

ROUND FOLD MAPS OF n -DIMENSIONAL MANIFOLDS INTO $(n - 1)$ -DIMENSIONAL EUCLIDEAN SPACE

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ABSTRACT. We determine those smooth closed n -dimensional manifolds with n greater than or equal to 4 which admit round fold maps into $(n - 1)$ -dimensional Euclidean space; i.e. fold maps whose critical value sets consist of disjoint spheres of dimension $n - 2$ isotopic to concentric spheres. We also classify such round fold maps up to a certain natural equivalence relation.

1. INTRODUCTION

Let M be a smooth closed manifold of dimension $n \geq 2$. A smooth map $f : M \rightarrow \mathbb{R}^p$ with $n \geq p \geq 1$ is called a *fold map* if it has only fold points as its singularities (for details, see §2). Note that fold points are the simplest singularities among those which appear generically [9] and that fold maps are natural generalizations of Morse functions.

In [20, 22, 23, 24], the second author considered the following smaller class of generic smooth maps. A fold map $f : M \rightarrow \mathbb{R}^p$ is *simple* if for every $q \in \mathbb{R}^p$, each component of $f^{-1}(q)$ contains at most one singular point. In particular, if $f|_{S(f)}$ is an embedding, then f is simple, where $S(f) \subset M$ denotes the set of singular points of f . Note that if f is a fold map, then $S(f)$ is a regular closed submanifold of M of dimension $p - 1$ and that $f|_{S(f)}$ is an immersion in general. In [24], the second author proved that a closed orientable 3-dimensional manifold M admits a fold map $f : M \rightarrow \mathbb{R}^2$ such that $f|_{S(f)}$ is an embedding if and only if M is a graph manifold, where a closed orientable 3-dimensional manifold is a graph manifold if it is the finite union of S^1 -bundles over compact surfaces attached along their torus boundaries. Thus, for example, if M is hyperbolic, then M never admits such a fold map, although every closed orientable 3-dimensional manifold admits a fold map into \mathbb{R}^2 by [16].

On the other hand, the first author introduced the notion of a round fold map [12, 13, 14]: a smooth map $f : M \rightarrow \mathbb{R}^p$ is a *round fold map* if it is a fold map and $f|_{S(f)}$ is an embedding onto the disjoint union of some concentric $(p - 1)$ -dimensional spheres in \mathbb{R}^p (for details, see §2). Round fold maps are naturally simple. As has been studied by the first author, round fold maps have various nice properties.

The main result of this paper is Theorem 2.5, which characterizes those smooth closed n -dimensional manifolds that admit round fold maps into \mathbb{R}^{n-1} for $n \geq 4$. We also classify such round fold maps up to C^∞ \mathcal{A} -equivalence (see Theorems 5.4 and 5.7).

The paper is organized as follows. In §2, we prepare several definitions concerning round fold maps together with some examples, and state our main theorem. In §3, we prove the main theorem mentioned above. The main ingredients are the celebrated results about the homotopy groups of diffeomorphism groups of compact surfaces [4, 5, 6, 10]. We also give a similar characterization of those smooth orientable closed connected n -dimensional manifolds that admit directed round fold maps into \mathbb{R}^{n-1} , where a fold map is *directed* if the number of regular fiber components increases toward the central region of \mathbb{R}^{n-1} (for details, see §3). In §4, we give some related results and remarks. Finally in §5, we classify the round fold maps

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as in Theorem 2.5 up to C^∞ \mathcal{A} -equivalence using Morse functions on compact surfaces. The key idea for the classification is to use some results about the homotopy type of the group of diffeomorphisms of compact surfaces that preserve a given Morse function, due to Maksymenko [17, 18].

Throughout the paper, all manifolds and maps between them are smooth of class C^∞ unless otherwise specified. For a space X , id_X denotes the identity map of X . The symbol “ \cong ” denotes a diffeomorphism between smooth manifolds.

2. ROUND FOLD MAPS

Let M be a closed n -dimensional manifold and $f : M \rightarrow \mathbb{R}^p$ a smooth map, where we assume $n \geq p \geq 2$. By the *codimension* of f we mean the integer $p - n \leq 0$.

Definition 2.1. A point $q \in M$ is a *singular point* of f if the rank of the differential $df_q : T_q M \rightarrow T_{f(q)} \mathbb{R}^p$ is strictly smaller than p . We denote by $S(f)$ the set of all singular points of f . A point $q \in S(f)$ is a *fold point* if f is represented by the map

$$(x_1, x_2, \dots, x_n) \mapsto (x_1, x_2, \dots, x_{p-1}, \pm x_p^2 \pm x_{p+1}^2 \pm \dots \pm x_n^2)$$

around the origin with respect to certain local coordinates around q and $f(q)$. Let λ be the number of negative signs appearing in the above expression. The integer

$$\max\{\lambda, n - p + 1 - \lambda\} \in \{[(n - p + 1)/2], [(n - p + 1)/2] + 1, \dots, n - p + 1\}$$

is called the *absolute index* of the fold point q , which is known to be well-defined, where for $x \in \mathbb{R}$, $\lceil x \rceil$ denotes the minimum integer greater than or equal to x . We call a point $q \in S(f)$ a *definite fold point* if its absolute index is equal to $n - p + 1$, otherwise an *indefinite fold point*.

A smooth map $f : M \rightarrow \mathbb{R}^p$ is called a *fold map* if it has only fold points as its singular points. Note that then $S(f)$ is a closed $(p - 1)$ -dimensional submanifold of M and that $f|_{S(f)}$ is an immersion.

Note also that if $p = n - 1$, then the absolute index of a fold point is equal either to 1 or to 2.

Definition 2.2. Let C be a finite disjoint union of embedded $(p - 1)$ -dimensional spheres in \mathbb{R}^p , $p \geq 2$. We say that C is *concentric* if each component bounds a p -dimensional disk in \mathbb{R}^p and for every pair c_0, c_1 of distinct components of C , exactly one of them, say c_i , is contained in the bounded region of $\mathbb{R}^p \setminus c_{1-i}$ (see Fig. 1 for $p = 2$). (In this case, we say that c_i (or c_{1-i}) is an *inner component* (resp. an *outer component*) with respect to c_{1-i} (resp. c_i).) In other words, C is isotopic to a set of concentric $(p - 1)$ -dimensional spheres in \mathbb{R}^p . (Note that the condition that each component of C bounds a p -dimensional disk is redundant for $p \neq 4$.)

Definition 2.3. We say that a smooth map $f : M \rightarrow \mathbb{R}^p$ of a closed n -dimensional manifold M into the p -dimensional Euclidean space is a *round fold map* if it is a fold map and $f|_{S(f)}$ is an embedding onto a concentric family of embedded spheres. Note that a round fold map is a simple stable map in the sense of [20, 22, 23, 24]. Note also that the outermost component of $f(S(f))$ consists of the images of definite fold points.

Example 2.4. Let F be a compact connected m -dimensional manifold possibly with boundary and $h : F \rightarrow [1/2, \infty)$ a Morse function such that $h(\partial F) = 1/2$ and that h has no critical point near the boundary. (Throughout the paper, a Morse function is a smooth function whose critical points are all non-degenerate and have distinct critical values.) Then for $p \geq 2$ we can construct a round fold map $f : M \rightarrow \mathbb{R}^p$ in such a way that M is the closed $(m + p - 1)$ -dimensional manifold $(\partial F \times D^p) \cup (F \times \partial D^p) = \partial(F \times D^p)$, that f restricted to $F \times \{x\}$ can be identified with the Morse function h to the half line emanating from the origin and passing through x for each $x \in \partial D^p$, and that f restricted to $\partial F \times D^p$ is the projection to the second factor multiplied

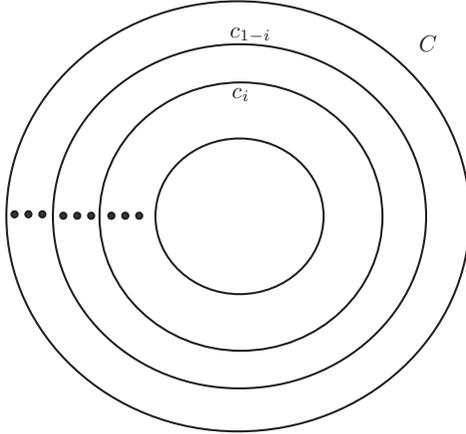


FIGURE 1. Family of concentric 1-dimensional spheres in \mathbb{R}^2

by $1/2$, where D^p is the unit disk in \mathbb{R}^p centered at the origin. Such a round fold map f is said to be the result of a *trivial open book construction* with respect to the Morse function h .

Note that the source manifold M has a natural open book structure with binding $\partial F \times \{0\}$.

Let S^{n-1} be the unit sphere centered at the origin in \mathbb{R}^n and $\gamma : S^{n-1} \rightarrow S^{n-1}$ the orientation reversing diffeomorphism defined by

$$\gamma(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots, x_n)$$

for $(x_1, x_2, \dots, x_n) \in S^{n-1}$. In the following, we denote the total space of the non-orientable S^{n-1} -bundle over S^1 with monodromy γ ,

$$[0, 1] \times S^{n-1} / (1, q) \sim (0, \gamma(q)),$$

by $S^1 \tilde{\times} S^{n-1}$.

Furthermore, we denote by $S^2 \tilde{\times} S^2$ the total space of the unique non-trivial S^2 -bundle over S^2 .

The main theorem of this paper is the following.

Theorem 2.5. *A closed connected n -dimensional manifold with $n \geq 4$ admits a round fold map into \mathbb{R}^{n-1} if and only if it is diffeomorphic to one of the following manifolds:*

- (1) *standard n -dimensional sphere S^n ,*
- (2) *a connected sum of finite numbers of copies of $S^1 \times S^{n-1}$ or $S^1 \tilde{\times} S^{n-1}$,*
- (3) *$S^{n-2} \times \Sigma$ for a closed connected surface Σ ,*
- (4) *$S^2 \tilde{\times} S^2$ for $n = 4$.*

3. PROOF OF THEOREM 2.5

In this section, we prove Theorem 2.5.

Proof of Theorem 2.5. Let $f : M \rightarrow \mathbb{R}^{n-1}$ be a round fold map of a closed connected n -dimensional manifold with $n \geq 4$. In the following, for $r > 0$, C_r denotes the $(n-2)$ -dimensional sphere of radius r centered at the origin in \mathbb{R}^{n-1} . We may assume that

$$f(S(f)) = \bigcup_{r=1}^s C_r$$

for some $s \geq 1$ by composing a diffeomorphism of \mathbb{R}^{n-1} if necessary. Set $K = f^{-1}(0)$, which is a closed submanifold of dimension 1 of M if it is not empty. Let D be the closed $(n-1)$ -dimensional disk centered at the origin with radius $1/2$. Then, $f^{-1}(D)$ is diffeomorphic to $K \times D$, which can be identified with a tubular neighborhood $N(K)$ of K in M . Furthermore, the composition $\rho = \pi \circ f : M \setminus \text{Int } N(K) \rightarrow S^{n-2}$ is a submersion, where $\pi : \mathbb{R}^{n-1} \setminus \text{Int } D \rightarrow S^{n-2}$ is the standard radial projection and $\rho|_{\partial N(K)} : \partial N(K) = K \times \partial D \rightarrow S^{n-2}$ corresponds to the projection to the second factor followed by a scalar multiplication. Hence, ρ is a smooth fiber bundle. In other words, M admits an open book structure with binding K . The fiber (or the page) is identified with $F = f^{-1}(J)$, where

$$J = [1/2, \infty) \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{n-2} = \mathbb{R}^{n-1},$$

and it is a compact surface possibly with boundary. As we are assuming that M is connected and $n \geq 4$, so is F . Note that $h = f|_F : F \rightarrow J$ is a Morse function with exactly s critical points.

Note that all these arguments work even when $K = \emptyset$. In this case, F is a closed connected surface and M is the total space of an F -bundle over S^{n-2} .

In the following, A denotes the annulus $S^1 \times [-1, 1]$, and P denotes the compact surface obtained from the 2-sphere by removing three open disks: in other words, P is a *pair of pants*. Furthermore, B denotes the compact surface obtained from the Möbius band with an open disk removed.

Set $M_r = f^{-1}(C_{[r-(1/2), r+(1/2)]})$, $r = 1, 2, \dots, s$, and $M_0 = f^{-1}(C_{[0, 1/2]})$, where

$$C_{[a, b]} = \{(x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid a \leq \sqrt{x_1^2 + x_2^2 + \dots + x_{n-1}^2} \leq b\}$$

for $0 \leq a < b$. Note that for $r \geq 1$, the map $\rho|_{M_r} : M_r \rightarrow S^{n-2}$ is a smooth fiber bundle and the fiber is diffeomorphic to D^2 , P or B together with a finite number of copies of A (see [25], for example).

Now, suppose that $n \geq 5$. Then by [4, 5, 6, 10], the identity component of the group of diffeomorphisms of D^2 , P , B and A all have vanishing homotopy groups of dimension $n-3$. Therefore, the above bundles are all trivial. Furthermore, for obtaining $M \setminus \text{Int } N(K)$, we need to glue the pieces by bundle maps with fiber a disjoint union of circles. Again, as the identity component of the group of diffeomorphisms of S^1 has vanishing homotopy group of dimension $n-2$, we see that the fiber bundle $\rho : M \setminus \text{Int } N(K) \rightarrow S^{n-2}$ is trivial. Moreover, the diffeomorphism used for attaching $N(K)$ to $M \setminus \text{Int } N(K)$ is again standard. As a result we see that M is diffeomorphic to the union

$$(K \times D^{n-1}) \cup_{\partial} (F \times S^{n-2}) = (\partial F \times D^{n-1}) \cup_{\partial} (F \times \partial D^{n-1}),$$

where the attaching diffeomorphism is the standard one. This implies that M is diffeomorphic to $\partial(F \times D^{n-1})$.

If F has no boundary, then M is diffeomorphic to $F \times S^{n-2}$, where F is a closed connected surface. If F has non-empty boundary, then $F \times D^{n-1}$ is diffeomorphic to an $(n+1)$ -dimensional manifold obtained by attaching some 1-handles to D^{n+1} . Therefore, its boundary is diffeomorphic to the connected sum of finite numbers of copies of $S^1 \times S^{n-1}$ or $S^1 \tilde{\times} S^{n-1}$.

Now suppose that $n = 4$. In this case, P -bundles and B -bundles over S^2 are all trivial, while D^2 -bundles and A -bundles may possibly be non-trivial. If F is a closed connected surface, then M is diffeomorphic to the total space of an F -bundle over S^2 . If F is diffeomorphic to S^2 , then we see that M is diffeomorphic either to $S^2 \times S^2$ or to $S^2 \tilde{\times} S^2$. If F is not diffeomorphic to S^2 , then a P -bundle piece or a B -bundle piece necessarily appears, and such a bundle must be trivial. Since the boundary S^1 -bundle (or S^1 -bundles) of an arbitrary non-trivial D^2 -bundle (resp. A -bundle) over S^2 is (resp. are) always non-trivial, no such non-trivial bundle appears. This implies that the F -bundle over S^2 must be trivial. Therefore, M is diffeomorphic to $S^2 \times \Sigma$ for a closed connected surface Σ .

If F has non-empty boundary, then as $N(K)$ is a trivial bundle over D , the boundary of $M \setminus \text{Int } N(K)$ is a trivial ∂F -bundle over S^2 . Then by [6], we see that the F -bundle over S^2 , $\rho : M \setminus \text{Int } N(K) \rightarrow S^2$, is trivial. Therefore, by an argument similar to the above, we see that M is diffeomorphic to the connected sum of finite numbers of copies of $S^1 \times S^3$ and $S^1 \widetilde{\times} S^3$.

Conversely, suppose that M is one of the manifolds listed in the theorem.

If M is the standard n -dimensional sphere S^n , then the standard projection $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n-1}$ restricted to the unit sphere S^n is a round fold map: in fact, it is a so-called special generic map (see [21], for example), and has only definite fold as its singularities.

If M is a connected sum of a copies of $S^1 \times S^{n-1}$ and b copies of $S^1 \widetilde{\times} S^{n-1}$, then let us consider the compact surface with boundary, say F , obtained from the 2-disk by attaching a orientable 1-handles and b non-orientable 1-handles along the boundary. Then,

$$(F \times S^{n-2}) \cup (\partial F \times D^{n-1}) = \partial(F \times D^{n-1})$$

admits a round fold map into \mathbb{R}^{n-1} , as is seen from the construction given in Example 2.4, and this manifold is diffeomorphic to M .

If $M = S^{n-2} \times \Sigma$ for a closed connected surface Σ , then it obviously admits a round fold map into \mathbb{R}^{n-1} , as is seen by using the construction given in Example 2.4 again.

Finally, if $n = 4$ and $M = S^2 \widetilde{\times} S^2$, then it admits a special generic map into \mathbb{R}^3 which is also a round fold map [21].

This completes the proof. \square

Now let us discuss directed round fold maps. Let $f : M \rightarrow \mathbb{R}^{n-1}$ be a round fold of a closed orientable n -dimensional manifold M . For a component c of $f(S(f))$, take a small arc $\alpha \cong [-1, 1]$ in \mathbb{R}^{n-1} that intersects $f(S(f))$ exactly at one point in c transversely. We also assume that the point $\alpha \cap f(S(f))$ is not an end point of α . Then, $f^{-1}(\alpha)$ is a compact surface with boundary $f^{-1}(a) \cup f^{-1}(b)$, which is diffeomorphic to a finite disjoint union of circles, where a and b are the end points of α . Furthermore, $f|_{f^{-1}(\alpha)} : f^{-1}(\alpha) \rightarrow \alpha$ can be regarded as a Morse function with exactly one critical point. As M is orientable, we see that $f^{-1}(\alpha)$ is diffeomorphic to the union of D^2 (or P) and a finite number of copies of A (see [25], for example). Therefore, the number of components of $f^{-1}(a)$ differs from that of $f^{-1}(b)$ exactly by one. If $f^{-1}(a)$ has more components than $f^{-1}(b)$, then we normally orient c from b to a : otherwise, we orient c from a to b . It is easily shown that this normal orientation is independent of the choice of α . In this way, each component of $f(S(f))$ is normally oriented. If the normal orientation points inward, then the component is said to be *inward-directed*: otherwise, *outward-directed*.

Definition 3.1. Let $f : M \rightarrow \mathbb{R}^{n-1}$ be a round fold map of a closed orientable n -dimensional manifold. We say that f is *directed* if all the components of $f(S(f))$ are inward-directed. It is easy to see that a round fold map f is directed if and only if the number of components of a regular fiber over a point in the innermost component of $\mathbb{R}^{n-1} \setminus f(S(f))$ coincides with the number of components of $S(f)$.

Then, as a corollary of the above proof of Theorem 2.5, we have the following.

Theorem 3.2. *A closed connected orientable n -dimensional manifold with $n \geq 4$ admits a directed round fold map into \mathbb{R}^{n-1} if and only if it is diffeomorphic to one of the following manifolds:*

- (1) *standard n -dimensional sphere S^n ,*
- (2) *connected sum of a finite number of copies of $S^1 \times S^{n-1}$.*

Proof. Suppose that $f : M \rightarrow \mathbb{R}^{n-1}$ is a directed round fold map of a closed connected orientable n -dimensional manifold. Then, the surface F given in the proof of Theorem 2.5 must be a compact orientable surface with non-empty boundary of genus zero, since f is directed. Then, we see that M must be diffeomorphic to S^n or to the connected sum of a finite number of copies

of $S^1 \times S^{n-1}$. Conversely, we see that S^n and the connected sum of a finite number of copies of $S^1 \times S^{n-1}$ admit directed round fold maps by using the open book construction described in Example 2.4 associated with an appropriate Morse function on a compact orientable surface of non-empty boundary of genus zero. (More precisely, we use a Morse function on such a surface such that the number of components of level sets increases as the level value decreases.) This completes the proof. \square

For example, for a closed connected orientable surface Σ , the manifold $S^{n-2} \times \Sigma$, $n \geq 4$, admits a round fold map into \mathbb{R}^{n-1} , but does not admit a directed one.

4. REMARKS AND RELATED RESULTS

As a direct corollary to Theorem 2.5, we immediately get the following.

Corollary 4.1. *A closed n -dimensional manifold with $n \geq 4$ homotopy equivalent to S^n admits a round fold map into \mathbb{R}^{n-1} if and only if it is diffeomorphic to the standard n -dimensional sphere.*

Note that by [7], every n -dimensional homotopy sphere admits a fold map into \mathbb{R}^{n-1} .

Remark 4.2. The manifolds appearing in Theorem 2.5 are all null-cobordant. In particular, their Stiefel–Whitney numbers all vanish, and their signatures all vanish when the dimension is divisible by 4 and the manifold is orientable. Compare this with [20, Proposition 3.12].

Remark 4.3. The fundamental groups of the manifolds appearing in Theorem 2.5 are either trivial, free, or a surface fundamental group.

Remark 4.4. In [22, Proposition 3.6], it has been given a topological characterization of simply connected closed 4-dimensional manifolds that admit simple fold maps into \mathbb{R}^3 . Compare this with our Theorem 2.5.

Remark 4.5. We have seen that if a closed connected n -dimensional manifold admits a round fold map into \mathbb{R}^{n-1} , then the manifold admits an open book structure over S^{n-2} .

On the other hand, as our proof shows, for $n \geq 4$, if a closed connected n -dimensional manifold M admits an open book structure over S^{n-2} with non-empty binding, then it must be diffeomorphic to one of the following manifolds:

- (1) standard n -dimensional sphere S^n ,
- (2) a connected sum of finite numbers of copies of $S^1 \times S^{n-1}$ or $S^1 \tilde{\times} S^{n-1}$.

In particular, the manifold M admits a round fold map into \mathbb{R}^{n-1} .

If M admits an open book structure over S^{n-2} with empty binding, then the manifold is the total space of a Σ -bundle over S^{n-2} for some closed connected surface Σ . Recall that by [4, 5, 6, 10], we have $\pi_{n-3}(\text{Diff}_0(\Sigma)) = 0$ except for the following cases, where $\text{Diff}_0(\Sigma)$ denotes the identity component of the group of diffeomorphisms of Σ :

- (a) $\Sigma = S^2$: $\pi_{n-3}(SO(3))$,
- (b) $\Sigma = T^2$ and $n = 4$: $\pi_1(T^2) \cong \mathbb{Z} \oplus \mathbb{Z}$,
- (c) $\Sigma = K^2$ and $n = 4$: $\pi_1(SO(2)) \cong \mathbb{Z}$,

where T^2 is the 2-dimensional torus and K^2 is the Klein bottle. In case (a), M is the total space of an S^2 -bundle over S^{n-2} . In case (b) (resp. (c)), M is the total space of a T^2 -bundle (resp. K^2 -bundle) over S^2 . Otherwise, M is diffeomorphic to $\Sigma \times S^{n-2}$ and it admits a round fold map into \mathbb{R}^{n-1} .

Remark 4.6. For $n = 3$, a characterization of closed *orientable* 3-dimensional manifolds that admit round fold maps into \mathbb{R}^2 has been obtained in [15]. For non-orientable 3-dimensional manifolds, such a characterization has not been known, as far as the authors know.

Those closed orientable 3-dimensional manifolds which admit directed round fold maps into \mathbb{R}^2 have also been characterized in [15].

Let $f : M \rightarrow \mathbb{R}^{n-1}$ be a round fold map of a closed connected n -dimensional manifold M . We denote by $n_0(f)$ (resp. $n_1(f)$) the number of connected components of $S(f)$ with absolute index 2 (resp. 1).

Proposition 4.7. *When $n = \dim M$ is even, the Euler characteristic of M is equal to*

$$2(n_0(f) - n_1(f)).$$

Proof. We may assume that $f(S(f))$ is a concentric family of spheres in \mathbb{R}^{n-1} . Then, $\ell \circ f : M \rightarrow \mathbb{R}$ is a Morse function on M , where ℓ is a projection to a real line. We see that the number of critical points of $\ell \circ f$ with even indices equals to $2n_0(f)$ and that of odd indices equals to $2n_1(f)$. Then, the result follows. \square

In the above proof, we see that the indices of the critical points of the Morse function $\ell \circ f$ are 0, 1, 2, $n-2$, $n-1$ or n . In particular, $b_3(M) = b_4(M) = \dots = b_{n-3}(M) = 0$ when $n \geq 6$, where $b_j(M)$ is the j -th Betti number of M (for any coefficient).

Remark 4.8. Proposition 4.7 is also a consequence of a theorem of Fukuda [8].

Example 4.9. Let us consider an arbitrary $(n-2)$ -dimensional closed connected manifold X which embeds into \mathbb{R}^{n-1} , $n \geq 4$, such that the closed connected n -dimensional manifold $M = \Sigma \times X$ is not diffeomorphic to any manifold listed in Theorem 2.5, where Σ is a closed connected surface not diffeomorphic to S^2 . (For example, consider $X = S^1 \times S^{n-3}$.) Then, $M = \Sigma \times X$ admits a fold map $f : M \rightarrow \mathbb{R}^{n-1}$ such that $f|_{S(f)}$ is an embedding onto a union of parallel copies of X embedded in \mathbb{R}^{n-1} . (For example, for an arbitrary Morse function $h : \Sigma \rightarrow \mathbb{R}$, consider the composition

$$M = \Sigma \times X \xrightarrow{h \times \text{id}_X} \mathbb{R} \times X \hookrightarrow \mathbb{R}^{n-1},$$

where the last map is an embedding.) However, M does not admit a round fold map into \mathbb{R}^{n-1} according to our Theorem 2.5.

Recall that for $n = 3$, if a closed orientable 3-dimensional manifold admits a simple fold map into \mathbb{R}^2 , then it also admits a round fold map into \mathbb{R}^2 as has been shown in [15]. The above example shows that this is not the case for $n \geq 4$ in general.

5. CLASSIFICATION OF ROUND FOLD MAPS

In this section, we consider the classification problem of round fold maps of closed n -dimensional manifolds into \mathbb{R}^{n-1} , $n \geq 4$.

We recall the following standard definition.

Definition 5.1. Let $f_i : M_i \rightarrow N_i$ be smooth maps of smooth manifolds, $i = 0, 1$. We say that f_0 and f_1 are C^∞ \mathcal{A} -equivalent if there exist diffeomorphisms $\psi : M_0 \rightarrow M_1$ and $\Psi : N_0 \rightarrow N_1$ such that $f_1 = \Psi \circ f_0 \circ \psi^{-1}$.

Furthermore, we say that f_0 and f_1 are C^∞ \mathcal{R} -equivalent if $N_0 = N_1$ and there exists a diffeomorphism $\psi : M_0 \rightarrow M_1$ such that $f_1 = f_0 \circ \psi^{-1}$.

For our classification of round fold maps up to \mathcal{A} -equivalence, we will need the following.

Definition 5.2. Let $f : M \rightarrow \mathbb{R}^{n-1}$ be a round fold map of a closed connected n -dimensional manifold. Then, there exists a diffeomorphism $\Phi : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ such that the critical value set of the round fold map $\Phi \circ f$ coincides with $\cup_{r=1}^s C_r$, where s is the number of connected components of $S(f)$. For $J = [1/2, \infty) \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{n-2} = \mathbb{R}^{n-1}$, we set $F = (\Phi \circ f)^{-1}(J)$, which is a compact surface and is connected if $n \geq 4$. Then the restriction $\Phi \circ f|_F : F \rightarrow J$ is a Morse function. We call F a *page* of f and the function $\Phi \circ f|_F$ a *page Morse function* associated with f .

Lemma 5.3. *Let $f : M \rightarrow \mathbb{R}^{n-1}$ be a round fold map of a closed connected n -dimensional manifold. Then, the page Morse functions associated with f are unique up to $C^\infty \mathcal{R}$ -equivalence.*

Proof. Let $\Phi_i : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$ be diffeomorphisms such that the critical value set of the round fold map $\Phi_i \circ f$ coincides with $\cup_{r=1}^s C_r$, where s is the number of connected components of $S(f)$, $i = 0, 1$. Set $J_0 = J$, $J_1 = \Phi_0 \circ \Phi_1^{-1}(J_0)$, $F_i = (\Phi_0 \circ f)^{-1}(J_i)$ and $h_i = \Phi_0 \circ f|_{F_i}$, $i = 0, 1$. Note that $h_0 : F_0 \rightarrow J_0$ is a page Morse function associated with f , while $F_1 = (\Phi_1 \circ f)^{-1}(J_0)$ is another page for f and $\Phi_1 \circ \Phi_0^{-1} \circ h_1 : F_1 \rightarrow J_0$ is another page Morse function associated with f .

By integrating a certain vector field tangent to $\cup_{r=1}^s C_r$, we can construct a smooth isotopy $H_t : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}$, $t \in [0, 1]$, with the following properties:

$$\begin{aligned} (5.1) \quad H_0 &= \text{id}_{\mathbb{R}^{n-1}}, \\ (5.2) \quad H_t(C_r) &= C_r, \quad r = 1, 2, \dots, s, \\ (5.3) \quad H_1(J_0) &= J_1, \\ (5.4) \quad H_1|_{J_0} &= \Phi_0 \circ \Phi_1^{-1}|_{J_0}. \end{aligned}$$

Then, by lifting the vector field generated by the isotopy $\{H_t\}_{t \in [0, 1]}$ with respect to $\Phi_0 \circ f$, we can construct a smooth isotopy $\varphi_t : M \rightarrow M$, $t \in [0, 1]$, such that the following holds:

$$\begin{aligned} (5.5) \quad \varphi_0 &= \text{id}_M, \\ (5.6) \quad \varphi_t(S(f)) &= S(f), \\ (5.7) \quad \varphi_1(F_0) &= F_1, \\ (5.8) \quad H_t \circ \Phi_0 \circ f &= \Phi_0 \circ f \circ \varphi_t, \quad t \in [0, 1]. \end{aligned}$$

Then, we see that

$$\begin{aligned} \Phi_0 \circ f \circ \varphi_1|_{F_0} &= H_1 \circ \Phi_0 \circ f|_{F_0} \\ &= \Phi_0 \circ \Phi_1^{-1} \circ \Phi_0 \circ f|_{F_0} \end{aligned}$$

by virtue of (5.8) and (5.4) above. This implies that

$$\Phi_1 \circ f \circ \varphi_1|_{F_0} = \Phi_0 \circ f|_{F_0}.$$

Hence, the two page Morse functions associated with f are $C^\infty \mathcal{R}$ -equivalent. \square

Then, we have the following classification theorem.

Theorem 5.4. *Let $f_i : M_i \rightarrow \mathbb{R}^{n-1}$ be round fold maps of closed connected n -dimensional manifolds with $n \geq 5$, $i = 0, 1$. Then f_0 and f_1 are $C^\infty \mathcal{A}$ -equivalent if and only if their page Morse functions are $C^\infty \mathcal{R}$ -equivalent.*

Proof. Necessity follows easily from Lemma 5.3.

Conversely, suppose that the page Morse functions of f_i , $i = 0, 1$, are $C^\infty \mathcal{R}$ -equivalent. We may assume that $f_i(S(f_i))$, $i = 0, 1$, are of the form $\cup_{r=1}^s C_r$. Set $J = [1/2, \infty) \times \{0\} \subset \mathbb{R} \times \mathbb{R}^{n-2}$, $F_i = f_i^{-1}(J)$ and $h_i = f_i|_{F_i} : F_i \rightarrow J$, $i = 0, 1$. By assumption, there exists a diffeomorphism $\varphi : F_0 \rightarrow F_1$ such that $f_0 = f_1 \circ \varphi$.

Let us consider the decomposition

$$(5.9) \quad \mathbb{R}^{n-1} = D \cup R \cup L,$$

where D is the closed disk in \mathbb{R}^{n-1} centered at the origin with radius $1/2$,

$$\begin{aligned} R &= ([0, \infty) \times \mathbb{R}^{n-2}) \setminus \text{Int } D \quad \text{and} \quad L = ((-\infty, 0] \times \mathbb{R}^{n-2}) \setminus \text{Int } D \\ &\subset \mathbb{R} \times \mathbb{R}^{n-2} = \mathbb{R}^{n-1}. \end{aligned}$$

Let us also consider the associated decomposition

$$M = f_i^{-1}(D) \cup f_i^{-1}(R) \cup f_i^{-1}(L),$$

$i = 0, 1$. As has been observed in the proof of Theorem 2.5, $f_i^{-1}(D)$ is either empty or the total space of a trivial fiber bundle over D , where the fiber is a finite union of circles, while $f_i^{-1}(R)$ and $f_i^{-1}(L)$ are the total spaces of locally trivial fiber bundles over the $(n-2)$ -dimensional disk, $i = 0, 1$.

Let $T_R : [1/2, \infty) \times D^{n-2} \rightarrow R$ (or $T_L : [1/2, \infty) \times D^{n-2} \rightarrow L$) be the diffeomorphism defined by $T_R(r, x) = rx$ (resp. $T_L(r, x) = -rx$), where we identify D^{n-2} with

$$\{(x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1} \mid x_1^2 + x_2^2 + \dots + x_{n-1}^2 = 1, x_1 \geq 0\} \subset S^{n-2}.$$

Then, the map

$$T_R^{-1} \circ f_i|_{f_i^{-1}(R)} : f_i^{-1}(R) \rightarrow [1/2, \infty) \times D^{n-2}$$

can be identified with a family of Morse functions parametrized by D^{n-2} , $i = 0, 1$. More precisely, to each $x \in D^{n-2}$ is associate the Morse function

$$\eta_1 \circ T_R^{-1} \circ f_i|_{F_{x,i}} : F_{x,i} \rightarrow [1/2, \infty),$$

where $F_{x,i} = f_i^{-1}(T_R([1/2, \infty) \times \{x\}])$ and $\eta_1 : [1/2, \infty) \times D^{n-2} \rightarrow [1/2, \infty)$ is the projection to the first factor. By using a vector field argument (see, for example, [1, 3]), we can construct a trivializing diffeomorphism $\tilde{\varphi}_i : F_i \times D^{n-2} \rightarrow f_i^{-1}(R)$ in such a way that we have the commutative diagram

$$\begin{array}{ccccc} f_i^{-1}(R) & \xleftarrow{\tilde{\varphi}_i} & F_i \times D^{n-2} & \xrightarrow{\tilde{\eta}_{2,i}} & D^{n-2} \\ \downarrow f_i & & \downarrow h_i \times \text{id}_{D^{n-2}} & & \downarrow \text{id}_{D^{n-2}} \\ R & \xleftarrow{T_R} & [1/2, \infty) \times D^{n-2} & \xrightarrow{\eta_2} & D^{n-2}, \end{array}$$

where $\eta_2 : [1/2, \infty) \times D^{n-2} \rightarrow D^{n-2}$ and $\tilde{\eta}_{2,i} : F_i \times D^{n-2} \rightarrow D^{n-2}$ are the projections to the second factors, $i = 0, 1$. A similar argument applies to $f_i^{-1}(L)$ as well.

By our assumption together with the above observation, we see that

$$f_0 : f_0^{-1}(R) \rightarrow R \quad \text{and} \quad f_0 : f_0^{-1}(L) \rightarrow L$$

are C^∞ \mathcal{R} -equivalent to

$$f_1 : f_1^{-1}(R) \rightarrow R \quad \text{and} \quad f_1 : f_1^{-1}(L) \rightarrow L,$$

respectively.

Now, through the above trivializations, the attaching diffeomorphisms between $f_i^{-1}(R)$ and $f_i^{-1}(L)$, $i = 0, 1$, are identified with families of diffeomorphisms of compact surfaces that preserve the page Morse functions, parametrized by $\partial D^{n-2} = S^{n-3}$. By a result of Maksymenko [17, 18], the space of such diffeomorphisms has vanishing $(n-3)$ -th homotopy group, as we are assuming $n \geq 5$. This implies that, by changing the above trivializations slightly near the attaching parts, we may assume that the attaching diffeomorphisms for f_0 and f_1 coincide with each other. Therefore, we can construct a diffeomorphism $f_0^{-1}(R \cup L) \rightarrow f_1^{-1}(R \cup L)$ that gives \mathcal{R} -equivalence between f_0 and f_1 over $R \cup L$.

Now, as f_0 and f_1 are trivial fiber bundles over D , it is easy to extend this diffeomorphism to a diffeomorphism $M_0 \rightarrow M_1$ that gives \mathcal{R} -equivalence between f_0 and f_1 over \mathbb{R}^{n-1} . This is because the group of diffeomorphisms of the circle has trivial $(n-2)$ -th homotopy group. This completes the proof. \square

According to the above proof, round fold maps as in Theorem 5.4 are always constructed through a trivial open book construction as in Example 2.4 up to \mathcal{A} -equivalence.

Remark 5.5. As the above proof shows, if the given round fold map f is of the standard form (i.e., if $f(S(f))$ is of the form $\cup_{r=1}^s C_r$), then its C^∞ \mathcal{R} -equivalence class is determined by the \mathcal{R} -equivalence class of its page Morse function.

Remark 5.6. As has been shown in [11] (see also [2, proof of Theorem 3.8]), two Morse functions on a compact connected surface are C^∞ \mathcal{A} -equivalent if and only if their associated functions on the Reeb graphs (with orientation reversing information when the source surface is non-orientable) are topologically equivalent. A Reeb graph is the quotient space of the source surface obtained by identifying each connected component of level sets to a point, and such a space is known to have the structure of a finite graph (see [19]). In particular, such Morse functions can be classified up to C^∞ \mathcal{A} -equivalence by using purely combinatorial objects. If we use appropriate functions on Reeb graphs, classification up to C^∞ \mathcal{R} -equivalence is also possible.

Now, let us consider the case where $n = 4$. In this case, we have the following.

Theorem 5.7. *Let $f_i : M \rightarrow \mathbb{R}^3$ be round fold maps of a closed connected 4-dimensional manifold M , $i = 0, 1$. Then f_0 and f_1 are C^∞ \mathcal{A} -equivalent if and only if exactly one of the following holds.*

- (1) *Both of f_0 and f_1 have indefinite fold points and the page Morse functions of f_0 and f_1 are \mathcal{R} -equivalent.*
- (2) *Both of f_0 and f_1 have only definite fold points as their singularities and M is diffeomorphic to S^4 .*
- (3) *Both of f_0 and f_1 have only definite fold points as their singularities, M is diffeomorphic to an S^2 -bundle over S^2 , and the self-intersection numbers of the components of $S(f_0)$ coincide with those of $S(f_1)$ up to order, with respect to a fixed orientation of M .*

Remark 5.8. In Theorem 5.7 (3), each of $S(f_i)$, $i = 0, 1$, consists exactly of two components and their self-intersection numbers are of the form $k_i, -k_i$ for some integer k_i . Note that M is diffeomorphic to $S^2 \times S^2$ if the self-intersection numbers are even and to $S^2 \tilde{\times} S^2$ if the self-intersection numbers are odd.

Proof of Theorem 5.7. Necessity is clear.

As to sufficiency, when both of f_0 and f_1 have indefinite fold points, the proof is the same as that of Theorem 5.4. This is because the space of the Morse functions preserving the page Morse function is contractible in this case [17, 18].

When both of f_0 and f_1 have only definite fold as their singularities, then as the page function, we have the following two possibilities: one is the Morse function on D^2 with exactly one critical point, which is the maximum point, and the other is the Morse function on S^2 with exactly two critical points, which are the minimum and the maximum points.

In the former case, the identity component of the group of diffeomorphisms of D^2 preserving the Morse function has the homotopy type of S^1 [17, 18]. However, if we use an attaching diffeomorphism for $f_i^{-1}(R)$ and $f_i^{-1}(L)$ corresponding to a non-trivial element of its fundamental group, then the boundary 3-dimensional manifold is not diffeomorphic to $S^2 \times S^1$, which is a contradiction as $f_i^{-1}(D)$ is a trivial circle bundle over D , $i = 0, 1$. (Here, we use the decomposition of \mathbb{R}^3 as in (5.9).) Therefore, in this case, f_0 and f_1 are necessarily \mathcal{A} -equivalent and M is diffeomorphic to S^4 .

In the latter case, $f_i^{-1}(D)$ is empty and $S(f_i)$ consists exactly of two components, $i = 0, 1$. Let S_i^+ (or S_i^-) denote the component of $S(f_i)$ which is mapped by f_i to the outer (resp. inner) component of $f_i(S(f_i))$ in \mathbb{R}^3 . The attaching diffeomorphism for $f_i^{-1}(R)$ and $f_i^{-1}(L)$ corresponds to an element of the fundamental group of the identity component of the group of diffeomorphisms of the circle, which is isomorphic to the infinite cyclic group \mathbb{Z} . Let $\sigma_i \in \mathbb{Z}$ denote the corresponding element, $i = 0, 1$. Here, we may assume that the self-intersection number of S_i^+ in M coincides with σ_i . Then, the self-intersection number of S_i^- in M are equal to $-\sigma_i$, $i = 0, 1$. Recall that by our assumption, we have $\sigma_0 = \pm\sigma_1$. If $\sigma_0 = \sigma_1$, then we can easily construct an orientation preserving self-diffeomorphism of M that gives the \mathcal{R} -equivalence between f_0 and f_1 . If $\sigma_0 = -\sigma_1$, then we consider an orientation reversing diffeomorphism of

S^2 that preserves the page Morse function for f_0 , and this gives rise to an orientation reversing self-diffeomorphism of M that preserves f_0 . Then, using this diffeomorphism, we may assume that $\sigma_0 = \sigma_1$. Therefore, in this case as well, we conclude that f_0 and f_1 are \mathcal{R} -equivalent. This completes the proof. \square

It would be a difficult but interesting problem to classify round fold maps on closed orientable 3-dimensional manifolds into \mathbb{R}^2 (refer to [15]) up to \mathcal{A} -equivalence. Note that for simple stable maps of closed orientable 3-dimensional manifolds into \mathbb{R}^2 , a classification result has been given in [24]: however, the classification is up to a certain equivalence strictly weaker than the \mathcal{A} -equivalence, and the result is given in terms of links in the 3-dimensional manifolds.

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