Journal *of* Singularities
Volume 25 (2022), 30-53

Proc. of 16th International Workshop on Singularities, São Carlos, 2020
DOI: 10.5427/jsing.2022.25b

ALGEBRAIC KNOTS ASSOCIATED WITH MILNOR FIBRATIONS

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Abstract. In this paper, after reviewing basic material on Milnor fibrations, we explain topological invariants of algebraic knots associated with isolated singularities of complex hypersurfaces. These invariants have their origins in knot theory and are very important for the classification of isolated singularities of complex hypersurfaces up to certain topological equivalence relations. These relations correspond to isotopy and cobordism of associated knots. We also discuss the existence of non-trivial examples of real Milnor fibrations and the fibered knot conjecture.

1. Introduction

This is a survey paper on the topology of isolated singularities of complex hypersurfaces and related topics on knots in general dimensions. (In fact, the contents of this article are mainly based on the mini-course delivered by the second author: “Topologia das singularidades e teoria de nós” (in Portuguese), IV Encontro de Singularidades no Nordeste, Departamento de Matemática da Universidade Federal da Paraíba, João Pessoa, Brazil, held during November 22–24, 2017.)

It is classically known, as *Milnor's fibration theorem* [34], that around a singular point of a complex hypersurface in $\mathbb{C}^{n+1}$, there is a fibration structure, and this is a fundamental material for studying the topology of the singularity. More precisely, if it is an isolated singularity, then the associated link is a codimension two submanifold of a small sphere of dimension $2n+1$ such that its complement fibers over the circle. Such a $(2n-1)$-dimensional submanifold is called an *algebraic knot* associated with the singularity. In this article we survey the study of the topology of such isolated singularities from the viewpoint of knot theory.

In Section 2, we recall Milnor's fibration theorem together with its relation to the topological type of a hypersurface singularity or that of a holomorphic function germ defining the singularity. Then, we recall several results about the decomposability of the algebraic knots. We also consider the case where the algebraic knot is a topological sphere. In Section 3, we recall the classification of algebraic knots up to isotopy by using Seifert forms in the case of $n \geq 3$. In Section 4, we study the topological types of Brieskorn–Pham polynomials using their Alexander polynomials. In Section 5, we introduce the notion of cobordism for algebraic knots, which is a relation weaker than the isotopy. We recall that for hypersurface singularities in $\mathbb{C}^2$, two algebraic knots are cobordant if and only if they are isotopic. However, for isolated hypersurface singularities in $\mathbb{C}^{n+1}$ with $n \geq 3$, this is not true in general. In Section 6, we recall the notion of algebraic cobordism for Seifert forms. In particular, we give a brief sketch of a proof for the fact that the Seifert forms for cobordant algebraic knots have metabolizers. In Section 7, we present several known results about algebraic knots defined by weighted homogeneous polynomials together with some explicit examples with interesting properties. We also give some related open questions. Finally, in Section 8, we consider Milnor fibrations associated with real polynomial mappings

2010 *Mathematics Subject Classification.* Primary 32S55, Secondary 57Q45, 57M25.
Key words and phrases. Knot, Milnor fibration, topological invariant, knot cobordism.

$\mathbb{R}^n \to \mathbb{R}^p$ with isolated singularities. We consider the problem to determine those dimension pairs (n, p) for which non-trivial examples exist. We also address a conjecture about fibered knots in S^3.

2. Algebraic knots associated with complex Milnor fibrations

In [34] Milnor showed that given a non-constant holomorphic function f defined on a small neighborhood U of the origin in $\mathbb{C}^{n+1}$ with $f(0) = 0$, there exists a small $\epsilon > 0$ such that

$$\phi_f = \frac{f}{|f|} : S_\epsilon^{2n+1} \setminus K_f \to S^1 \tag{2.1}$$

is the projection of a smooth locally trivial fiber bundle, where S_ϵ^{2n+1} is the $(2n+1)$-dimensional sphere in $\mathbb{C}^{n+1}$ centered at the origin with radius ϵ, and

$$K_f = f^{-1}(0) \cap S_\epsilon^{2n+1}$$

is called the *link* of the singularity at the origin. We call the fiber bundle ϕ_f the *Milnor fibration* associated with f.

Remark 2.1. It is known that there exists an $\epsilon_0 > 0$ such that for all ϵ with $0 < \epsilon \leq \epsilon_0$, the above statement holds for ϵ and that the associated fibrations are all smoothly equivalent. Such a positive real number ϵ_0 is called a *Milnor radius* for f at the origin.

The link plays a fundamental role in the study of the local topology of the hypersurface $V = f^{-1}(0)$ near the origin. More precisely, we have the following (see Figure 1).

Theorem 2.2 ([34, 16])**.** *The intersection of V with a small ball B_ϵ^{2n+2} centered at the origin with radius ϵ is homeomorphic to the cone over $K_f = V \cap S_\epsilon^{2n+1}$.*

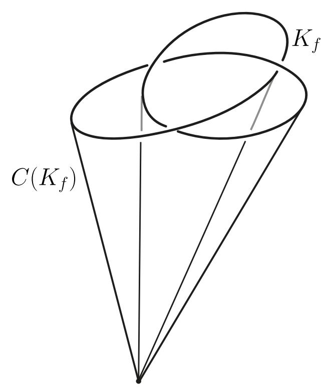

Figure 1. The topology of V within B_ϵ^{2n+2}, where $C(K_f)$ denotes the cone over K_f.

Let $F_\theta = \phi_f^{-1}(e^{i\theta})$ be the fiber of (2.1), where $e^{i\theta} \in S^1$. It is a real $2n$-dimensional parallelizable manifold. Using Morse theory, Milnor proved that F_θ has the homotopy type of a finite CW complex of dimension n and that the link K_f is $(n-2)$-connected, that is, $\pi_j(K_f) = 0$ for all $j \leq n-2$. As the fibers are all diffeomorphic, we sometimes denote a fiber by F_f and call it the *Milnor fiber* associated with f.

Denote by $\Sigma_f = \{z \in U \subset \mathbb{C}^{n+1} | \nabla f(z) = 0\}$ the set of critical points of f, or the singular locus of f, where

$$\nabla f = \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_{n+1}} \right).$$

In the case that $0 \in \Sigma_f$ is an isolated point[1], Milnor gave further details about the topology of the fiber and the link. More precisely, in such a case the fiber F_θ has the homotopy type of a wedge of n-dimensional spheres $S^n \vee \cdots \vee S^n$, also known as *Milnor's bouquet of spheres*, with μ_f copies of S^n attached to a single common point. The number μ_f is called the *Milnor number* of f at the origin. This number is also given by the topological degree of the mapping

$$\frac{\nabla f}{\|\nabla f\|} : S_\epsilon^{2n+1} \to S^{2n+1},$$

and is also known to be equal to the dimension of

$$\mathcal{O}_{n+1} \Big/ \left(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \ldots, \frac{\partial f}{\partial z_{n+1}} \right)$$

over $\mathbb{C}$, where $\mathcal{O}_{n+1}$ denotes the $\mathbb{C}$-algebra of holomorphic function germs at the origin in $\mathbb{C}^{n+1}$.

Milnor also proved that for all $\epsilon > 0$ small enough, the manifold

$$(f^{-1}(0) \setminus \{0\}) \cap B_\epsilon^{2n+2}$$

transversely intersects S_ϵ^{2n+1} and thus K_f is a $(2n-1)$-dimensional smooth manifold. The codimension two (oriented) submanifold K_f of S_ϵ^{2n+1} is called the *algebraic knot* associated with f at the origin. Furthermore, each fiber F_θ can be considered as the interior of a smooth compact manifold with boundary $\overline{F_\theta} = F_\theta \cup K_f$. Thus in a neighborhood of the link K_f all fibers fit around their common boundary K_f like an open book structure, as illustrated in Figure 2. In this sense, the algebraic knot K_f is a *fibered knot*.

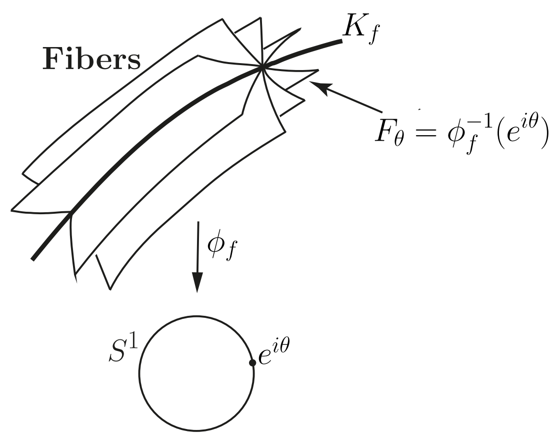

FIGURE 2. Open book structure.

Example 2.3. Consider the polynomial

$$g(z_1, z_2) = z_1^3 - z_2^2$$

in two variables, with an isolated critical point at the origin. Then, for $\epsilon > 0$, there exist uniquely $r_1, r_2 > 0$ such that $r_1^3 = r_2^2$ and $r_1^2 + r_2^2 = \epsilon^2$, and the link

$$K_g = \{(r_1 e^{2\pi i t}, r_2 e^{3\pi i t}) \mid t \in \mathbb{R}\}$$

is a trefoil knot in the torus $S_{r_1}^1 \times S_{r_2}^1 \subset S_\epsilon^3$, see Figure 3.

[1]In this case we say that 0 is an *isolated singularity*, or an *isolated critical point*, of f.

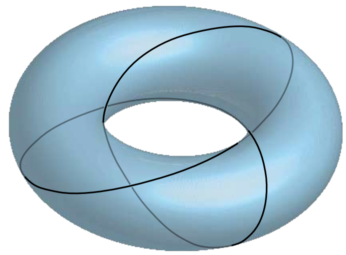

FIGURE 3. Algebraic knot K_g sits on a torus.

In this case the closure of the fiber $\overline{F}_\theta$ is a Seifert surface of K_g and has the homotopy type of a wedge of 1-dimensional spheres with $\mu_g = 2$. We refer to Figure 4 as an illustration.

FIGURE 4. Homotopy type of the Milnor fiber.

For topological equivalence relations for isolated complex hypersurface singularities, the following is known.

Theorem 2.4 ([29, 41, 45]). *Let $f, g \in \mathbb{C}[z_1, z_2, \ldots, z_{n+1}]$ be polynomials with $f(0) = g(0) = 0$ having isolated singularities at the origin. Then, the following statements are all equivalent, where ϕ_f and ϕ_g are the Milnor fibrations for f and g, respectively.*

(1) *There exists a homeomorphism germ $h : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that*
$$h(f^{-1}(0)) = g^{-1}(0).$$
(2) *There exist homeomorphism germs $h : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ and $H : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $f = H \circ g \circ h$.*
(3) *(S^{2n+1}, K_f) and (S^{2n+1}, K_g) are homeomorphic.*
(4) *(S^{2n+1}, K_f) and (S^{2n+1}, K_g) are diffeomorphic.*
(5) *There exist homeomorphisms $h' : (S^{2n+1}, K_f) \to (S^{2n+1}, K_g)$ and $H' : S^1 \to S^1$ such that the diagram*
$$\begin{array}{ccc} S^{2n+1} \setminus K_f & \xrightarrow{h'} & S^{2n+1} \setminus K_g \\ \phi_f \downarrow & & \downarrow \phi_g \\ S^1 & \xrightarrow{H'} & S^1 \end{array}$$
commutes.

(6) *There exist diffeomorphisms* $h' : (S^{2n+1}, K_f) \to (S^{2n+1}, K_g)$ *and* $H' : S^1 \to S^1$ *such that the diagram*

$$\begin{array}{ccc} S^{2n+1} \setminus K_f & \xrightarrow{h'} & S^{2n+1} \setminus K_g \\ \phi_f \downarrow & & \downarrow \phi_g \\ S^1 & \xrightarrow{H'} & S^1 \end{array}$$

commutes.

Remark 2.5. It is known that a holomorphic function germ with an isolated critical point at the origin is always topologically equivalent to a polynomial function germ (for example, see [18, Proposition (6.39)] or [49]). Therefore, in order to study the topology of an isolated complex hypersurface singularity, we may assume that the hypersurface is defined by a polynomial function.

2.1. **Case of** $n = 1$**.** In this subsection, let us consider the two variable case. Suppose that $f = f(z_1, z_2)$ is locally irreducible at the origin. In this case, f has an isolated singularity at the origin and K_f is connected. We can solve the equation $f(z_1, z_2) = 0$ in the form of the so-called Puiseux expansion:

$$\begin{cases} z_1 = w^{a_0}, \ 0 < a_0 \\ z_2 = \lambda_1 w^{a_1} + \lambda_2 w^{a_2} + \cdots, \quad 0 < a_1 < a_2 < \cdots \end{cases}$$

Based on such a description, Brauner showed the following.

Theorem 2.6 ([13])**.** *If* $f = f(z_1, z_2)$ *is locally irreducible at the origin, then the algebraic knot* K_f *is an iterated torus knot.*

In particular K_f is a prime knot (see [50]), where a knot is *prime* if it is not isotopic to the connected sum of two non-trivial knots. A schematic picture explaining what is an iterated torus knot can be found in Figure 5.

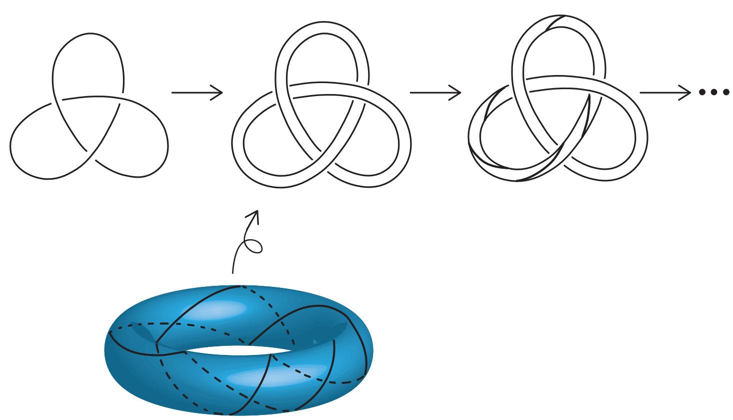

Figure 5. Construction of iterated torus knots.

Remark 2.7. It should not be forgotten that Wirtinger has essentially contributed a lot in the topological study of algebraic knots. For details, see [23].

Note that for $n \geq 2$, the following is known.

Theorem 2.8 ([33, 43])**.** *For* $n \geq 2$*, there exist decomposable algebraic knots.*

Note that a knot is *decomposable* if it is isotopic to the connected sum of two non-trivial knots. A schematic picture of a decomposable knot can be found in Figure 6.

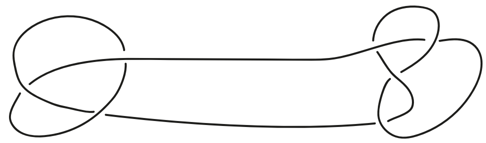

FIGURE 6. An example of a decomposable knot.

2.2. **Case of $n = 2$.** Let us consider the three variable case, $f = f(z_1, z_2, z_3)$, with (at most) an isolated singularity at the origin. In this case K_f is 3-dimensional. Then, using resolution of singularities, Mumford showed the following.

Theorem 2.9 ([36]). *The fundamental group $\pi_1(K_f)$ is trivial if and only if $f^{-1}(0)$ is not singular at the origin.*

In particular, for the 3-dimensional link K_f, it is simply connected if and only if it is homeomorphic to S^3, which gives the solution to the Poincaré conjecture for algebraic links. Note that in general, the 3-dimensional manifold K_f is a so-called Waldhausen graph manifold (for example, see [37]) and this was essential in the above theorem.

2.3. **Case of $n \geq 3$.** Suppose $n = 2m \geq 4$ and consider

$$f_k = z_1^2 + \cdots + z_{2m-1}^2 + z_{2m}^3 + z_{2m+1}^{6k-1},$$

for $k \geq 1$. This class of polynomials shows that the situation is quite different from Theorem 2.9 for higher dimensions.

Theorem 2.10 ([14, 26]). *The link K_{f_k} is homeomorphic to the $(4m-1)$-dimensional sphere S^{4m-1}. Furthermore, we have*

$$\{K_{f_k} \,|\, k = 1, 2, \ldots\} = bP_{4m},$$

where bP_{4m} is the group of $(4m-1)$-dimensional homotopy spheres which are boundaries of parallelizable manifolds of dimension $4m$.

Note that the group bP_{4m} is finite cyclic and is non-trivial in general: for example, $bP_8 \cong \mathbb{Z}/28\mathbb{Z}$, $bP_{12} \cong \mathbb{Z}/992\mathbb{Z}$, $bP_{16} \cong \mathbb{Z}/8128\mathbb{Z}$, etc. (see [28]).

3. CLASSIFICATION OF ALGEBRAIC KNOTS

In this section, we discuss several invariants of algebraic knots and their classification up to isotopy.

Suppose that f has an isolated critical point at the origin. As $f^{-1}(0) \cap (B_\epsilon^{2n+2} \setminus \{0\})$ is a complex manifold, it has a natural orientation, and K_f inherits an orientation as its boundary. Then, the Milnor fiber F_f also has a natural orientation so that the oriented boundary of its closure coincides with K_f. Note also that S_ϵ^{2n+1} is also oriented as the boundary of B_ϵ^{2n+2}. Then, we have a natural normal orientation for F_f in S_ϵ^{2n+1}.

Definition 3.1. The *Seifert form* of the fiber F_f is the pairing

$$L_f : H_n(F_f; \mathbb{Z}) \times H_n(F_f; \mathbb{Z}) \to \mathbb{Z}$$

given by taking the linking number $L_f(\alpha, \beta) = \mathrm{lk}(a_+, b)$, where

- a and b are n-cycles representing the homology classes α and β, respectively,
- a_+ indicates a translate of a in the positive normal direction to F_f (see Figure 7).

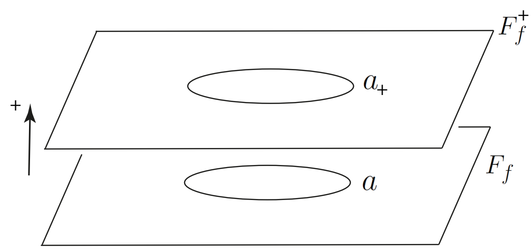

FIGURE 7. Translate a_+ of a in the positive normal direction to F_f, where F_f^+ denotes a parallel translate of F_f in the positive normal direction.

Example 3.2. Consider $g(z_1, z_2) = z_1^3 - z_2^2$ as in Example 2.3. In this case $H_1(F_g; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ has two generators, $\alpha = [a]$ and $\beta = [b]$ (see Figure 8), and the associated matrix of the Seifert form is given by

$$\begin{pmatrix} \ell_{11} & \ell_{12} \\ \ell_{21} & \ell_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

where the entries in the matrix are given by the linking numbers

$$\ell_{11} = \mathrm{lk}(a_+, a),\ \ell_{12} = \mathrm{lk}(a_+, b),\ \ell_{21} = \mathrm{lk}(b_+, a),\ \ell_{22} = \mathrm{lk}(b_+, b).$$

(For the linking numbers, see Figure 9.)

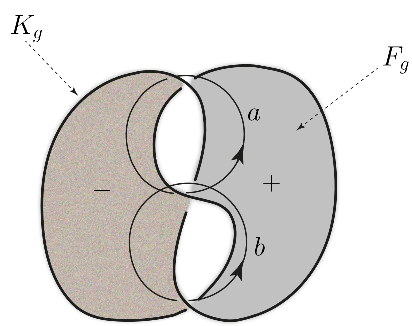

FIGURE 8. 1-cycles representing generators of $H_1(F_g; \mathbb{Z})$.

We have the following classification theorem of algebraic knots for $n \geq 3$.

Theorem 3.3 ([21, 27])**.** *For $n \geq 3$, two algebraic knots K_f and K_g are isotopic if and only if their Seifert forms L_f and L_g are isomorphic, i.e. if and only if there exists an isomorphism $\varphi : H_n(F_f; \mathbb{Z}) \to H_n(F_g; \mathbb{Z})$ such that $L_f(x, y) = L_g(\varphi(x), \varphi(y))$ holds for all $x, y \in H_n(F_f; \mathbb{Z})$.*

Note that for $n = 1$, when f and g are locally irreducible, the above theorem also holds.

Theorem 3.3 seems to be very strong: however, the problem is that the computation of the Seifert form for a given isolated singularity of a complex hypersurface is in general extremely difficult. This is the main reason why the above theorem has not been used so far, unfortunately.

Furthermore, for two or three variable case, the above theorem does not hold in general as follows.

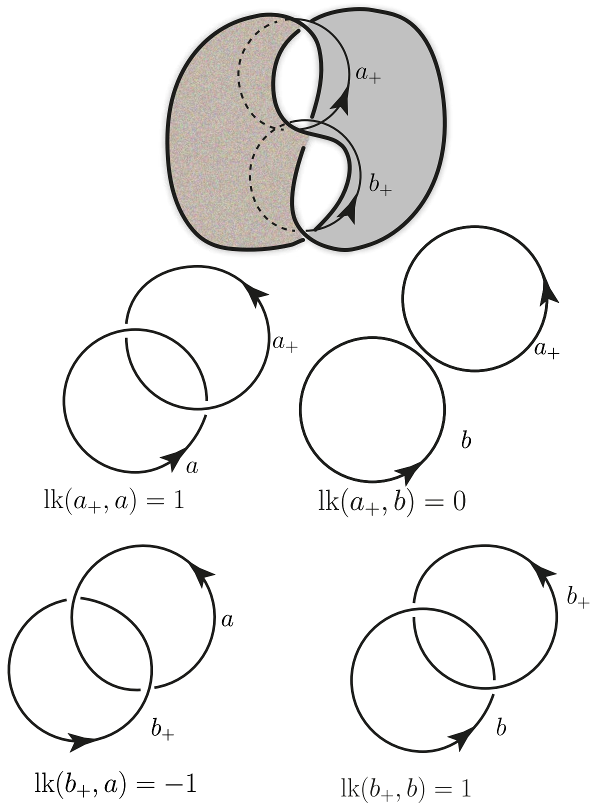

FIGURE 9. Linking numbers.

Theorem 3.4 ([19, 5]). *For $n = 1, 2$, there exist f and g such that L_f and L_g are isomorphic but that the algebraic knots K_f and K_g are not isotopic.*

When $n = 1$ and f is locally irreducible, the following invariant is very effective.

Definition 3.5. Suppose that f has an isolated critical point at the origin in $\mathbb{C}^{n+1}$. Then, the polynomial $\Delta_f(t) = \det(tL_f + (-1)^n L_f^T)$ in t is called the *Alexander polynomial* of the algebraic knot K_f, where L_f is identified with the representation matrix of the Seifert form with respect to a fixed basis, and L_f^T denotes its transpose.

Then, in the two variable case, we have the following.

Theorem 3.6 ([30, 55]). *If $n = 1$, then we have the following.*

(1) *Let f and g be locally irreducible. Then, the algebraic knots K_f and K_g are isotopic if and only if $\Delta_f(t) = \pm\Delta_g(t)$.*
(2) *When f is not necessarily locally irreducible, the isotopy class of K_f is completely determined by the isotopy classes of the connected components and their linking numbers.*

4. BRIESKORN–PHAM POLYNOMIALS

In this section, we consider the following restricted class of complex polynomials and give more detailed results.

Definition 4.1. Let $a_1, a_2, \dots, a_{n+1}$ be integers greater than or equal to 2. The polynomial $f(z_1, z_2, \dots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$ is called a *Brieskorn–Pham polynomial*. Note that it has an isolated critical point at the origin in $\mathbb{C}^{n+1}$. The integers $a_1, a_2, \dots, a_{n+1}$ are called the *exponents*.

Yoshinaga–Suzuki showed the following.

Theorem 4.2 ([53])**.** *Let f and g be Brieskorn–Pham polynomials of $n+1$ variables. Then the following three are equivalent to each other.*

(1) *The algebraic knots K_f and K_g are isotopic.*
(2) *The exponents of f and g coincide up to order.*
(3) *The Alexander polynomials satisfy $\Delta_f(t) = \pm\Delta_g(t)$.*

For Brieskorn–Pham polynomials, their Seifert forms have been calculated (see [48]). We have $L_f = A_{a_1} \otimes A_{a_2} \otimes \cdots \otimes A_{a_{n+1}}$, where for an integer $a \geq 2$, A_a is the integral bilinear form on the free abelian group of rank $a-1$, which is represented by the following $(a-1)\times(a-1)$ matrix:

$$A_a = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 & 1 \end{pmatrix}.$$

In Section 7, we will give explicit examples of polynomials which are not topologically equivalent to Brieskorn–Pham polynomials. We will also give more results and examples of Brieskorn–Pham polynomials there.

For the classification of Seifert forms over the real numbers, we refer to [12].

5. Cobordism of algebraic knots

In the following, an *m-dimensional knot* refers to a smooth closed oriented m-dimensional manifold embedded in S^{m+2}. Two such knots K and K' are considered to be equivalent if they are orientation preservingly isotopic to each other. In this case, we write $K \sim_{\mathrm{iso}} K'$. In this section, we consider a weaker relation as follows.

Definition 5.1 ([24])**.** Let K_0 and K_1 be m-dimensional knots in S^{m+2}. We say that K_0 and K_1 are *cobordant* if there exists a compact oriented $(m+1)$-dimensional submanifold X of $S^{m+2} \times [0,1]$ such that the following conditions are satisfied.

(1) The manifold X is diffeomorphic to $K_0 \times [0,1]$.
(2) The manifold X intersects $S^{m+2} \times \{0,1\}$ transversely and we have

$$X \cap (S^{m+2} \times \{0,1\}) = \partial X = (K_0 \times \{0\}) \cup (-K_1 \times \{1\}),$$

where $-K_1$ denotes the knot K_1 with the orientation reversed.

In this case, we write $K_0 \sim_{\mathrm{cob}} K_1$. The embedded manifold X is called a *cobordism* between K_0 and K_1. For a schematic picture, we refer to Figure 10.

Remark 5.2. If K_0 and K_1 are isotopic, then they are cobordant (see Figure 11). In general, the converse is not valid: Figure 12 illustrates this situation schematically.

Suppose that f and g have isolated singularities at the origin. In general, if the algebraic knots K_f and K_g are cobordant, then the topology of the singularities of f and g are somehow related (see Figure 13).

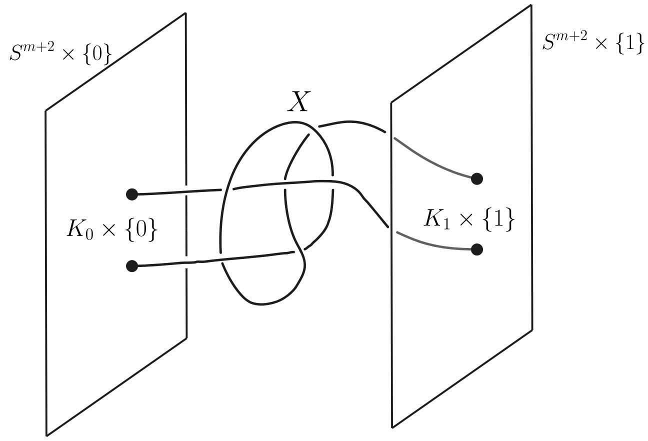

FIGURE 10. A cobordism.

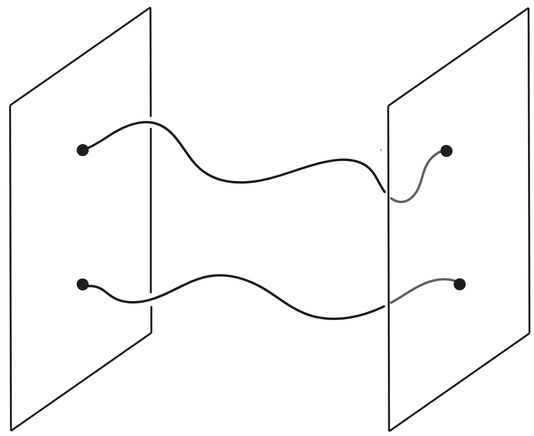

FIGURE 11. Isotopy trace gives a cobordism.

For example, suppose that f_t, $t \in [0,1]$, is a μ-constant family of holomorphic functions with isolated singularities at the origin in $\mathbb{C}^{n+1}$. This means that the Milnor number μ_{f_t} of f_t at the origin is independent of t. Then, we have the following.

Proposition 5.3. *The algebraic knots K_{f_0} and K_{f_1} are cobordant to each other, provided that $n \neq 2$.*

Proof. We have only to show that for each $T \in [0,1]$, there exists a $\delta > 0$ such that K_{f_t} are all cobordant for $t \in [0,1] \cap [T-\delta, T+\delta]$.

Let $\epsilon > 0$ be a Milnor radius for f_T. Then, there exists a $\delta > 0$ such that the sphere S^{2n+1}_ϵ intersects $(f_t)^{-1}(0)$ transversely for all $t \in [0,1] \cap [T-\delta, T+\delta]$. For such a fixed t, let $\epsilon' > 0$ be a Milnor radius for f_t such that $0 < \epsilon' < \epsilon$. Then, we see that the knots K_{f_T} and $K'_{f_t} = (f_t)^{-1}(0) \cap S^{2n+1}_\epsilon$ are isotopic to each other. Furthermore, we see that

$$X = (f_t)^{-1}(0) \cap (B^{2n+2}_\epsilon \setminus \operatorname{Int} B^{2n+2}_{\epsilon'})$$

gives a cobordism between K'_{f_t} and K_{f_t}. This can be seen by an argument as in [31]. We refer to Figure 14. Hence, K_{f_t} and K_{f_T} are cobordant to each other. $\square$

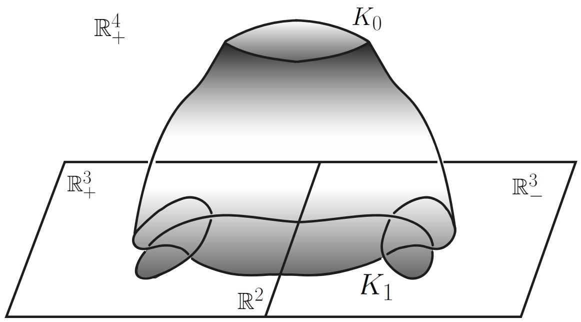

FIGURE 12. Knots K_0 and K_1 are cobordant, but are not isotopic. Here, K_1 is the connected sum of a trefoil knot and its mirror image, $\mathbb{R}^N_+$ (resp. $\mathbb{R}^N_-$) denotes the upper (resp. lower) half space of $\mathbb{R}^N$, the cobordism is embedded in $\mathbb{R}^3 \times [0,1] \subset \mathbb{R}^4_+$, and $\mathbb{R}^3$ is assumed to be embedded in S^3.

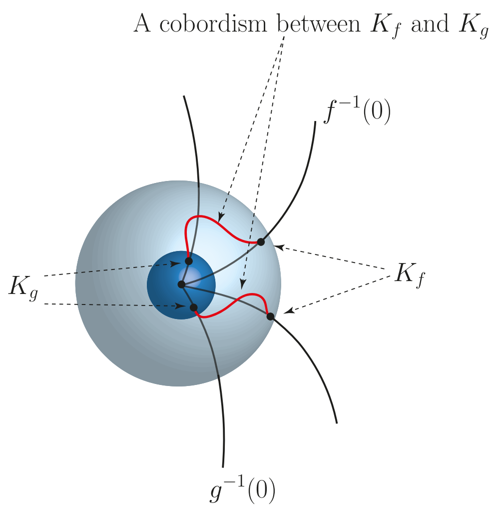

FIGURE 13. Isolated hypersurface singularities whose associated algebraic knots are cobordant.

Note that according to [31], K_{f_0} and K_{f_1} are, in fact, isotopic for $n \neq 2$. When $n = 2$, the problem is still open as far as the authors know. One of the strategies for a positive solution would be to first show that K_{f_0} and K_{f_1} are cobordant using results of [8, 37] and then show that K_{f_0} and K_{f_1} are isotopic (see Remark 5.9). Here, we note that for $n = 2$, it is still an open question whether K_{f_0} and K_{f_1} are diffeomorphic or not: we still do not even know if they have isomorphic fundamental groups or not.

5.1. **Case of $n = 1$.** Suppose that $f(z_1, z_2)$ is locally irreducible at the origin. Note that then, it has an isolated singularity at the origin. For the Milnor fibration as in (2.1), we have a smooth one parameter family of diffeomorphisms $h_t : \phi_f^{-1}(1) \to \phi_f^{-1}(e^{2\pi i t})$ between the Milnor fibers, $0 \leq t \leq 1$. Note that h_0 is the identity and h_1 is called a *geometric monodromy* of the Milnor fibration.

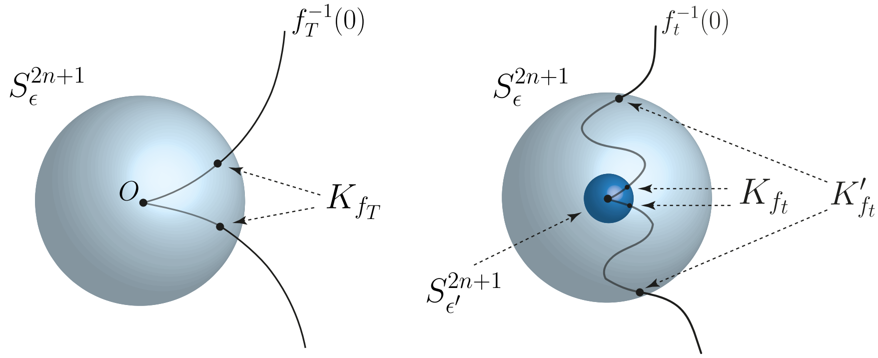

FIGURE 14. $K_{f_t} \sim_{\text{cob}} K'_{f_t} \sim_{\text{iso}} K_{f_T}$

Proposition 5.4 ([34]). *We have $\Delta_f(t) = \pm\det(tI_\mu - (h_1)_*)$, where μ is the Milnor number, I_μ is the $\mu \times \mu$ identity matrix, and $(h_1)_* : H_1(F_0;\mathbb{Z}) \to H_1(F_0;\mathbb{Z})$ is the isomorphism induced by the geometric monodromy on $F_0 = \phi_f^{-1}(1)$.*

Theorem 5.5 ([30]). *Suppose that $f(z_1, z_2)$ and $g(z_1, z_2)$ are locally irreducible at the origin. Then, the following three are equivalent to each other.*

(1) *The algebraic knots K_f and K_g are isotopic.*
(2) *The algebraic knots K_f and K_g are cobordant.*
(3) *They have the same Alexander polynomials up to sign: $\Delta_f(t) = \pm\Delta_g(t)$.*

We give a brief sketch of the proof of Theorem 5.5. As explained before, K_f and K_g are iterated torus knots and their types are completely characterized by their Puiseux pairs. According to [30], one can show that the Alexander polynomial of K_f (or K_g) determines the associated Puiseux pairs of f (resp. g). Thus, if $\Delta_f(t) = \pm\Delta_g(t)$, then K_f and K_g are isotopic. The converse is also true, since the Alexander polynomial is a topological invariant.

If K_f and K_g are isotopic, then they are obviously cobordant as explained before. Now, if K_f and K_g are cobordant, then according to Fox–Milnor [25], we have

$$\Delta_f(t) \cdot \Delta_g(t) = \pm t^m p(t)p(t^{-1})$$

for some integer m and some polynomial $p(t)$. Using the fact that the Alexander polynomial of an algebraic knot is a product of cyclotomic polynomials and that some of their zeros are simple, one can then show that $\Delta_f(t) = \pm\Delta_g(t)$. Hence K_f and K_g are isotopic.

Remark 5.6. By Theorems 3.6 and 5.5, for $n = 1$, even for hypersurface singularities which may not necessarily be locally irreducible, two algebraic knots are isotopic if and only if they are cobordant.

5.2. **Case of $n \geq 3$.** Suppose that f and g have isolated singularities at the origin in $\mathbb{C}^{n+1}$. In this subsection, we assume that $n \geq 3$.

Question 5.7 ([22]). Let K_f and K_g be homeomorphic to S^{2n-1}. If K_f and K_g are cobordant, then are K_f and K_g isotopic?

The answer to the above question is negative as follows.

Theorem 5.8 ([20]). *For all $n \geq 3$, there exist f and g such that K_f and K_g are homeomorphic to S^{2n-1}, and that they are cobordant but are not isotopic.*

Remark 5.9. For $n = 2$, Question 5.7 does not make sense (see Theorem 2.9). If we do not assume that the algebraic links are topological spheres, then we do not know if a result like Theorem 5.8 holds or not for $n = 2$.

6. Algebraic cobordism

In this section, we introduce the notion of algebraic cobordism for Seifert forms which corresponds to that of (geometric) cobordism of knots.

Let G_i be a free abelian group with finite rank, $i = 0, 1$. Let

$$L_i : G_i \times G_i \to \mathbb{Z}$$

be a bilinear form over $\mathbb{Z}$. We put

$$L = L_0 \oplus (-L_1) : G \times G \to \mathbb{Z},$$

where $G = G_0 \oplus G_1$.

Definition 6.1. Suppose that $m = \operatorname{rank} G$ is an even integer. A direct summand $M \subset G$ is called a *metabolizer* if M has rank equal to $m/2$ and L vanishes on M, that is $L(x, y) = 0$ for all $x, y \in M$.

Now, suppose that f and g have isolated singularities at the origin in $\mathbb{C}^{n+1}$. For the associated algebraic knots, we have the following.

Proposition 6.2 ([7])**.** *If K_f and K_g are cobordant, then there exists a metabolizer M for $L = L_f \oplus (-L_g)$.*

Proof. Let $X \subset S^{2n+1} \times [0, 1]$ be a cobordism between

$$K_f \subset S^{2n+1} \times \{0\} \quad \text{and} \quad K_g \subset S^{2n+1} \times \{1\}.$$

Then, we can show that there exists a compact orientable $(2n + 1)$-dimensional manifold $V \subset S^{2n+1} \times [0, 1]$ such that $\partial V = (F_f \times \{0\}) \cup X \cup ((-F_g) \times \{1\})$ (see Figure 15). This can be constructed by a standard argument as follows. By computing the 1st cohomology group, we can show that there exists a smooth map

$$(S^{2n+1} \times (-\varepsilon, 1 + \varepsilon)) \setminus ((F_f \times \{0\}) \cup X \cup ((-F_g) \times \{1\})) \to S^1$$

which is standard near the submanifold for some $\varepsilon > 0$. Then, we take an appropriate regular value of this smooth mapping and consider its inverse image by the map. Its closure gives V as desired.

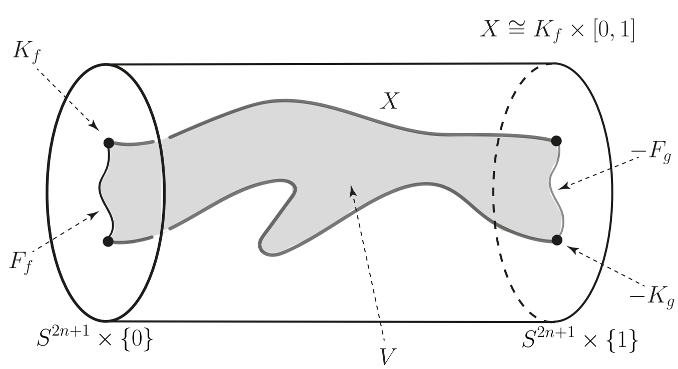

Figure 15. The manifold V.

Since F_f and F_g are $(n-1)$-connected and $K_f \cong K_g$ are $(n-2)$-connected, we can show that ∂V is $(n-1)$-connected.

In the following, homology groups are with integer coefficients. Consider the homology exact sequence of the pair $(V, \partial V)$:

$$0 \to H_{2n+1}(V) \to H_{2n+1}(V, \partial V) \to H_{2n}(\partial V) \to \cdots \to H_{n+1}(V, \partial V) \to \operatorname{Ker} j \to 0,$$

where $j : H_n(\partial V) \to H_n(V)$ is the homomorphism induced by the inclusion $\partial V \hookrightarrow V$. Then, by considering the alternating sum of the ranks of the above groups, which vanishes, together with the Poincaré duality, we obtain $\operatorname{rank}(\operatorname{Ker} j) = b_n(\partial V)/2$, where b_n denotes the n-th Betti number.

Consider

$$H_n(F_f) \oplus H_n(-F_g) \xrightarrow{\lambda} H_n(\partial V) \xrightarrow{j} H_n(V),$$

where λ is the homomorphism induced by the inclusions. Let us consider the subgroup

$$\lambda^{-1}((\operatorname{Ker} j)^\wedge) \subset H_n(F_f) \oplus H_n(-F_g),$$

where $(\operatorname{Ker} j)^\wedge$ is the smallest direct summand of $H_n(\partial V)$ that contains $\operatorname{Ker} j$. (In other words, $(\operatorname{Ker} j)^\wedge$ is the smallest primitive subgroup containing $\operatorname{Ker} j$.) Let $\alpha = [a]$ and $\beta = [b]$ be elements of $\lambda^{-1}((\operatorname{Ker} j)^\wedge)$, where a and b are n-cycles representing α and β, respectively. We may assume that both a and b are n-cycles of $F_f \cup (-F_g)$. Then, as a non-zero integral multiple of $\lambda(a)$ (resp. $\lambda(b)$) bounds an $(n+1)$-chain $\tilde{a}$ (resp. $\tilde{b}$) in V, we can show that $\operatorname{lk}(a_+, b) = 0$ by using $\tilde{a}_+$ and $\tilde{b}$ as they do not intersect with each other (see Figure 16), where $\tilde{a}_+$ is a translate of $\tilde{a}$ in the positive normal direction to V. Hence, we have $L(\alpha, \beta) = 0$.

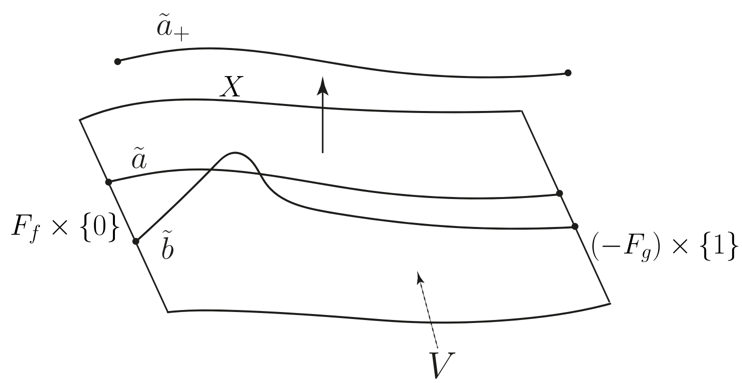

FIGURE 16. The $(n+1)$-chains $\tilde{a}_+$ and $\tilde{b}$ do not have intersections.

Then, by a bit more effort, we can find a metabolizer $M \subset \lambda^{-1}((\operatorname{Ker} j)^\wedge)$ (for details, see [7]). This completes the proof. □

Definition 6.3. Suppose that f and g have isolated singularities at the origin in $\mathbb{C}^{n+1}$ and consider the associated Seifert forms L_f and L_g of K_f and K_g, respectively. In the following, we set $G = H_n(F_f; \mathbb{Z}) \oplus H_n(-F_g; \mathbb{Z})$, $G^* = \operatorname{Hom}_{\mathbb{Z}}(G, \mathbb{Z})$, $L = L_f \oplus (-L_g)$, $S_f = L_f + (-1)^n L_f^T$, $S_g = L_g + (-1)^n L_g^T$, and $S = L + (-1)^n L^T$. Furthermore, for the adjoint $S^* : G \to G^*$ of S, we consider the quotient map $q : G \to \overline{G} = G/\operatorname{Ker} S^*$, and for a subgroup $M \subset G$, we set $\overline{M} = q(M) \subset \overline{G}$. We say that L_f and L_g are *algebraically cobordant* if there exists a metabolizer $M \subset H_n(F_f; \mathbb{Z}) \oplus H_n(-F_g; \mathbb{Z})$ for $L = L_f \oplus (-L_g)$ satisfying the conditions (i) and (ii) below.

(i) The subgroup $\overline{M}$ is pure (and hence primitive) in $\overline{G}$, that is, $\overline{G}/\overline{M}$ is torsion-free.

(ii) There exist isomorphisms

$$\varphi : \operatorname{Ker} S_f^* \to \operatorname{Ker} S_g^* \text{ and } \theta : \operatorname{Tors}(\operatorname{Coker} S_f^*) \to \operatorname{Tors}(\operatorname{Coker} S_g^*)$$

such that

(ii-1) $M \cap \operatorname{Ker} S^* = \{(x, \varphi(x)) \mid x \in \operatorname{Ker} S_f^*\}$,

(ii-2) for the projection $d : G^* \to \operatorname{Coker} S^* = G^*/\operatorname{Im} S^*$, we have

$$d(S^*(M)^\wedge) = \{(x, \theta(x)) \mid x \in \operatorname{Tors}(\operatorname{Coker} S_f^*)\},$$

where S_f^* and S_g^* are the adjoints of S_f and S_g, respectively, and $S^*(M)^\wedge$ is the smallest direct summand of G^* containing $S^*(M)$.

The notion of algebraic cobordism is very important as the following theorem shows.

Theorem 6.4 ([7]). *If $n \geq 3$, then K_f and K_g are cobordant if and only if the Seifert forms L_f and L_g associated with K_f and K_g, respectively, are algebraically cobordant.*

However, the practical problem is that it is usually very difficult to tell if two given Seifert forms are algebraically cobordant or not. In this sense, the following weaker notion sometimes plays an important role.

Definition 6.5. We say that Seifert forms L_f and L_g are *Witt equivalent over* $\mathbb{R}$ if there exists a metabolizer for $(L_f \otimes \mathbb{R}) \oplus (-L_g \otimes \mathbb{R})$, where the notion of a metabolizer for forms over $\mathbb{R}$ can be defined in the same way as in the case of integral bilinear forms (see Definition 6.1).

According to Proposition 6.2, we obviously have the following.

Proposition 6.6. *If two algebraic knots K_f and K_g are cobordant, then the Seifert forms L_f and L_g are Witt equivalent over $\mathbb{R}$.*

In the following section, we will see that Proposition 6.6 is useful for certain purposes.

7. Weighted homogeneous polynomials

In this section, we present some results about the topology of the following important class of polynomials.

Definition 7.1. A polynomial $f \in \mathbb{C}[z_1, z_2, \ldots, z_{n+1}]$ is *weighted homogeneous* if there exist positive rational numbers $(w_1, w_2, \ldots, w_{n+1})$, called *weights*, such that for every monomial $cz_1^{k_1} z_2^{k_2} \cdots z_{n+1}^{k_{n+1}}$, $c \neq 0$, of f, we have $\displaystyle\sum_{j=1}^{n+1} \frac{k_j}{w_j} = 1$.

Example 7.2. Here are some explicit examples of weighted homogeneous polynomials, together with their weights.

(1) $z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$, with weights $(a_1, a_2, \ldots, a_{n+1})$. (This is also called a *Brieskorn–Pham polynomial*. See Definition 4.1.)

(2) $z_1^{a_1} + z_2^{a_2} + z_2 z_3^{a_3}$, with weights $(a_1, a_2, a_2 a_3/(a_2 - 1))$.

(3) $f(z_1, z_2) = z_1^2 z_2 + z_1 z_2^6$ and $g(z_1, z_2) = z_1^3 z_2 + z_1 z_2^4$, with weights $(11/5, 11)$ and $(11/3, 11/2)$, respectively. (They have distinct weights, while it is known that they have the same Alexander polynomials: $\Delta_f(t) = \Delta_g(t) = (t-1)(t^{11} - 1)$ [54].)

(4) Consider the weighted homogeneous polynomials

$$F(z_1, z_2, \cdots, z_{n+1}) = z_1^2 z_2 + z_1 z_2^6 + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_{n+1}^2,$$
$$G(z_1, z_2, \ldots, z_{n+1}) = z_1^3 z_2 + z_1 z_2^4 + z_3^3 + z_4^{13} + z_5^2 + \cdots + z_{n+1}^2,$$

$n \geq 3$, with weights

$$(11/5, 11, 3, 13, 2, \dots, 2) \text{ and } (11/3, 11/2, 3, 13, 2, \dots, 2),$$

respectively. Since they are the same type of suspensions of the polynomials as in (3) above, they have the same Alexander polynomials. More precisely, by using a formula due to Milnor and Orlik [35], we get

$$\begin{aligned}\Delta_F(t) &= \Delta_G(t) \\ &= \frac{(t^{286} + (-1)^{n+1}t^{143} + 1)(t^{26} + (-1)^{n+1}t^{13} + 1)}{(t^{22} + (-1)^{n+1}t^{11} + 1)(t^2 + (-1)^{n+1}t + 1)}.\end{aligned}$$

Then, since we have $\Delta_F(1) = \Delta_G(1) = \pm 1$, we see that both K_F and K_G are homeomorphic to S^{2n-1} [34]. However, using a result of Steenbrink [52], we see that the signatures of the intersection forms of their Milnor fibers are distinct. Therefore, F and G do not have the same topological type and also at least one of them does not have the topological type of a Brieskorn–Pham polynomial (see [44]). Note that this kind of an example does not exist for the cases of 2 and 3 variables. More precisely, if f is a weighted homogeneous polynomial with an isolated singularity at the origin in $\mathbb{C}^2$ or $\mathbb{C}^3$ such that K_f is a circle or a homology 3-sphere, i.e. $\Delta_f(1) = \pm 1$, then f is topologically equivalent to a Brieskorn–Pham polynomial ([54, 44]).

(5) The weighted homogeneous polynomials

$$f(z_1, z_2, z_3) = z_1^5 + z_2^{31} + z_2 z_3^{75} \quad \text{and} \quad g(z_1, z_2, z_3) = z_1^7 + z_2^{11} + z_3^{154}$$

have weights $(5, 31, 155/2)$ and $(7, 11, 154)$, respectively. Then, by using results of [39], we can compute the Seifert invariants of the 3-dimensional manifolds K_f and K_g, and then we see that they are diffeomorphic to each other. Furthermore, by using a formula obtained in [35], we have that the Milnor numbers of f and g coincide. We also see that the signatures of their Milnor fibers coincide by using a formula obtained in [52]: however, K_f and K_g are not cobordant. In fact, the Alexander polynomials $\Delta_f(t)$ and $\Delta_g(t)$, which can be computed by using a result obtained in [35], do not satisfy the Fox–Milnor condition.

The following shows that the weights are analytic invariants of weighted homogeneous singularities.

Proposition 7.3 ([47])**.** *Suppose that f is a weighted homogeneous polynomial with an isolated singularity at the origin in $\mathbb{C}^{n+1}$. Then, the weights $(w_1, w_2, \dots, w_{n+1})$ can be chosen so that $w_j \geq 2$ for all j. Furthermore, under this condition, the weights are invariant under analytic change of coordinates up to order.*

Let f be a weighted homogeneous polynomial with an isolated singularity at the origin in $\mathbb{C}^{n+1}$ with weights $(w_1, w_2, \dots, w_{n+1})$ such that $w_j \geq 2$ for all j. We define

$$P_f(t) = \prod_{j=1}^{n+1} \frac{t - t^{1/w_j}}{t^{1/w_j} - 1},$$

which is known to be a polynomial in $\mathbb{Z}[t^{1/m}]$ for some m [51]. The following proposition shows that $P_f(t)$ encodes all the information on the weights.

Proposition 7.4 ([51])**.** *Suppose that f and g are weighted homogeneous polynomials with isolated singularities at the origin in $\mathbb{C}^{n+1}$. Then, f and g have the same weights up to order if and only if $P_f(t) = P_g(t)$.*

Related to the cobordism of algebraic knots defined by weighted homogeneous polynomials, the following is known.

Theorem 7.5 ([9])**.** *Suppose that f and g are weighted homogeneous polynomials with isolated singularities at the origin in $\mathbb{C}^{n+1}$. Then, their Seifert forms L_f and L_g are Witt equivalent to each other over $\mathbb{R}$ if and only if $P_f(t) \equiv P_g(t) \mod t+1$.*

Here we give a sketch of proof of the above theorem.

By considering the suspensions $f(z)+z_{n+2}^2$ and $g(z)+z_{n+2}^2$ if necessary, we may assume that n is even. For the Milnor fiber F_f of f, let us consider the decomposition

$$H^n(F_f;\mathbb{C}) = \bigoplus_{\lambda} H^n(F_f;\mathbb{C})_\lambda,$$

where $H^n(F_f;\mathbb{C})_\lambda$ is the eigenspace of the algebraic monodromy h^* for the eigenvalue λ and $h: F_f \to F_f$ is the geometric monodromy. Recall that for the Seifert form L_f for the algebraic knot K_f, $S_f = L_f + L_f^T$ gives the *intersection form* for F_f. Furthermore, the intersection form on $H^n(F_f;\mathbb{C})$ decomposes as the orthogonal direct sum of $S_f|_{H^n(F_f;\mathbb{C})_\lambda}$. For each λ, we set $\sigma_\lambda(f) = a_\lambda - b_\lambda$, which is called the *equivariant signature*, where a_λ and b_λ are the numbers of positive and negative eigenvalues, respectively, of $S_f|_{H^n(F_f;\mathbb{C})_\lambda}$.

Lemma 7.6 ([51])**.** *The Seifert forms L_f and L_g are Witt equivalent over $\mathbb{R}$ if and only if $\sigma_\lambda(f) = \sigma_\lambda(g)$ for all λ.*

Now, suppose that L_f and L_g are Witt equivalent over $\mathbb{R}$. We have $P_f(t) = P_f^0(t) + P_f^1(t)$, where for $P_f(t) = \sum c_\alpha t^\alpha$, we set

$$\begin{aligned} P_f^0(t) &= \sum_{\lfloor\alpha\rfloor:\ \text{even}} c_\alpha t^\alpha, \\ P_f^1(t) &= \sum_{\lfloor\beta\rfloor:\ \text{odd}} c_\beta t^\beta, \end{aligned}$$

where for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the largest integer not exceeding x. Similarly, we also have $P_g(t) = P_g^0(t) + P_g^1(t)$. According to [51], we have

$$\sigma_\lambda(f) = \sum_{\substack{\lambda=\exp(-2\pi i\alpha) \\ \lfloor\alpha\rfloor:\ \text{even}}} c_\alpha - \sum_{\substack{\lambda=\exp(-2\pi i\alpha), \\ \lfloor\alpha\rfloor:\ \text{odd}}} c_\alpha$$

for $\lambda \neq 1$, and a similar formula holds also for $\sigma_\lambda(g)$. Since $\sigma_\lambda(f) = \sigma_\lambda(g)$, we have

$$\begin{aligned} tP_f^0(t) - P_f^1(t) &\equiv tP_g^0(t) - P_g^1(t) \mod t^2-1, \\ tP_f^1(t) - P_f^0(t) &\equiv tP_g^1(t) - P_g^0(t) \mod t^2-1. \end{aligned}$$

Thus, we have $(t-1)P_f(t) \equiv (t-1)P_g(t) \mod t^2-1$ and therefore, we have $P_f(t) \equiv P_g(t) \mod t+1$.

Conversely, if $P_f(t) \equiv P_g(t) \mod t+1$, then we have $(t-1)P_f(t) \equiv (t-1)P_g(t) \mod t^2-1$. Then, we can show that $\sigma_\lambda(f) = \sigma_\lambda(g)$ for all λ, and by Lemma 7.6, the Seifert forms L_f and L_g are Witt equivalent over $\mathbb{R}$. This completes the proof of Theorem 7.5.

Remark 7.7. It is known that $L_f \otimes \mathbb{R} \cong L_g \otimes \mathbb{R}$ over $\mathbb{R}$ if and only if $P_f(t) \equiv P_g(t) \mod t^2-1$ [46].

As a consequence of Theorem 7.5, we have the following.

Corollary 7.8 ([9])**.** *Consider the Brieskorn–Pham polynomials*

$$\begin{aligned} f(z_1, z_2, \ldots, z_{n+1}) &= z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_n+1} \quad \textit{and} \\ g(z_1, z_2, \ldots, z_{n+1}) &= z_1^{b_1} + z_2^{b_2} + \cdots + z_{n+1}^{b_n+1} \end{aligned}$$

with $a_j \geq 2$ and $b_j \geq 2$ for all j. Then, the Seifert forms L_f and L_g are Witt equivalent over $\mathbb{R}$ if and only if

$$\prod_{j=1}^{n+1} \cot\left(\frac{\pi \ell}{2a_j}\right) = \prod_{j=1}^{n+1} \cot\left(\frac{\pi \ell}{2b_j}\right)$$

for all odd integers ℓ.

Proof. There exists a positive integer m such that $P_f(t) = Q_f(s)$ and $P_g(t) = Q_g(s)$ for some polynomials $Q_f(s)$ and $Q_g(s)$ in $s = t^{1/m}$. We have that $P_f(t) \equiv P_g(t) \mod t+1$ if and only if $Q_f(\xi) = Q_g(\xi)$ for all $\xi \in \mathbb{C}$ with $\xi^m = -1$. Note that $\xi = \exp(\pi i \ell/m)$ for odd integers ℓ, and we have

$$\frac{-1 - \exp(\pi i \ell/a_j)}{\exp(\pi i \ell/a_j) - 1} = i \cot\left(\frac{\pi \ell}{2a_j}\right).$$

Then, the result easily follows. □

Question 7.9. If we have

$$\prod_{j=1}^{n+1} \cot\left(\frac{\pi \ell}{2a_j}\right) = \prod_{j=1}^{n+1} \cot\left(\frac{\pi \ell}{2b_j}\right)$$

for all odd integers ℓ, do we have $a_j = b_j$ for all j up to order?

This question has an affirmative answer for $n = 1$ and $n = 2$ (see [9]). More precisely, for $n = 1$, we have the following.

Proposition 7.10 ([9])**.** *Let f and g be weighted homogeneous polynomials with isolated singularities at the origin in $\mathbb{C}^2$. Then, the Seifert forms L_f and L_g are Witt equivalent over $\mathbb{R}$ if and only if f and g have the same weights up to order.*

For $n = 2$, we have the following.

Proposition 7.11 ([9])**.** *Consider the Brieskorn–Pham polynomials $f(z_1, z_2, z_3) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}$ and $g(z_1, z_2, z_3) = z_1^{b_1} + z_2^{b_2} + z_3^{b_3}$ in 3 variables with $a_j \geq 2$ and $b_j \geq 2$ for all j. Then, the Seifert forms L_f and L_g are Witt equivalent over $\mathbb{R}$ if and only if $a_j = b_j$ for all j up to order.*

These results lead to the following question.

Question 7.12. Are the exponents of Brieskorn–Pham polynomials cobordism invariants? Compare this with Theorem 4.2.

In some special cases, the answer is affirmative as follows.

Theorem 7.13 ([9])**.** *Let f and g be Brieskorn–Pham polynomials with isolated singularities in $\mathbb{C}^{n+1}$. We assume that for each of f and g, no exponent is a multiple of the other. Then K_f and K_g are cobordant if and only if they have the same exponents up to order.*

Definition 7.14. Let f be a polynomial in $\mathbb{C}[z_1, z_2, \ldots, z_{n+1}]$ with $f(0) = 0$. Then, the minimum degree of the monomials in f is called the *multiplicity* of f at 0, denoted by $\mathrm{m}(f)$.

Example 7.15. For example, for $f(z_1, z_2, \ldots, z_{n+1}) = z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}}$, we have $\mathrm{m}(f) = \min\{a_1, a_2, \ldots, a_{n+1}\}$.

The following is well known as the Zariski Conjecture.

Conjecture 7.16 ([55])**.** *Suppose that f and g have isolated singularities at the origin in $\mathbb{C}^{n+1}$. If K_f and K_g are isotopic, then $\mathrm{m}(f) = \mathrm{m}(g)$. In other words, the multiplicity is a topological invariant for isolated complex hypersurface singularities.*

We can also ask the following question, which is still open as far as the authors know.

Question 7.17. Suppose that f and g have isolated singularities at the origin in $\mathbb{C}^{n+1}$. If K_f and K_g are cobordant, then do we have $\mathrm{m}(f) = \mathrm{m}(g)$?

If f and g are Brieskorn–Pham polynomials, the answer to Question 7.17 is affirmative in some cases as follows.

Proposition 7.18 ([9]). *Consider the Brieskorn–Pham polynomials*

$$\begin{aligned} f(z_1, z_2, \dots, z_{n+1}) &= z_1^{a_1} + z_2^{a_2} + \cdots + z_{n+1}^{a_{n+1}} \text{ and} \\ g(z_1, z_2, \dots, z_{n+1}) &= z_1^{b_1} + z_2^{b_2} + \cdots + z_{n+1}^{b_{n+1}} \end{aligned}$$

such that $a_j \geq 2$, $b_j \geq 2$ for all j and that $a_j \neq a_k$ and $b_j \neq b_k$ for $j \neq k$. If K_f and K_g are cobordant, then we have $\mathrm{m}(f) = \mathrm{m}(g)$.

8. Real Milnor fibrations

In the real setting, Milnor considered a real polynomial mapping $f : U \subset \mathbb{R}^n \to \mathbb{R}^p$ with $f(0) = 0$, $n \geq p \geq 2$ such that in some open neighborhood U of the origin $0 \in \mathbb{R}^n$, we have $\Sigma_f \cap U = \{0\}$, where

$$\Sigma_f = \{x \in U \,|\, \mathrm{rank}\, Jf(x) \text{ fails to be maximal}\},$$

and $Jf(x)$ denotes the Jacobian matrix of f at x. This means that 0 is an isolated singular point of the mapping f.

Milnor showed the existence of fiber bundle structures for real maps with isolated singular points as follows.

Theorem 8.1 ([34, Theorem 11.2]). *There exists an $\epsilon_0 > 0$ small enough such that for all ϵ with $0 < \epsilon \leq \epsilon_0$, there exists an η_0 with $0 < \eta_0 \ll \epsilon$ such that the restriction map*

$$f_| : f^{-1}(S_\eta^{p-1}) \cap B_\epsilon^n \to S_\eta^{p-1} \tag{8.1}$$

is the projection of a smooth locally trivial fiber bundle for all η with $0 < \eta \leq \eta_0$, where B_ϵ^n denotes the n-dimensional closed ball centered at the origin of radius ϵ in $\mathbb{R}^n$, and S_η^{p-1} denotes the sphere of radius η centered at the origin in $\mathbb{R}^p$.

We denote by $K_f = f^{-1}(0) \cap S_\epsilon^{n-1}$, the *link* of the singularity at the origin, where S_ϵ^{n-1} is the sphere of radius ϵ centered at the origin in $\mathbb{R}^n$. The isotopy class of the oriented submanifold K_f of S^{n-1} is called the *real algebraic knot* associated with f at the origin.

Remark 8.2 ([34, p. 99]). The complement $S_\epsilon^{n-1} \setminus K_f$ also fibers over S_η^{p-1}, each fiber F_f being the interior of a compact manifold $\overline{F}_f$ bounded by K_f. In fact, Milnor showed that $f^{-1}(B_\eta^p) \cap B_\epsilon^n$ is diffeomorphic to an n-dimensional ball and that $S_\epsilon^{n-1} \setminus K_f$ is diffeomorphic to $\partial(f^{-1}(B_\eta^p) \cap B_\epsilon^n) \setminus K_f$, after smoothing the corners. Milnor also showed that F_f is $(p-2)$-connected, provided that the link K_f is not empty.

As Milnor points out, it is difficult to find explicit examples of polynomial mappings with an isolated singular point at the origin as above. Then, Milnor in [34, p.100] posed the following question.

Question 8.3. For which dimensions $n \geq p \geq 2$ do non-trivial examples exist?

According to Milnor, the projection $f(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_p)$ is trivial. In general, for a map $f : U \subset \mathbb{R}^n \to \mathbb{R}^p$, $n \geq p \geq 2$, with an isolated singular point at the origin, Milnor proposed the following definition: "An example will be called *trivial* if the fiber $\overline{F}_f$ of (8.1) is diffeomorphic to the closed ball B^{n-p}".

In [17] Church and Lamotke answered Milnor's question (Question 8.3) in most cases in the following way.

Theorem 8.4 ([17, p. 149])**.**

(a) *For $0 \leq n-p \leq 2$, non-trivial examples occur precisely for the dimensions (n,p) in the set $\{(2,2),(4,3),(4,2)\}$.*
(b) *For $n-p \geq 4$, non-trivial examples occur for all (n,p).*
(c) *For $n-p=3$, all examples are trivial except for $(n,p)=(5,2)$, $(8,5)$ and possibly $(6,3)$.*

For the pair $(5,2)$ we have the following characterization of triviality.

Proposition 8.5. *Let $f:(\mathbb{R}^5,0)\to(\mathbb{R}^2,0)$ be a real polynomial map germ with an isolated singularity at origin. Then, f is trivial if and only if the associated real algebraic knot is an unknotted 2-sphere in S^4.*

Proof. If f is trivial, then the knot K_f, which is isotopic to the boundary $\partial\overline{F}_f$ of the closure of the Milnor fiber F_f, which is diffeomorphic to B^3, must be an unknotted 2-sphere.

Conversely, suppose that K_f is an unknotted 2-sphere. Consider the following piece of the homotopy long exact sequence of the Milnor fibration $F_f \hookrightarrow S^4\setminus K_f \to S^1$:

$$\pi_2(S^1)\to\pi_1(F_f)\to\pi_1(S^4\setminus K_f)\to\pi_1(S^1)\to\pi_0(F_f). \tag{8.2}$$

Note that $\overline{F}_f$ and F_f have the same homotopy type. Therefore, $\pi_0(\overline{F}_f)\cong\pi_0(F_f)$ vanishes, since F_f is connected. Moreover, since K_f is an unknotted 2-sphere, we have $\pi_1(S^4\setminus K_f)\cong\mathbb{Z}$. Thus we can write (8.2) as follows:

$$0\to\pi_1(F_f)\to\mathbb{Z}\to\mathbb{Z}\to 0.$$

Then, we have that $\pi_1(F_f)$ vanishes and therefore F_f is simply connected.

Let $N=\overline{F}_f\cup_{K_f}B^3$ be the 3-dimensional closed manifold obtained by attaching a 3-ball to $\overline{F}_f$ along the 2-sphere boundary. We have, by the van Kampen theorem, $\pi_1(N)\cong\pi_1(\overline{F}_f)\cong\pi_1(F_f)$, which vanishes. Then, by the Poincaré conjecture (proved by Perelman), we conclude that N is diffeomorphic to the 3-sphere and consequently $\overline{F}_f$ is diffeomorphic to a 3-ball, and hence, f is trivial. □

According to Church–Lamotke's Theorem (Theorem 8.4), if $n-p=3$ Milnor's question (Question 8.3) remained open only for the dimension pair $(n,p)=(6,3)$. The answer to this case was given in [3, Section 3] as follows.

Theorem 8.6 ([3])**.** *For each integer $r>0$, there exists a polynomial mapping*

$$f:(\mathbb{R}^6,0)\to(\mathbb{R}^3,0)$$

with an isolated singularity at the origin such that the link K_f has $2r+1$ connected components. In particular, f is non-trivial.

It is an interesting but difficult problem to find an explicit example of polynomials as in the above theorem.

Now let us consider the realization problem of fibered knots.

Definition 8.7. A closed oriented submanifold M of dimension $n-p-1$ of S^{n-1} is called a *fibered knot* if the following conditions hold.

(1) The normal disk bundle $N(M)$ of M in S^{n-1} is trivial and we have a trivialization $\tau: N(M)\to M\times D^p$.

(2) There is a smooth locally trivial fiber bundle $\pi : S^{n-1} \setminus M \to S^{p-1}$ such that $\pi|_{N(M)\setminus M}$ coincides with the composition

$$N(M) \setminus M \xrightarrow{\tau|_{N(M)\setminus M}} M \times (D^p \setminus \{0\}) \xrightarrow{p_2} D^p \setminus \{0\} \xrightarrow{\rho} S^{p-1},$$

where ρ is the radial projection defined by $\rho(x) = x/||x||$, $x \in D^p \setminus \{0\}$, and p_2 is the projection to the second factor.

Such a structure is often called an *open book structure* or a *spinnable structure* as well in the literature. The manifold pair (S^{n-1}, M) is often called an *NS-pair*, where "NS" stands for Neuwirth–Stallings.

Note that by Theorem 8.1, a real polynomial mapping with an isolated singular point at the origin gives rise to a fibered knot. Then, we have the following natural problem.

Problem 8.8. Characterize those fibered knots which arise as real algebraic knots.

Related to the above problem, the following is known.

Theorem 8.9 ([2])**.** *Let M be an arbitrary $(n-p-1)$-dimensional closed submanifold of S^{n-1} with $n \geq p \geq 2$. We assume that it bounds a compact $(n-p)$-dimensional submanifold of S^{n-1} with trivial normal bundle. Then, there exists a polynomial mapping $f : \mathbb{R}^n \to \mathbb{R}^p$ with $f(0) = 0$ such that $f^{-1}(0) \cap S^{n-1}_\epsilon$ is isotopic to M for sufficiently small $\epsilon > 0$ and that $f^{-1}(0)$ has an isolated singular point at the origin.*

Note that in the above theorem, in $f^{-1}(0) \cap \Sigma_f$, the origin is isolated. Therefore, f may not have an isolated singular point.

We have the following conjecture.

Conjecture 8.10 ([6])**.** *Every fibered knot in S^3 is realized as the algebraic knot associated with a polynomial mapping $\mathbb{R}^4 \to \mathbb{R}^2$ with an isolated singular point at the origin.*

As far as the authors know, the above conjecture is still open. Let us present below a very short account of the conjecture that may motivate the reader's interest. The authors would like to apologize if some important contributions are not cited or mentioned here in this short review.

In [32] Looijenga introduced a topological construction aiming to answer Milnor's questions concerning the existence of non-trivial examples, as described in the paragraph just after Question 8.3, and the existence of real polynomial map germs $(\mathbb{R}^{2m}, 0) \to (\mathbb{R}^2, 0)$ that are not topologically equivalent to holomorphic function germs.

With this purpose he started with an $(\ell - q - 1)$-dimensional fibered knot K in S^ℓ and considered the connected sum $(S^\ell, K)\sharp((-1)^{\ell-1}S^\ell, (-1)^{\ell-q}K)$, which was shown to be associated with a real polynomial mapping $f : \mathbb{R}^{\ell+1} \to \mathbb{R}^{q+1}$ with an isolated singularity at the origin, up to isotopy. He then applied the construction to the figure eight knot K, and obtained a real polynomial map $\mathbb{R}^4 \to \mathbb{R}^2$ with an isolated singularity at origin whose associated real algebraic knot is isotopic to $K\sharp K$. It turns out that $K\sharp K$ cannot appear as the algebraic knot associated with a holomorphic function germ, since its Alexander polynomial is not a product of cyclotomic polynomials (see [15]). Furthermore, he showed the existence of such examples in higher dimensions by performing the spinning construction for K an even number of times and then by using the connected sum construction as above. This guarantees the existence of a real polynomial map germ $(\mathbb{R}^{2m}, 0) \to (\mathbb{R}^2, 0)$ with an isolated singularity at the origin, for each $m \geq 2$, which is not topologically equivalent to a holomorphic function germ $(\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$.

Since the fibered knots constructed in this way are all real algebraic knots, one might consider them as the first examples toward Conjecture 8.10, even before it was stated.

A'Campo [1] proved the vanishing of the Lefschetz number for the geometric monodromy of the fibered knot associated with a holomorphic function germ with an isolated singularity. He calculated the Lefschetz number for the real polynomial

$$f(u, v, z_1, z_2, \ldots, z_m) = uv(\bar{u} + \bar{v}) + z_1^2 + z_2^2 + \cdots + z_m^2$$

and concluded that f cannot be topologically equivalent to a holomorphic function germ from $(\mathbb{C}^{m+2}, 0)$ to $(\mathbb{C}, 0)$.

It seems that the concept of real algebraic knots as currently used today was introduced by Perron in [40], where it is shown that the figure eight knot admits a realization as the real algebraic knot K_f associated with a polynomial map $f : \mathbb{R}^4 \to \mathbb{R}^2$ with an isolated singularity at the origin. The construction of such an f, however, was done in a non-standard way. Rudolph [42] exhibited the complex (but not holomorphic) polynomial

$$f(u, v) = u^3 - 3(v\bar{v})^2(1 + v^2 - \bar{v}^2)u - 2(v^2 + \bar{v}^2)$$

with an isolated singularity at the origin such that the associated real algebraic knot is also the figure eight knot.

Surprisingly, these complex polynomials as presented by A'Campo and Rudolph are included in a wider class of polynomials later called *mixed polynomials* and studied extensively by Oka [38]. This class was also studied by other mathematicians including Seade, Cisneros-Molina, Ruas, Pichon, Tibăr, Chen, Ribeiro, etc.

More recently Bode and Dennis [11, 10] built a family of mixed polynomials related to braid parametrizations of links in S^3, where the idea for the construction is similar to that by Perron [40, pp. 443–445]. Their construction shows, in a direct way, a close relationship between links in S^3 and links of mixed semi-algebraic singularities with special properties, like radial actions. In particular, under good conditions, their family gives polynomial mappings with an isolated singularity at the origin, providing in this way a partial answer to Conjecture 8.10.

Finally, Araújo dos Santos and Sanchez Quiceno [4] addressed the question of clarifying the connection between certain classes of mixed singularities and Conjecture 8.10. For this purpose, they considered the product $p = fg$ of mixed polynomials f and $g : \mathbb{C}^2 \to \mathbb{C}$, and under general conditions they studied conditions for p to have an isolated singularity at the origin, even when the link K_f or K_g associated with f or g, respectively, is not fibered. See [4, Example 3.7] for details. These studies suggest that Conjecture 8.10 may be approached by mixed polynomial singularities.

Acknowledgment

The second author would like to express his deep gratitude to the organizers of "IV Encontro de Singularidades no Nordeste" held at Departamento de Matemática da Universidade Federal da Paraíba, João Pessoa, Brazil, during November 22–24, 2017, for giving him an opportunity to give a mini-course on the topology of algebraic knots. The third author would like to thank the Organizing Committee of the 16th International Workshop on Real and Complex Singularities for giving her the opportunity to give a talk on the Milnor fibration and its topology. The authors would like to express thanks to the anonymous referee for important comments which drastically improved the presentation of the paper. They would also like to thank Eder Leandro Sanchez Quiceno for helpful discussions. The first author thanks Fapesp/Tematic project process: 2019/21181-0. The second author has been supported by JSPS KAKENHI Grant Number JP17H06128.

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