus, since $S \subset E$, we obviously have that γ is not contained in S. Therefore, again by Proposition 15 we conclude that q is a saddle-node having $S \subset E$ as weak separatrix, which finishes the proof.

5. Resolution of the set of characteristic curves

Let $\mathcal{F}$ be a holomorphic foliation in (V, p), where p is a regular point of the complex surface V. Let γ be a characteristic curve of $\mathcal{F}$. From now on, a characteristic curve will be understood as a class of equivalent curves, so we use the same symbol γ to denote the class of characteristic curves having a flat contact with γ. By Proposition 14 there are two kinds of characteristic curves:

(1) *The characteristic curves having a flat contact with formal separatrices of $\mathcal{F}$.* Since two different separatrices have not a flat contact, this set of characteristic curves can be

identified with the set $\hat{\text{Sep}}(\mathcal{F})$ of formal separatrices of $\mathcal{F}$. This set can be infinite if the foliation is dicritical, but the subset of purely formal separatrices is always finite.

(2) *The tangent characteristic curves.* These characteristic curves are associated to tangent saddle-nodes in the resolution of $\mathcal{F}$, so there are finitely many of them.

Definition 20. Let $\mathcal{F}$ be a non-reduced foliation at (V, p). Let $\mathrm{C}(\mathcal{F})$ be the set of characteristic curves of $\mathcal{F}$ and let $\gamma_1, \ldots, \gamma_k$ be the tangent ones. Let π be a finite composition of successive blow-ups at $0 \in V$. We say that π is a resolution of $\mathrm{C}(\mathcal{F})$ if the following properties hold:

(1) π is a desingularization of $\hat{\text{Sep}}(\mathcal{F})$;
(2) for each $j = 1, \ldots k$, the curve γ_j has flat contact with some component of the exceptional divisor at a point p_j;

Moreover, we say that π is a minimal resolution if any other resolution is obtained from π by doing additional blow-ups.

Remark 21. In fact, if π is a resolution of $\mathrm{C}(\mathcal{F})$, we necessarily have that the points $p_1, \ldots, p_k$ above are pairwise distinct. Suppose for example that $p_1 = p_2$. As a first case γ_1 and γ_2 have flat contact with the same component of the exceptional divisor through p_1, which means that $\gamma_1 = \gamma_2$, a contradiction. In the second case γ_1 and γ_2 have flat contact with two diferent components D_1 and D_2 of the exceptional divisor through p_1. This is also impossible but we need to use Theorem 22 below: π is necessarily a resolution of the foliation, so p_1 is a reduced singularity and, by Proposition 15, we conclude that $\gamma_1 \subset D_1$ or $\gamma_2 \subset D_2$, which is impossible.

Now, we can state our main theorem.

Theorem 22. *Let $\mathcal{F}$ be a non-reduced foliation at (V, p) and let $\mathrm{C}(\mathcal{F})$ be its set of characteristic curves. Let π be a finite composition of successive blow-ups at p. Then π is a resolution of $\mathcal{F}$ if and only if π is a resolution of $C(\mathcal{F})$. Clearly the same assertion holds if we replace the term "resolution" by "minimal resolution".*

Before starting the proof of Theorem 22, we need the following two lemmas:

Lemma 23. *Let $\mathcal{F}$ be a singularity at (V, p) having a formal non-singular separatrix S through p. Then $\mathrm{C}(\mathcal{F})$ has some element different from S.*

Lemma 24. *Let $\mathcal{F}$ be a singularity at (V, p) having two transverse formal non-singular separatrices S_1 and S_2. Then, either $\mathcal{F}$ is reduced or $\mathrm{C}(\mathcal{F})$ has some element different from S_1 and S_2.*

Proof of Lemmas 23 and 24. Lemmas 23 and 24 are clearly true for reduced singularities. Suppose that they hold for foliations whose resolution is achieved with at most $k \geq 0$ blow-ups. Let $\mathcal{F}$ be as in Lemma 23 and suppose that the resolution of $\mathcal{F}$ is achieved with $k+1$ blow-ups. Do a blow-up at p, let D be the exceptional divisor and let $\tilde{\mathcal{F}}$ be the strict transform of $\mathcal{F}$. Then S is a formal non-singular separatrix of $\tilde{\mathcal{F}}$, which is transverse to D at a point $\tilde{p}$. We can assume that D is invariant by $\tilde{\mathcal{F}}$, otherwise the foliation would be dicritical and the set $\mathrm{C}(\mathcal{F})$ would be infinite. Suppose that there exists a singularity of $\tilde{\mathcal{F}}$ at a point $q \in D \backslash \{\tilde{p}\}$. Then, by the inductive hypothesis, Lemma 23 holds for q and therefore we find a characteristic curve γ of $\tilde{\mathcal{F}}$ at q different from (D, q). Then γ defines a characteristic curve of $\mathcal{F}$ different from S. Thus, we can assume that $\tilde{p}$ is the only singularity of $\tilde{\mathcal{F}}$ in D. By the inductive hypothesis, Lemma 24 holds for $\tilde{p}$ and therefore we have the following two cases:

(1) $\tilde{\mathcal{F}}$ is non-reduced at $\tilde{p}$. In this case, by the inductive hypothesis we have that Lemma 24 holds for $\tilde{p}$ and therefore we find a characteristic curve γ of $\tilde{\mathcal{F}}$ at $\tilde{p}$ different from S and $(D, \tilde{p})$. Therefore γ defines a characteristic curve of $\mathcal{F}$ different from S.
(2) $\tilde{\mathcal{F}}$ is reduced at $\tilde{p}$. Then, since $\tilde{p}$ is the only singularity of $\tilde{\mathcal{F}}$ in D we have that the holonomy associated to $(D, \tilde{p})$ is trivial and the Camacho-Sad index $\mathrm{CS}(\tilde{\mathcal{F}}, D, \tilde{p})$ is equal

to -1. Recall the following well known property: if we have a foliation of algebraic multiplicity ν at p and the blow-up of p produces an invariant exceptional divisor with k singularities which are non-degenerate, then $k = \nu + 1$. Thus, if $\tilde{p}$ were non-degenerate, the algebraic multiplicity at p is zero and therefore $\mathcal{F}$ is a regular at p — a contradiction. Then assume that $\tilde{p}$ is a saddle-node. Since the Camacho-Sad index of the strong separatrix of a saddle-node is null, we deduce that $(D, \tilde{p})$ is the weak separatrix of $\tilde{p}$. Then, as we have seen in Example 8, there exists a a curve γ outside D representing the characteristic curve defined by $(D, \tilde{p})$ at $(\tilde{\mathcal{F}}, \tilde{p})$. Then γ defines a characteristic curve of $(\mathcal{F}, p)$ different from S.

Suppose now that $\mathcal{F}$ is a singularity as in Lemma 24 whose resolution process is done with $k+1$ blow-ups. As before, let D be the exceptional divisor of the blow-up at p —clearly we can again assume that D is invariant by $\tilde{\mathcal{F}}$— and let $\tilde{\mathcal{F}}$ be the strict transform of $\mathcal{F}$. Then S_1 and S_2 become transverse to D at points $\tilde{p}_1$ and $\tilde{p}_2$, respectively. Suppose that there exists a singularity of $\tilde{\mathcal{F}}$ at a point $q \in D\backslash\{\tilde{p}_1, \tilde{p}_2\}$. Then, by the inductive hypothesis we find a characteristic curve at q different from (D, q), which gives a characteristic curve of $\mathcal{F}$ different from S_1 and S_2. Therefore we assume that $\tilde{p}_1$ and $\tilde{p}_2$ are the only singularities of $\tilde{\mathcal{F}}$ in D. By the inductive hypothesis, Lemma 24 holds for $\tilde{p}_1$ and $\tilde{p}_2$. Then, if $\tilde{p}_1$ or $\tilde{p}_2$ is non-reduced we find a characteristic curve γ of $\tilde{\mathcal{F}}$ different from S_1, S_2 and D, which give us a characteristic curve of $\mathcal{F}$ different from S_1 and S_2. Thus, we assume that $\tilde{p}_1$ and $\tilde{p}_2$ are reduced. If $\tilde{p}_1$ and $\tilde{p}_2$ were non-degenerate, we easily see that $\mathcal{F}$ is necessarily a reduced non-degenerate singularity, so we can suppose that $\tilde{p}_1$ is a saddle-node. If $(D, \tilde{p}_1)$ is the weak separatrix of $\tilde{p}_1$, by Example 8 there exists a curve γ outside D representing the characteristic curve defined by $(D, \tilde{p}_1)$. Then γ defines a characteristic curve of $\mathcal{F}$ different from S_1 and S_2. Thus we can assume that $(D, \tilde{p}_1)$ is the strong separatrix of $\tilde{p}_1$. Since

$$\mathrm{CS}(\tilde{\mathcal{F}}, D, \tilde{p}_1) + \mathrm{CS}(\tilde{\mathcal{F}}, D, \tilde{p}_2) = -1,$$

we have that $\mathrm{CS}(\tilde{\mathcal{F}}, D, \tilde{p}_2) = -1$. Then, if $\tilde{p}_2$ were a saddle-node, $(D, \tilde{p}_2)$ is its weak separatrix and, as above, we find a curve γ outside D representing the characteristic curve defined by $(D, \tilde{p}_2)$, which defines a characteristic curve of $\mathcal{F}$ different from S_1 and S_2. Then we can assume that $\tilde{p}_2$ is non-degenerate and therefore we deduce that $\mathcal{F}$ is a saddle-node, that is, $\mathcal{F}$ is reduced. □

Proof of Theorem 22. Assume that π is a resolution of $\mathcal{F}$. Then clearly π is a resolution of the set of formal separatrices of $\mathcal{F}$. Let γ be a tangent characteristic curve of $\mathcal{F}$. Let E be the exceptional divisor of π and denote by $\tilde{\mathcal{F}}$ the strict transform of $\mathcal{F}$ by π. By Lemma 17 the curve γ tends to a singularity $q \in E$ of the foliation $\tilde{\mathcal{F}}$, so γ is a characteristic curve of $\tilde{\mathcal{F}}$ at q. Thus, by Proposition 15 the curve γ has flat contact with a separatrix $\tilde{S}$ of $\tilde{\mathcal{F}}$ at q. Since γ has a flat contact with no separatrix of $\mathcal{F}$, we necessarily have that $\tilde{S}$ is contained in the exceptional divisor E. Therefore π is a resolution of $\mathrm{C}(\mathcal{F})$. Assume now that π is a resolution of $\mathrm{C}(\mathcal{F})$ and let q be any point in E. Suppose first that $\tilde{\mathcal{F}}$ is regular at q and let L be the corresponding leaf passing through q. We must prove that L is either contained in E or transverse to E at q. If L is not contained in E and is tangent to E, then (L, q) define a separatrix of $\mathcal{F}$, which clearly is not desingularized — this is a contradiction. Suppose now that q is a singularity of $\tilde{\mathcal{F}}$ and assume first that q is a corner point of the exceptional divisor E, that is, q is the intersection of two components D_1 and D_2 of E. In view of Lemma 24, in order to prove that q is reduced will be sufficient to show that there are no characteristic curves of $(\tilde{\mathcal{F}}, q)$ different from (D_1, q) and (D_2, q). Suppose that there exist one such characteristic curve and denote it by γ. Then γ defines a characteristic curve of $(\mathcal{F}, p)$. Clearly γ can not be a formal separatrix of $\mathcal{F}$ because π is a resolution of $\hat{\mathrm{Sep}}(\mathcal{F})$ and γ tends to the corner point q. Then γ is a tangent characteristic curve of $\mathcal{F}$ and, since π is a resolution of $\mathrm{C}(\mathcal{F})$, we have that γ has a flat contact to some component of the exceptional divisor at q — either D_1 or D_2. Then, as a characteristic curve of $(\tilde{\mathcal{F}}, q)$, the curve γ is equivalent to (D_1, q) or (D_2, q), which is a contradiction. Assume now that

q is not a corner point, so q is contained in a unique component D of the exceptional divisor E. By Lemma 23 there exists a characteristic curve γ of $(\tilde{\mathcal{F}}, q)$ different from (D, q) —that means that γ has not a flat contact with D at q. Then γ defines a characteristic curve of $(\mathcal{F}, p)$ which is not a tangent one, that is, γ has a flat contact with some formal separatrix S of $(\mathcal{F}, p)$. Thus, since π is a resolution of $\mathrm{C}(\mathcal{F})$, we have that S is formal non-singular separatrix of $(\tilde{\mathcal{F}}, q)$ which is transverse to D. Then, if we suppose that q is non-reduced, it follows from Lemma 24 that there exists a characteristic curve γ' of $(\tilde{\mathcal{F}}, q)$ different from S and (D, q). Then γ' defines a characteristic curve of $(\mathcal{F}, p)$. If γ' were a formal separatrix of $(\mathcal{F}, p)$, since γ' tends to q and π is a resolution of $\mathrm{C}(\mathcal{F})$, we will have necessarily $\gamma' = S$, which is a contradiction. Then γ' is a tangent characteristic curve of $\mathcal{F}$ and, since π is a resolution of $\mathrm{C}(\mathcal{F})$, we have that γ' has a flat contact with (D, q), which is again a contradiction. □

6. Smooth conjugations

The main application of Theorem 1 in [1] is the following equiresolution theorem.

Theorem 25. *Let $\mathcal{F}$ and $\mathcal{F}'$ be topologically equivalent germs of holomorphic foliations at $(\mathbb{C}^2, 0)$. Suppose that $\mathcal{F}$ is a generalized curve. Then $\mathcal{F}'$ is also a generalized curve. Besides, $\mathcal{F}$ and $\mathcal{F}'$ are equisingular, that is, they have isomorphic resolutions.*

After proving that $\mathcal{F}'$ is also a generalized curve [1], the proof of Theorem 25 finishes as follows: since the sets $\mathrm{Sep}(\mathcal{F})$ and $\mathrm{Sep}(\mathcal{F}')$ of convergent separatrices are topologically equivalent, we have that they are equisingular [9], so the equisingularity of the foliations follows from Theorem 1. The main difficulty to extend Theorem 25 beyond the class of generalized curves is that, in general, a resolution of $\mathrm{Sep}(\mathcal{F})$ is not necessarily a resolution of $\mathcal{F}$. In view of Theorem 2, a generalization of Theorem 25 involve the problem of proving that topologically equivalent foliations have equisingular sets of characterictic curves. Of course this approach put the initial task of proving that a characteristic curve is a topological invariant, which is a difficult problem. In this section we discus the more accessible case of smooth equivalences of foliations. This problem has been already studied in [6] for the class of second type foliations. Recall that these are the foliations that after resolution have no saddle-node with weak separatrix contained in the exceptional divisor, that is, the foliations having no tangent saddle-nodes after resolution. Then, if $\mathcal{F}$ is a second type singularity, the set $\mathrm{C}(\mathcal{F})$ contains no tangent characteristic curves and therefore we have

$$\mathrm{C}(\mathcal{F}) = \hat{\mathrm{Sep}}(\mathcal{F}).$$

Thus, in [6] the following theorem is proved:

Theorem 26. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two germs of holomorphic foliations at $(\mathbb{C}^2, 0)$ equivalent by a germ of C^∞ diffeomorphism. Then $\hat{\mathrm{Sep}}(\mathcal{F})$ and $\hat{\mathrm{Sep}}(\mathcal{F}')$ are equisingular.*

Since a second type foliation has the same resolution as its set of formal separatrices, Theorem 26 easily implies the following equiresolution Theorem [6]:

Theorem 27. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two germs of holomorphic foliations at $(\mathbb{C}^2, 0)$ equivalent by a germ of C^∞ diffeomorphism. Suppose that $\mathcal{F}$ is a foliation of second type. Then $\mathcal{F}'$ is of second type. Moreover, $\mathcal{F}$ and $\mathcal{F}'$ are equisingular.*

The term "characteristic curve" is already introduced in [6] in a more restrictive way: a characteristic curve in [6] is an end $\gamma(t)$ of the foliation having a non-null Taylor series at $t = 0$. These characteristic curves corresponds to our non-tangent characteristic curves —the characteristic curves becoming from formal separatrices. In our nomenclature, the key result in the proof of Theorem 26 can be stated as follows:

Theorem 28. [6]*Let $\mathcal{F}$ and $\mathcal{F}'$ be two germs of holomorphic foliations at $(\mathbb{C}^2, 0)$ equivalent by a germ h of C^∞ diffeomorphism. Let γ be a non-tangent characteristic curve of $\mathcal{F}$. Then $h(\gamma)$ is a non-tangent characteristic curve of $\mathcal{F}'$.*

Thus, in order to extend Theorem 27 to all foliations, the following plausible conjecture can have a key role:

Conjecture 29. *Let $\mathcal{F}$ and $\mathcal{F}'$ be two germs of holomorphic foliations at $(\mathbb{C}^2, 0)$ equivalent by a germ h of C^∞ diffeomorphism. Let γ be a tangent characteristic curve of $\mathcal{F}$. Then $h(\gamma)$ is a tangent characteristic curve of $\mathcal{F}'$.*

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