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ZARISKI MULTIPLES ASSOCIATED WITH QUARTIC CURVES

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Abstract. We investigate reducible plane curves whose irreducible components are a general smooth quartic curve, some of its bitangents, and some of its 4-tangent conics. We show that the deformation types of plane curves of this type coincide with the homeomorphism types. The number of deformation types grows as the 62nd power of the degree of the plane curves when the degree tends to infinity. Thus we obtain Zariski multiples of large sizes.

1. Introduction

By a *plane curve*, we mean a reduced, possibly reducible, complex projective plane curve. We say that two plane curves C and C' have the same *combinatorial type* if there exist tubular neighbourhoods $T(C)$ of C and $T(C')$ of C' such that $(T(C), C)$ and $(T(C'), C')$ are homeomorphic, whereas we say that C and C' have the same *homeomorphism type* if $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are homeomorphic.

A *Zariski N-ple* is a set of plane curves $\{C_1, \dots, C_N\}$ such that the curves C_i have the same combinatorial type, but their homeomorphism types are pairwise different. The notion of Zariski N-ples was introduced by Artal-Bartolo [1] in reviving a classical example of 6-cuspidal curves of degree 6 due to Zariski. Since then, this notion has been studied by many people from various points of view. Some of the tools that have been used in this investigation are: Alexander polynomials, characteristic varieties, fundamental groups of complements, topological invariants of branched coverings, and so on. See the survey paper [3]. Recently, Bannai et al. [4, 5, 6] have investigated Zariski N-ples such that each member is a union of a smooth quartic curve and some of its bitangents.

In this paper, we introduce 4-tangent conics of a smooth quartic curve, and consider Zariski N-ples $Z_1, \dots, Z_N$ such that each Z_i is a union of a smooth quartic curve, some of its bitangents, and some of its 4-tangent conics.

Let Q be a smooth quartic curve. A *bitangent* of Q is a line whose intersection multiplicity with Q is even at each intersection point. It is well known that every smooth quartic curve has exactly 28 bitangents. We say that a bitangent $\bar{l}$ of Q is *ordinary* if $\bar{l}$ is tangent to Q at distinct 2 points. A smooth conic $\bar{c} \subset \mathbb{P}^2$ is called a *4-tangent conic of Q* if $\bar{c}$ is tangent to Q at 4 distinct points. Every smooth quartic curve has 63 one-dimensional connected families of 4-tangent conics (see Theorem 4.1).

Definition 1.1. Let m and n be non-negative integers such that $m \le 28$. We say that a plane curve Z is a *$\mathcal{Q}^{(m,n)}$-curve* if Z is of the form

$$Z = Q + \bar{l}_1 + \cdots + \bar{l}_m + \bar{c}_1 + \cdots + \bar{c}_n, \tag{1.1}$$

where Q is a smooth quartic curve, $\bar{l}_1, \dots, \bar{l}_m$ are distinct bitangents of Q, and $\bar{c}_1, \dots, \bar{c}_n$ are distinct 4-tangent conics of Q, and they satisfy that

(i) the bitangents $\bar{l}_1, \dots, \bar{l}_m$ are ordinary,
(ii) the intersection of any three of $Q, \bar{l}_1, \dots, \bar{l}_m, \bar{c}_1, \dots, \bar{c}_n$ is empty, and
(iii) the intersection of any two of $\bar{l}_1, \dots, \bar{l}_m, \bar{c}_1, \dots, \bar{c}_n$ is transverse.

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m	1	2	3	4	5	6	7	8	9	10	11	12	13	14
N	1	1	2	3	5	10	16	23	37	54	70	90	101	103
G	1	1	2	3	5	9	16	23	37	54	70	90	101	103

TABLE 1.1. $N^{(m,0)} = N^{(28-m,0)}$

n	1	2	3	4	5	6	7	8	9	10
N	1	3	9	30	112	501	2483	13791	81404	490750
G	1	3	7	22	71	306	1585	9831	64790	425252

TABLE 1.2. $N^{(0,n)}$

(m,n)	$(1,1)$	$(1,2)$	$(2,1)$	$(1,3)$	$(2,2)$	$(3,1)$	$(1,4)$	$(2,3)$	$(3,2)$	$(4,1)$
N	2	8	4	33	23	9	162	132	66	20
G	2	8	3	30	17	8	140	95	57	17

(m,n)	$(1,5)$	$(2,4)$	$(3,3)$	$(4,2)$	$(5,1)$	$(1,6)$	$(2,5)$	$(3,4)$	$(4,3)$	$(5,2)$	$(6,1)$
N	901	889	508	190	45	5674	6503	4348	1854	531	103
G	753	670	430	164	42	4829	5259	3812	1649	501	96

TABLE 1.3. $N^{(m,n)}$

It is obvious that any two $\mathcal{Q}^{(m,n)}$-curves have the same combinatorial type. We construct a non-singular variety $\mathcal{Z}^{(m,n)}$ parameterizing all $\mathcal{Q}^{(m,n)}$-curves in Section 5.

Definition 1.2. We say that two $\mathcal{Q}^{(m,n)}$-curves have the same *deformation type* if they belong to the same connected component of the parameter space $\mathcal{Z}^{(m,n)}$.

It is obvious that $\mathcal{Q}^{(m,n)}$-curves of the same deformation type have the same homeomorphism type. Our main results are the following:

Theorem 1.3. *If two $\mathcal{Q}^{(m,n)}$-curves have the same homeomorphism type, then they have the same deformation type.*

We put

$$d^{(m,n)} := \binom{28}{m} \cdot \binom{n+62}{62}, \tag{1.2}$$

which grows as $O(n^{62})$ when $n \to \infty$.

Theorem 1.4. *The number $N^{(m,n)}$ of deformation types of $\mathcal{Q}^{(m,n)}$-curves satisfies*

$$d^{(m,n)}/1451520 \leq N^{(m,n)} \leq d^{(m,n)}. \tag{1.3}$$

The main ingredient of the proof of these results is the monodromy argument of Harris [9] (see Theorem 3.1). This argument converts the problem of enumerating deformation types of $\mathcal{Q}^{(m,n)}$-curves to an easy combinatorial problem of counting orbits of an action of the Weyl group $W(E_7)$ on a certain finite set. In Tables 1.1, 1.2, 1.3, we give a list of $N^{(m,n)}$ for some (m,n). See Section 5.2 for more detail.

To each $\mathcal{Q}^{(m,n)}$-curve Z, we associate a discrete invariant $g(Z)$, which we call an *intersection graph*. This invariant is similar to the *splitting graph* defined in [19]. Note that each of the bitangents $\bar{l}_1, \dots, \bar{l}_m$ and a 4-tangent conics $\bar{c}_1, \dots, \bar{c}_n$ of the smooth quartic curve $Q \subset Z$ splits by the double covering $Y \to \mathbb{P}^2$ branched along Q. This data $g(Z)$ describes how the irreducible components of these pull-backs intersect on Y. See Section 8 for the precise definition. In Tables, we also present the number G of non-isomorphic intersection graphs obtained from $\mathcal{Q}^{(m,n)}$-curves. When $n = 0$, the intersection graph $g(Z)$ is the two-graph studied in [5], in which Bannai and Ohno studied $\mathcal{Q}^{(m,0)}$-curves for $m \le 6$, and enumerated their homeomorphism types that can be distinguished by the two-graphs. See Sections 9.1 and 9.2 for the details.

The 4-tangent conics of a smooth quartic curve Q are related to the 2-torsion points of the Jacobian of Q (see Remark 4.3). A similar idea applied to plane cubic curves enabled us to construct in [15] certain equisingular families of plane curves with many connected components. In [15], it was also shown that these connected components cannot be distinguished by the fundamental groups of the complements, because they are all abelian. Then it was shown in [8] and [18] that the homeomorphism types of distinct connected components of these families can be distinguished by the invariant called *linking numbers*.

The embedding topology of reducible plane curves whose irreducible components are tangent to each other was also investigated by Artal Bartolo, Cogolludo, and Tokunaga in [2] from the view point of dihedral covering of the plane branched along the curve. In this case, the complement can have a non-abelian fundamental group. See [2, Corollary 1].

It would be an interesting problem to study the fundamental groups of the complements of $\mathcal{Q}^{(m,n)}$-curves, and their related invariants such as linking numbers and/or (non-)existence of finite coverings of the plane with prescribed Galois groups.

Via the cyclic covering of the plane of degree 4 branched along a smooth quartic curve, the geometry of $\mathcal{Q}^{(m,n)}$-curves is related to the geometry of $K3$ surfaces. By considering the double covering branched along a singular sextic curve and employing Torelli theorem for complex $K3$ surfaces, we have investigated in [16] Zariski N-ples of plane curves of degree 6 with only simple singularities. We expect that a similar idea can be applied to Zariski N-ples associated with singular quartic curves.

This paper is organized as follows. In Section 2, we introduce cyclic coverings $X_u \to Y_u \to \mathbb{P}^2$ branched over a smooth quartic curve Q_u, and fix some notation. In Section 3, we recall the result of Harris [9]. In Section 4, we construct the family of 4-tangent conics. In Section 5, we construct the space $\mathcal{Z}^{(m,n)}$ parameterizing all $\mathcal{Q}^{(m,n)}$-curves, and prove Theorems 1.3 and 1.4. In Section 6, we further study the family of 4-tangent conics in detail. The geometry of the $K3$ surface X_u is closely investigated. In Section 7, we study the configurations of lifts in Y_u of bitangents and 4-tangent conics, and we define the intersection graph $g(Z)$ in Section 8. In Section 9, we examine some examples for small m and n.

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2. Coverings of $\mathbb{P}^2$

For a positive integer d, we put

$$\Gamma(d) := H^0(\mathbb{P}^2, \mathcal{O}(d)).$$

Let $\mathcal{U}$ denote the space of smooth quartic curves, which is a Zariski open subset of $\mathbb{P}_*(\Gamma(4))$. Let u be a point of $\mathcal{U}$. We denote by $Q_u \subset \mathbb{P}^2$ the smooth quartic curve corresponding to the point u. We consider the following branched coverings:

$$\gamma_u \colon X_u \xrightarrow{\eta_u} Y_u \xrightarrow{\pi_u} \mathbb{P}^2,$$

where π_u is the double covering of $\mathbb{P}^2$ branched along Q_u, η_u is the double covering of Y_u branched along $\pi_u^{-1}(Q_u)$, and $\gamma_u = \pi_u \circ \eta_u$ is the cyclic covering of degree 4 of $\mathbb{P}^2$ branched along Q_u. We put

$$SY_u := H^2(Y_u, \mathbb{Z}),$$

$\mathbb{P}^2$	Y_u	X_u
$\bar{l}$	$\pi_u^*(\bar{l}) = l + l'$	$\gamma_u^*(\bar{l}) = \tilde{l} + \tilde{l}'$
	$\langle l, l\rangle_Y = \langle l', l'\rangle_Y = -1$	$\langle \tilde{l}, \tilde{l}\rangle_X = \langle \tilde{l}', \tilde{l}'\rangle_X = -2$
	$\langle l, l'\rangle_Y = 2$	$\langle \tilde{l}, \tilde{l}'\rangle_X = 4$
	$\langle h_u, l\rangle_Y = \langle h_u, l'\rangle_Y = 1$	$\langle \tilde{h}_u, \tilde{l}\rangle_X = \langle \tilde{h}_u, \tilde{l}'\rangle_X = 2$
$\bar{c}$	$\pi_u^*(\bar{c}) = c + c'$	$\gamma_u^*(\bar{c}) = \tilde{c} + \tilde{c}'$
	$\langle c, c\rangle_Y = \langle c', c'\rangle_Y = 0$	$\langle \tilde{c}, \tilde{c}\rangle_X = \langle \tilde{c}', \tilde{c}'\rangle_X = 0$
	$\langle c, c'\rangle_Y = 4$	$\langle \tilde{c}, \tilde{c}'\rangle_X = 8$
	$\langle h_u, c\rangle_Y = \langle h_u, c'\rangle_Y = 2$	$\langle \tilde{h}_u, \tilde{c}\rangle_X = \langle \tilde{h}_u, \tilde{c}'\rangle_X = 4$

TABLE 2.1. Pull-backs of bitangents and 4-tangent conics

which is a unimodular lattice of rank 8 with the cup-product $\langle\ ,\ \rangle_Y$. Let $h_u \in SY_u$ be the class of the pull-back of a line on $\mathbb{P}^2$ by π_u. It is well known that Y_u is a del Pezzo surface of degree 2 with the anti-canonical class h_u. (See [7, Chapters 6 and 8] about del Pezzo surfaces.) On the other hand, the surface X_u is a $K3$ surface. Let $\langle\ ,\ \rangle_X$ denote the cup product of $H^2(X_u, \mathbb{Z})$, and let $\tilde{h}_u$ be the class $\eta_u^*(h_u)$. Then $\tilde{h}_u$ is an ample class of degree $\langle \tilde{h}_u, \tilde{h}_u\rangle_X = 4$.

It is classically known that every smooth quartic curve Q_u has exactly 28 bitangents. Moreover, if u is general in $\mathcal{U}$, all bitangents $\bar{l}$ of Q_u are ordinary, that is, $\bar{l}$ is tangent to Q_u at two distinct points.

Definition 2.1. A reduced conic $\bar{c} \subset \mathbb{P}^2$ is called a *splitting conic* of Q_u if the intersection multiplicity of Q_u and $\bar{c}$ is even at each intersection point.

A smooth conic $\bar{c}$ is splitting if and only if $\pi_u^*(\bar{c}) \subset Y_u$ has two irreducible components. A singular reduced conic $\bar{c}$ is splitting if and only if $\bar{c}$ is a union of two distinct bitangents.

It is easy to see that a smooth conic $\bar{c} = \{g = 0\}$ defined by $g \in \Gamma(2)$ is a splitting conic of $Q_u = \{\varphi = 0\}$ defined by $\varphi \in \Gamma(4)$ if and only if there exist polynomials $f \in \Gamma(2)$ and $q \in \Gamma(2)$ such that $\varphi = fg + q^2$. By an easy dimension counting, we see the following:

Lemma 2.2. *Suppose that u is general in $\mathcal{U}$. Let $\bar{c} \subset \mathbb{P}^2$ be a smooth splitting conic of Q_u. Then the intersection multiplicities of Q_u and $\bar{c}$ are either* $(2,2,2,2)$ *or* $(4,2,2)$. □

Definition 2.3. A smooth splitting conic $\bar{c} \subset \mathbb{P}^2$ of Q_u is called a 4-*tangent conic* (resp. a 3-*tangent conic*) of Q_u if the intersection multiplicities of Q_u and $\bar{c}$ are $(2,2,2,2)$ (resp. $(4,2,2)$).

The following is easy to verify. The results are summarized in Table 2.1.

Proposition 2.4. (1) *Let $\bar{l}$ be an ordinary bitangent of Q_u. Then $\pi_u^*(\bar{l})$ is a union of two smooth rational curves l and l' on Y_u with self-intersection -1 that intersect at two points transversely, and $\gamma_u^*(\bar{l})$ is a union of two smooth rational curves $\tilde{l}$ and $\tilde{l}'$ on X_u with self-intersection -2 that intersect at two points with intersection multiplicities* $(2,2)$.

(2) *Let $\bar{c}$ be a* 4*-tangent conic of Q_u. Then $\pi_u^*(\bar{c})$ is a union of two smooth rational curves c and c' on Y_u with self-intersection* 0 *that intersect at four points transversely, and $\gamma_u^*(\bar{c})$ is a union of two smooth elliptic curves $\tilde{c}$ and $\tilde{c}'$ on X_u with self-intersection* 0 *that intersect at four points with intersection multiplicities* $(2,2,2,2)$. □

Definition 2.5. A curve l on Y_u is called a Y*-lift* of a bitangent $\bar{l}$ of Q_u if π_u maps l to $\bar{l}$ isomorphically. We also say that a curve c on Y_u is a Y*-lift* of a splitting conic $\bar{c}$ of Q_u if π_u maps c to $\bar{c}$ isomorphically.

3. Monodromy

Let u be a point of $\mathcal{U}$. It is well known that the lattice $SY_u = H^2(Y_u, \mathbb{Z})$ is isomorphic to the lattice of rank 8 whose Gram matrix is the diagonal matrix $\mathrm{diag}(1, -1, \ldots, -1)$, and that the orthogonal complement

$$\Sigma_u := (\mathbb{Z}h_u \hookrightarrow SY_u)^\perp$$

of the ample class h_u in SY_u is isomorphic to the negative-definite root lattice of type E_7. The deck transformation

$$\iota_u \colon Y_u \to Y_u$$

of $\pi_u \colon Y_u \to \mathbb{P}^2$ acts on Σ_u as -1. Note that the group $\mathrm{O}(\Sigma_u)$ of isometries of Σ_u is equal to the Weyl group $W(E_7)$, which is of order 2903040. Hence there exists an injective homomorphism

$$\mathrm{O}(SY_u, h_u) := \{\, g \in \mathrm{O}(SY_u) \mid h_u^g = h_u \,\} \;\hookrightarrow\; W(E_7). \tag{3.1}$$

It is easy to check that the action on Σ_u of each of the standard generators of $W(E_7)$ lifts to an isometry of SY_u that fixes h_u. Hence the homomorphism (3.1) is in fact an isomorphism. The family of lattices $\{\, SY_u \mid u \in \mathcal{U} \,\}$ forms a locally constant system

$$\mathcal{SY} \to \mathcal{U}.$$

Let b be a general point of $\mathcal{U}$, which will serve as a base point of $\mathcal{U}$. The monodromy action of $\pi_1(\mathcal{U}, b)$ on the lattice SY_b preserves $h_b \in SY_b$.

Theorem 3.1 (Harris [9])**.** *The monodromy homomorphism*

$$\pi_1(\mathcal{U}, b) \to \mathrm{O}(SY_b, h_b) \cong W(E_7) \tag{3.2}$$

is surjective.

The original statement in [9] is not on the monodromy action on the lattice SY_b, but on the Galois group $W(E_7)/\{\pm 1\} \cong \mathrm{GO}_6(\mathbb{F}_2)$ of bitangents of Q_b. Moreover the proof in [9] is via the proof of a similar result on cubic surfaces with E_7 replaced by E_6. Hence we give a direct and simple proof of Theorem 3.1 below.

For the proof, we prepare some more notation, which will be used throughout this paper. We denote by $\overline{L}_u$ the set of bitangents of Q_u, and L_u the set of Y-lifts of bitangents of Q_u. Let $\Sigma_u^\vee$ denote the dual lattice of Σ_u. By identifying $l \in L_u$ with its class $[l] \in SY_u$, we have an identification

$$\begin{aligned} L_u \quad &= \quad \{\, v \in SY_u \mid \langle v, h_u \rangle_Y = 1,\ \langle v, v \rangle_Y = -1 \,\} \\ &\cong \quad \{\, v \in \Sigma_u^\vee \mid \langle v, v \rangle_Y = -3/2 \,\}, \end{aligned} \tag{3.3}$$

where the second bijection is obtained by the orthogonal projection $SY_u \to \Sigma_u^\vee$. We put $\overline{SY}_u := SY_u/\langle \iota_u \rangle$, and consider the commutative diagram

$$\begin{array}{ccc} L_u & \hookrightarrow & SY_u \\ \downarrow & & \downarrow \\ \overline{L}_u & \hookrightarrow & \overline{SY}_u, \end{array} \tag{3.4}$$

where vertical arrows are quotient maps by the involution ι_u. Since the action of $\pi_1(\mathcal{U}, b)$ on SY_b commutes with ι_b, we have a monodromy action of $\pi_1(\mathcal{U}, b)$ on $\overline{SY}_b$. Thus we obtain a diagram

$$\begin{array}{ccc} \mathcal{L} & \hookrightarrow & \mathcal{SY} \\ \downarrow & & \downarrow \\ \overline{\mathcal{L}} & \hookrightarrow & \overline{\mathcal{SY}} \end{array} \tag{3.5}$$

of locally constant systems over $\mathcal{U}$ parameterizing the diagram (3.4) over $\mathcal{U}$, where vertical arrows are quotient maps by the family of involutions

$$\iota_{\mathcal{U}} := \{\, \iota_u \mid u \in \mathcal{U} \,\}.$$

Note that $\overline{\mathcal{L}}$ is the space parameterizing all bitangents of smooth quartic curves.

Proof of Theorem 3.1. Let $L_u^{\{7\}}$ (resp. $L_u^{[7]}$) be the set of non-ordered 7-tuples $\{l_1, \dots, l_7\}$ (resp. ordered 7-tuples $[l_1, \dots, l_7]$) of elements $l_1, \dots, l_7 \in L_u$ such that $\langle l_i, l_j \rangle_Y = 0$ for $i \neq j$. By (3.3), we can enumerate all elements of $L_u^{\{7\}}$. It turns out that $|L_u^{\{7\}}| = 576$, and hence

$$|L_u^{[7]}| = 576 \cdot 7! = 2903040 = |W(E_7)|. \tag{3.6}$$

(See also Remark 6.6.) For 7-tuples $\lambda = [l_1, \dots, l_7]$ and $\lambda' = [l'_1, \dots, l'_7]$ in $L_u^{[7]}$, there exists a unique isometry $g_{\lambda,\lambda'} \in \mathrm{O}(SY_u \otimes \mathbb{Q})$ such that

$$g_{\lambda,\lambda'}(h_u) = h_u, \quad g_{\lambda,\lambda'}(l_i) = l'_i \quad (i = 1, \dots, 7).$$

It is enough show that, when $u = b$, these elements $g_{\lambda,\lambda'}$ are contained in the image of the monodromy (3.2). Indeed, by (3.6), this claim implies that these isometries $g_{\lambda,\lambda'}$ constitute the whole group $\mathrm{O}(SY_b, h_b) \cong W(E_7)$. To prove this claim, it is enough to show that $\pi_1(\mathcal{U}, b)$ acts on $L_b^{[7]}$ transitively by the monodromy, or equivalently, to show that the total space $\mathcal{L}^{[7]}$ of the locally constant system

$$\mathcal{L}^{[7]} \to \mathcal{U}$$

obtained from the family $\{\, L_u^{[7]} \mid u \in \mathcal{U} \,\}$ is connected.

Let $\lambda = [l_1, \dots, l_7]$ be a point of $\mathcal{L}^{[7]}$ over $u \in \mathcal{U}$. Contracting the (-1)-curves $l_1, \dots, l_7$, we obtain a birational morphism

$$\mathrm{bl}_\lambda \colon Y_u \to \mathbf{P}_\lambda$$

to a projective plane $\mathbf{P}_\lambda$. We put $\beta_\lambda := [\mathrm{bl}_\lambda(l_1), \dots, \mathrm{bl}_\lambda(l_7)]$. Conversely, we fix a projective plane $\mathbf{P}$, and let $\mathbf{P}^{[7]}$ denote the set of ordered 7-tuples $[p_1, \dots, p_7]$ of distinct points of $\mathbf{P}$. For a general point $\beta = [p_1, \dots, p_7]$ of $\mathbf{P}^{[7]}$, let

$$\mathrm{bl}^\beta \colon Y^\beta \to \mathbf{P}$$

be the blowing-up at the points $p_1, \dots, p_7$. Then Y^β is a del Pezzo surface of degree 2, and the complete linear system of the anti-canonical divisor on Y^β gives a double covering $Y^\beta \to \mathbb{P}^2$ branched along a smooth quartic curve Q^β such that each of the 7 exceptional curves over $p_1, \dots, p_7$ is a Y-lift of a bitangent of Q^β. Hence there exist a point $\lambda \in \mathcal{L}^{[7]}$ and an isomorphism $\mathbf{P}_\lambda \cong \mathbf{P}$ that maps β_λ to β.

We put

$$\mathcal{I} := \left\{ (\lambda, \gamma, \beta) \;\middle|\; \begin{array}{l} \lambda \in \mathcal{L}^{[7]},\ \beta \in \mathbf{P}^{[7]}, \text{ and } \gamma \text{ is an isomorphism} \\ \mathbf{P}_\lambda \cong \mathbf{P} \text{ that maps } \beta_\lambda \text{ to } \beta \end{array} \right\}.$$

Then the projection $\mathcal{I} \to \mathcal{L}^{[7]}$ is surjective with fibers isomorphic to $\mathrm{PGL}_3(\mathbb{C})$, whereas the projection $\mathcal{I} \to \mathbf{P}^{[7]}$ is dominant with fibers isomorphic to $\mathrm{PGL}_3(\mathbb{C})$. Since $\mathbf{P}^{[7]}$ and $\mathrm{PGL}_3(\mathbb{C})$ are connected, we see that $\mathcal{L}^{[7]}$ is connected. □

4. Family of 4-tangent conics

In this section, we construct a space $\overline{\mathcal{C}}$ parameterizing all 4-tangent conics of smooth quartic curves.

Let $\overline{C}_u$ denote the set of 4-tangent conics of Q_u, and let C_u be the set of Y-lifts of 4-tangent conics of Q_u. We put

$$F_u := \{\, v \in SY_u \mid \langle v, h_u \rangle_Y = 2, \langle v, v \rangle_Y = 0 \,\} \;\cong\; \{\, v \in \Sigma_u \mid \langle v, v \rangle_Y = -2 \,\}, \tag{4.1}$$

where the second bijection is given by the orthogonal projection $SY_u \to \Sigma_u^\vee$. As was shown in Table 2.1, we have $[c] \in F_u$ for any $c \in C_u$. Note that $|F_u| = 126$, the number of roots of the root lattice Σ_u of type E_7. We put

$$\overline{F}_u := F_u / \langle \iota_u \rangle \quad \subset \quad \overline{SY}_u.$$

Then we have a commutative diagram

$$\begin{array}{ccccc} C_u & \stackrel{\Phi_u}{\longrightarrow} & F_u & \hookrightarrow & SY_u \\ \downarrow & & \downarrow & & \downarrow \\ \overline{C}_u & \stackrel{\overline{\Phi}_u}{\longrightarrow} & \overline{F}_u & \hookrightarrow & \overline{SY}_u \end{array} \tag{4.2}$$

where $\Phi_u\colon C_u \to F_u$ is given by $c \mapsto [c] \in SY_u$, and the vertical arrows are quotient by the involution $\iota_u\colon Y_u \to Y_u$. We have locally constant systems $\mathcal{F} \to \mathcal{U}$ and $\overline{\mathcal{F}} \to \mathcal{U}$ obtained from the families $\{\, F_u \mid u \in \mathcal{U} \,\}$ and $\{\, \overline{F}_u \mid u \in \mathcal{U} \,\}$.

Theorem 4.1. *There exists a commutative diagram*

$$\begin{array}{ccccc} \mathcal{C} & \stackrel{\Phi_{\mathcal{U}}}{\longrightarrow} & \mathcal{F} & \hookrightarrow & \mathcal{SY} \\ \downarrow & & \downarrow & & \downarrow \\ \overline{\mathcal{C}} & \stackrel{\overline{\Phi}_{\mathcal{U}}}{\longrightarrow} & \overline{\mathcal{F}} & \hookrightarrow & \overline{\mathcal{SY}} \end{array} \tag{4.3}$$

of morphisms over $\mathcal{U}$ that parameterizes the diagrams (4.2) over $\mathcal{U}$. The morphisms $\Phi_{\mathcal{U}}\colon \mathcal{C} \to \mathcal{F}$ and $\overline{\Phi}_{\mathcal{U}}\colon \overline{\mathcal{C}} \to \overline{\mathcal{F}}$ are smooth and surjective, and every fiber of them is a Zariski open subset of $\mathbb{P}^1$.

For the proof, we use the double covering $\eta_u\colon X_u \to Y_u$ of Y_u by the $K3$ surface X_u. We consider the Néron-Severi lattice

$$SX_u := H^2(X_u, \mathbb{Z}) \cap H^{1,1}(X_u)$$

with the intersection form $\langle\, ,\, \rangle_X$. Then η_u induces a embedding of the lattice

$$\eta_u^*\colon SY_u(2) \hookrightarrow SX_u,$$

where $SY_u(2)$ is the lattice obtained from SY_u by multiplying the intersection form by 2. Let $j_u\colon X_u \to X_u$ be a generator of the cyclic group $\mathrm{Gal}(X_u/\mathbb{P}^2)$ of order 4. Then $\eta_u\colon X_u \to Y_u$ is the quotient morphism by j_u^2. Hence j_u^2 acts on the image of $\eta_u^*\colon SY_u(2) \hookrightarrow SX_u$ trivially.

Proof of Theorem 4.1. Note that the family of involutions $\iota_{\mathcal{U}} = \{\, \iota_u \mid u \in \mathcal{U} \,\}$ acts on $\mathcal{F}$ over $\mathcal{U}$ without fixed points. Hence, if the parameterizing space $\Phi_{\mathcal{U}}\colon \mathcal{C} \to \mathcal{F}$ of $\Phi_u\colon C_u \to F_u$ is constructed, then $\overline{\Phi}_{\mathcal{U}}\colon \overline{\mathcal{C}} \to \overline{\mathcal{F}}$ is constructed as a quotient of $\Phi_{\mathcal{U}}\colon \mathcal{C} \to \mathcal{F}$ by $\iota_{\mathcal{U}}$.

Let u be an arbitrary point of $\mathcal{U}$, and let v be an element of $F_u \subset SY_u$. We put

$$\tilde{v} := \eta_u^*(v) \in SX_u.$$

We can easily confirm that there exist exactly 6 pairs $\{l_i, l_i'\}$ $(i = 1, \dots, 6)$ of Y-lifts of bitangents of Q_u such that $\langle l_i, l_i' \rangle_Y = 1$ and $v = [l_i] + [l_i']$, and that these 12 curves $l_1, l_1', \dots, l_6, l_6'$ are distinct. Hence the complete linear system on the $K3$ surface X_u corresponding to $\tilde{v} \in SX_u$ has no fixed components. The class $\tilde{v}$ is primitive in SX_u with $\langle \tilde{v}, \tilde{v} \rangle_X = 0$ and $\langle \tilde{h}_u, \tilde{v} \rangle_X = 4$. Therefore there exists an elliptic fibration on X_u such that the class of a fiber is equal to $\tilde{v}$. We denote this elliptic fibration by $\phi_v\colon X_u \to \mathbb{P}^1$.

If $c \in C_u$, then $\eta_u^*(c)$ is an elliptic curve, and hence $\eta_u^*(c)$ is a smooth fiber of an elliptic fibration of $\phi_v\colon X_u \to \mathbb{P}^1$, where $v = [c]$. Conversely, suppose that $(u, v) \in \mathcal{F}$, and let f be a smooth fiber of the elliptic fibration $\phi_v\colon X_u \to \mathbb{P}^1$. We denote by $a \subset \mathbb{P}^2$ the plane curve $\gamma_u(f)$ with the reduced structure. Let d be the degree of a, and δ the mapping degree of $\gamma_u|f\colon f \to a$. Since $\langle \tilde{h}_u, f \rangle_X = 4$ and $\gamma_u\colon X_u \to \mathbb{P}^2$ is Galois, we have $(d, \delta) = (1, 4)$, $(2, 2)$, or $(4, 1)$. If $(d, \delta) = (1, 4)$, then $f = \gamma_u^{-1}(a)$ is invariant under the action of $\mathrm{Gal}(X_u/\mathbb{P}^2) = \langle j_u \rangle$, and hence the class $[f] \in SX_u$ is a non-zero multiple of $\tilde{h}_u$, which contradicts $\langle f, f \rangle_X = 0$. If $(d, \delta) = (4, 1)$, then f, $j_u(f)$, $j_u^2(f)$, $j_u^3(f)$ are distinct curves that intersect over the points of $a \cap Q_u$. On the other hand, since $[f] = \eta_u^*(v) \in \mathrm{Im}\, \eta_u^*$, we have $j_u^{*2}([f]) = [f]$. This contradicts $\langle f, f \rangle_X = 0$. Hence $(d, \delta) = (2, 2)$, and we see that a is a smooth splitting conic. Note that a is 4-tangent, because otherwise f would be singular. Thus we have proved that $c \mapsto \eta_u^*(c)$ gives a bijection from C_u to the union of the sets of smooth fibers of elliptic fibrations $\phi_v\colon X_u \to \mathbb{P}^1$, where v runs through F_u.

Let $\mathcal{X} \to \mathcal{U}$ be the universal family of $\{ X_u \mid u \in \mathcal{U} \}$, and let $\pi_{\mathcal{F}}\colon \mathcal{F} \times_{\mathcal{U}} \mathcal{X} \to \mathcal{F}$ be the pull-back of $\mathcal{X} \to \mathcal{U}$ by $\mathcal{F} \to \mathcal{U}$. Let $\mathcal{M}$ be a line bundle on $\mathcal{F} \times_{\mathcal{U}} \mathcal{X}$ such that the class $[\mathcal{M}|X_u] \in SX_u$ of the line bundle $\mathcal{M}|X_u$ on $\pi_{\mathcal{F}}^{-1}(u, v) = X_u$ is equal to $v \in F_u$. Then $\pi_{\mathcal{F}*}\mathcal{M} \to \mathcal{F}$ is a vector bundle of rank 2. The fiber over $(u, v) \in \mathcal{F}$ of the $\mathbb{P}^1$-bundle $\mathbb{P}_*(\pi_{\mathcal{F}*}\mathcal{M}) \to \mathcal{F}$ is identified with the base curve of the elliptic fibration $\phi_v\colon X_u \to \mathbb{P}^1$. We can construct $\mathcal{C}$ as the open subset of $\mathbb{P}_*(\pi_{\mathcal{F}*}\mathcal{M})$ consisting of non-critical points of $\phi_v\colon X_u \to \mathbb{P}^1$. □

The non-singular varieties $\mathcal{C}$ and $\overline{\mathcal{C}}$ parameterize all pairs (u, c) and $(u, \bar{c})$, respectively, where $u \in \mathcal{U}$ and $c \in C_u$, $\bar{c} \in \overline{C}_u$. Since Φ_u and $\overline{\Phi}_u$ have connected fibers, we can regard F_u as the set of connected families of Y-lifts of 4-tangent conics of Q_u, and $\overline{F}_u$ as the set of connected families of 4-tangent conics. The following observation obtained in the proof of Theorem 4.1 will be used in the next section.

Proposition 4.2. *Every connected family $[c] \in F_u$ of Y-lifts of splitting conics is a pencil with no base points.* □

Remark 4.3. A line section Λ_u of $Q_u \subset \mathbb{P}^2$ is a canonical class of the genus 3 curve Q_u. Let $\mathrm{Pic}^0(Q_u)$ be the Picard group of line bundles of degree 0 of Q_u, and let $\mathrm{Pic}^0(Q_u)[2]$ be the subgroup of 2-torsion points of $\mathrm{Pic}^0(Q_u)$. For a 4-tangent conic $\bar{c}$ of Q_u, let $\Theta_u(\bar{c})$ be the reduced part of the divisor $\bar{c} \cap Q_u$ of Q_u. Then the class of the divisor $\Theta_u(\bar{c}) - \Lambda_u$ of degree 0 is a point of $\mathrm{Pic}^0(Q_u)[2] - \{0\}$, and this correspondence gives a bijection $\overline{F}_u \cong \mathrm{Pic}^0(Q_u)[2] - \{0\}$.

5. Proof of the main results

In this section, we construct the space $\mathcal{Z}^{(m,n)}$ parameterizing all $\mathcal{Q}^{(m,n)}$-curves, and prove Theorems 1.3 and 1.4.

5.1. **Deformation types.** We fix some notation. For a set A, let $S^k(A)$ denote the symmetric product $A^k/\mathfrak{S}_k$, where $A^k = A \times \cdots \times A$ (k times), and let $S_0^k(A)$ denote the complement in $S^k(A)$ of the image of the big diagonal in A^k.

For a morphism $\mathcal{A} \to \mathcal{U}$, let $\mathcal{S}^k(\mathcal{A})$ denote the symmetric product $\mathcal{A}^k/\mathfrak{S}_k$, where

$$\mathcal{A}^k := \mathcal{A} \times_{\mathcal{U}} \cdots \times_{\mathcal{U}} \mathcal{A} \quad (k \text{ times}),$$

and let $\mathcal{S}_0^k(\mathcal{A})$ denote the complement in $\mathcal{S}^k(\mathcal{A})$ of the image of the big diagonal in $\mathcal{A}^k$. Note that, if $\mathcal{A}$ is smooth over $\mathcal{U}$ with relative dimension 1, then $\mathcal{S}^k(\mathcal{A})$ is smooth over $\mathcal{U}$ with relative dimension k.

Recall that $\overline{\mathcal{L}} \to \mathcal{U}$ and $\overline{\mathcal{C}} \to \mathcal{U}$ are the spaces parameterizing all bitangents and all 4-tangent conics of smooth quartic curves, respectively. We put

$$\mathcal{Z}'^{(m,n)} := \mathcal{S}_0^m(\overline{\mathcal{L}}) \times_{\mathcal{U}} \mathcal{S}_0^n(\overline{\mathcal{C}}),$$

which is the space parameterizing all curves Z', where Z' is a union of a smooth quartic curve Q, m distinct bitangents of Q, and n distinct 4-tangent conics of Q. Now we can construct the parameter space

$$\varpi\colon \mathcal{Z}^{(m,n)} \to \mathcal{U}$$

of $\mathcal{Q}^{(m,n)}$-curves as the open subvariety of $\mathcal{Z}'^{(m,n)}$ consisting of points corresponding to plane curves Z' satisfying conditions (i), (ii), (iii) in Definition 1.1. For a point $\zeta \in \mathcal{Z}^{(m,n)}$, we denote by Z_ζ the $\mathcal{Q}^{(m,n)}$-curve corresponding to ζ.

For $u \in \mathcal{U}$, we put

$$P_u^{(m,n)} := S_0^m(\overline{L}_u) \times S^n(\overline{F}_u) \quad \subset \quad S_0^m(\overline{SY}_u) \times S^n(\overline{SY}_u).$$

The size of $P_u^{(m,n)}$ is equal to $d^{(m,n)}$ defined by (1.2). Then we obtain a finite étale covering

$$\rho\colon \mathcal{P}^{(m,n)} := \mathcal{S}_0^m(\overline{\mathcal{L}}) \times_{\mathcal{U}} \mathcal{S}^n(\overline{\mathcal{F}}) \to \mathcal{U}$$

of degree $d^{(m,n)}$ parameterizing the family

$$\{\, P_u^{(m,n)} \mid u \in \mathcal{U} \,\}.$$

Using $\overline{\Phi}_{\mathcal{U}}\colon \overline{\mathcal{C}} \to \overline{\mathcal{F}}$ in Theorem 4.1, we have a morphism $\theta'\colon \mathcal{Z}'^{(m,n)} \to \mathcal{P}^{(m,n)}$. Restricting θ' to the open subvariety $\mathcal{Z}^{(m,n)} \subset \mathcal{Z}'^{(m,n)}$, we obtain a morphism

$$\theta\colon \mathcal{Z}^{(m,n)} \to \mathcal{P}^{(m,n)},$$

which maps $\zeta \in \mathcal{Z}^{(m,n)}$ to

$$\theta(\zeta) := (\{\bar{l}_1, \dots, \bar{l}_m\}, [[\bar{c}_1], \dots, [\bar{c}_n]]) \ \in\ P_{\varpi(\zeta)}^{(m,n)}, \tag{5.1}$$

where Z_ζ has the irreducible components as in (1.1). Thus we obtain the following commutative diagram.

$$\begin{tikzcd} \mathcal{Z}^{(m,n)} \arrow[rr, "\theta"] \arrow[dr, "\varpi"'] & & \mathcal{P}^{(m,n)} \arrow[dl, "\rho"] \\ & \mathcal{U} & \end{tikzcd}$$

To investigate the image of θ, we put

$$\mathcal{U}' := \left\{ \ u \in \mathcal{U} \ \middle|\ \begin{array}{l} \text{every bitangent of } Q_u \text{ is ordinary, and their union} \\ \text{has only ordinary nodes as its singularities} \end{array} \right\},$$

which is a Zariski open dense subset of $\mathcal{U}$.

Lemma 5.1. *The morphism $\theta\colon \mathcal{Z}^{(m,n)} \to \mathcal{P}^{(m,n)}$ is smooth with each non-empty fiber being an irreducible variety of dimension n. The image of θ contains $\rho^{-1}(\mathcal{U}')$. In particular, the morphism θ is dominant.*

Proof. Since $\overline{\Phi}_{\mathcal{U}}\colon \overline{\mathcal{C}} \to \overline{\mathcal{F}}$ is smooth and surjective with each fiber being a Zariski open subset of $\mathbb{P}^1$, the morphism $\theta'\colon \mathcal{Z}'^{(m,n)} \to \mathcal{P}^{(m,n)}$ is smooth and surjective with each fiber being an irreducible variety of dimension n. Suppose that $u \in \mathcal{U}'$, and let $p := (\{\bar{l}_1, \dots, \bar{l}_m\}, [[\bar{c}_1], \dots, [\bar{c}_n]])$ be a point of $P_u^{(m,n)}$. By Proposition 4.2 and Bertini's theorem, if we choose each 4-tangent conic $\bar{c}'_j$ in the connected family $[\bar{c}_j] \in \overline{F}_u$ generally, the curve $Q_u + \sum \bar{l}_i + \sum \bar{c}'_j$ satisfies conditions (ii) and (iii) in Definition 1.1. Hence $\theta^{-1}(p) = \theta'^{-1}(p) \cap \mathcal{Z}^{(m,n)}$ is non-empty. □

Proof of Theorem 1.4. By Lemma 5.1, the connected components of $\mathcal{Z}^{(m,n)}$ are in bijective correspondence with the connected components of $\mathcal{P}^{(m,n)}$, and hence with the $\pi_1(\mathcal{U}, b)$-orbits in $P_b^{(m,n)}$. By Theorem 3.1, the number $N^{(m,n)}$ of $\pi_1(\mathcal{U}, b)$-orbits in $P_b^{(m,n)}$ satisfies (1.3), because $|W(E_7)/\{\pm 1\}| = 1451520$. □

5.2. **Computation of $N^{(m,n)}$.** Recall that Σ_b is a negative-definite root lattice of type E_7. Let Σ be the negative-definite root lattice of type E_7 with the standard basis, and let $\Sigma^\vee$ be its dual. According to (3.3) and (4.1), we define the subsets

$$\overline{L} := \{\, v \in \Sigma^\vee \mid \langle v, v \rangle = -3/2 \,\}/\{\pm 1\}$$

of $\overline{\Sigma}^\vee := \Sigma^\vee/\{\pm 1\}$, and

$$\overline{F} := \{\, v \in \Sigma \mid \langle v, v \rangle = -2 \,\}/\{\pm 1\}$$

of $\overline{\Sigma} := \Sigma/\{\pm 1\}$. We then put

$$P^{(m,n)} := S_0^m(\overline{L}) \times S^n(\overline{F}).$$

The group $W(E_7)$ is generated by seven standard reflections. The permutations on $\overline{L}$ and on $\overline{F}$ induced by these generators are easily calculated. Hence the permutations on $P^{(m,n)}$ induced by these generators are also calculated. Thus we can compute the orbit decomposition of $P^{(m,n)}$ by $W(E_7)$, and obtain the number $N^{(m,n)}$ of deformation types of $\mathcal{Q}^{(m,n)}$-curves.

Example 5.2. The size $d^{(4,0)}$ of $P^{(4,0)}$ is 20475. The group $W(E_7)$ decomposes this set into three orbits of sizes $315, 5040, 15120$. Hence $N^{(4,0)} = 3$.

Example 5.3. The size $d^{(0,4)}$ of $P^{(0,4)}$ is 720720. The group $W(E_7)$ decomposes this set into 30 orbits as follows:

$$720720 = 63 + 945 \times 3 + 1008 \times 2 + 1890 + 2016 + 3780 \times 2 + 5040 \times 2 + 10080 + 11340 + 15120 \times 5 + 22680 + 30240 \times 5 + 45360 \times 2 + 90720 + 120960 \times 2.$$

Hence $N^{(0,4)} = 30$.

Example 5.4. The size $d^{(2,2)}$ of $P^{(2,2)}$ is 762048. The group $W(E_7)$ decomposes this set into 23 orbits as follows:

$$762048 = 378 + 1890 + 3780 \times 3 + 6048 + 7560 \times 2 + 12096 \times 2 + 15120 + 22680 + 30240 \times 3 + 45360 \times 2 + 60480 \times 4 + 120960 \times 2.$$

Hence $N^{(2,2)} = 23$.

Remark 5.5. For the computation, we used `GAP` [20], which is good at computations of permutation groups.

5.3. **Real quartic curves.**

Definition 5.6. Note that $H^2(\mathbb{P}^2, \mathbb{Z}) \cong \mathbb{Z}$ has a canonical generator, that is, the class of a line. Let C and C' be plane curves with the same homeomorphism type. A homeomorphism $\sigma\colon (\mathbb{P}^2, C) \xrightarrow{\sim} (\mathbb{P}^2, C')$ is said to be *orientation-preserving* (resp. *orientation-reversing*) if the action of σ on $H^2(\mathbb{P}^2, \mathbb{Z})$ is the identity (resp. the multiplication by -1).

Example 5.7. Suppose that $\mathcal{Q}^{(m,n)}$-curves Z_ζ and $Z_{\zeta'}$ are of the same deformation type. Let $\alpha\colon I \to \mathcal{Z}^{(m,n)}$ be a path from ζ to ζ', where $I := [0,1] \subset \mathbb{R}$. By the parallel transport along α, we obtain a homeomorphism $\alpha_*\colon (\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_{\zeta'})$. It is obvious that α_* is orientation-preserving.

Proposition 5.8. *Every $\mathcal{Q}^{(m,n)}$-curve Z_ζ admits an orientation-reversing self-homeomorphism $(\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_\zeta)$.*

For the proof of Proposition 5.8, we use a classical result on real quartic curves. We give a structure of the $\mathbb{R}$-scheme to $\mathbb{P}^2$. We denote by $\Gamma_\mathbb{R}(d)$ the space of homogeneous polynomials of degree d on $\mathbb{P}^2$ with real coefficients, and consider the real projective space $\mathbb{P}_*(\Gamma_\mathbb{R}(4))$ as a closed subset of $\mathbb{P}_*(\Gamma(4))$. We then put

$$\mathcal{U}_\mathbb{R} := \mathcal{U} \cap \mathbb{P}_*(\Gamma_\mathbb{R}(4)).$$

The topological types of smooth real quartic curves are classified by Zeuthen and Klein, and the result is summarized in [12, Theorem 1.7]. Using this result, we obtain the following:

Theorem 5.9 (Zeuthen (1873) and Klein (1876))**.** *There exists a unique connected component $\mathcal{U}_{\mathbb{R},4}$ of $\mathcal{U}_\mathbb{R}$ consisting of points $u \in \mathcal{U}_\mathbb{R}$ such that the real plane curve $Q_u(\mathbb{R})$ is a union of 4 ovals. If $u \in \mathcal{U}_{\mathbb{R},4}$, the ovals in $Q_u(\mathbb{R})$ are pairwise non-nested, and every bitangent of Q_u is defined over $\mathbb{R}$.* □

Remark 5.10. For beautiful pictures of real plane quartic curves with real 28 bitangents, see [11] and [14, Section 10.5]. These pictures are in fact defined over $\mathbb{Q}$, and were obtained by the theory of Mordell-Weil lattices.

For an algebraic variety V defined over $\mathbb{R}$, we denote by $H^*(V, \mathbb{Z})$ the cohomology ring of the topological space $V(\mathbb{C})$ of $\mathbb{C}$-valued points of V, by

$$\tau_V \colon V(\mathbb{C}) \xrightarrow{\sim} V(\mathbb{C})$$

the self-homeomorphism of $V(\mathbb{C})$ obtained by the complex conjugation, and by $V_{\mathbb{C}}$ the variety $V \otimes_{\mathbb{R}} \mathbb{C}$ defined over $\mathbb{C}$. Let S be an algebraic surface defined over $\mathbb{R}$, and let C be a reduced irreducible curve on $S_{\mathbb{C}}$. Then there exists a unique reduced irreducible curve C' on $S_{\mathbb{C}}$ such that τ_S induces an orientation-reversing homeomorphism $C(\mathbb{C}) \xrightarrow{\sim} C'(\mathbb{C})$. We denote this curve C' by $\tau_S[C]$. Then we have

$$[\tau_S[C]] = -\tau_S^*([C])$$

in $H^2(S, \mathbb{Z})$. If C is also defined over $\mathbb{R}$, then $\tau_S[C] = C$, and hence $\tau_S^*([C]) = -[C]$. If $H^2(S, \mathbb{Z})$ is generated by classes of curves defined over $\mathbb{R}$, then τ_S acts on $H^2(S, \mathbb{Z})$ as the multiplication by -1, and hence, for any curve C (not necessarily defined over $\mathbb{R}$), we have $[\tau_S[C]] = [C]$ in $H^2(S, \mathbb{Z})$.

Lemma 5.11. *Let r be a point of $\mathcal{U}_{\mathbb{R},4}$. If $\bar{c}$ is a 4-tangent conic of Q_r, then the 4-tangent conic $\tau_{\mathbb{P}^2}[\bar{c}]$ of $\tau_{\mathbb{P}^2}[Q_r] = Q_r$ is in the same connected family as $\bar{c}$.*

Proof. Note that, for $\varphi \in \Gamma_{\mathbb{R}}(4)$ and $x \in \mathbb{P}^2(\mathbb{R})$, the sign of $\varphi(x)$ is well-defined, because $\lambda^4 > 0$ for any $\lambda \in \mathbb{R}^{\times}$. We choose a defining equation $\varphi \in \Gamma_{\mathbb{R}}(4)$ of Q_r in such a way that $\varphi(x) > 0$ for any point x of $\mathbb{P}^2(\mathbb{R})$ in the outside of the ovals of $Q_r(\mathbb{R})$. We let Y_r be defined over $\mathbb{R}$ by $w^2 = \varphi$, and consider the self-homeomorphism $\tau_Y \colon Y_r(\mathbb{C}) \xrightarrow{\sim} Y_r(\mathbb{C})$ given by the complex conjugation. For any bitangent $\bar{l}$ of Q_r, each of its Y-lifts l satisfies $\tau_Y[l] = l$, because $\varphi(x) \geq 0$ for any point x of $\bar{l}(\mathbb{R})$. Since the classes of these curves l span $SY_r = H^2(Y_r, \mathbb{Z})$, we see that τ_Y acts on SY_r as the multiplication by -1. Therefore, for any curve C on Y_r, we have $[\tau_Y[C]] = [C]$. In particular, if $c \subset Y_r$ is a Y-lift of $\bar{c}$, then $\tau_Y[c]$ is a Y-lift of the 4-tangent conic $\tau_{\mathbb{P}^2}[\bar{c}]$. Then $[\tau_Y[c]] = [c]$ in SY_r implies that $\tau_{\mathbb{P}^2}[\bar{c}]$ and $\bar{c}$ belong to the same connected family of 4-tangent conics. □

Proof of Proposition 5.8. Since $\mathcal{U}_{\mathbb{R},4}$ is open in $\mathbb{P}_*(\Gamma_{\mathbb{R}}(4))$, it follows that $\mathcal{U}_{\mathbb{R},4}$ is Zariski dense in $\mathcal{U}$, and hence there exists a point $r \in \mathcal{U}_{\mathbb{R},4} \cap \mathcal{U}'$. By Lemma 5.1, we see that $\varpi^{-1}(r)$ intersects every connected component of $\mathcal{Z}^{(m,n)}$. Let ξ be a point of $\varpi^{-1}(r)$ that belongs to the same connected component as ζ, and let

$$Z_\xi = Q_r + \bar{l}'_1 + \cdots + \bar{l}'_m + \bar{c}'_1 + \cdots + \bar{c}'_n$$

be the decomposition of Z_ξ into irreducible components. Remark that Q_r and all of its bitangents are defined over $\mathbb{R}$ by the definition of $\mathcal{U}_{\mathbb{R},4}$ (see Theorem 5.9). We choose a path $\alpha \colon I \to \mathcal{Z}^{(m,n)}$ from ζ to ξ. Then we obtain an orientation-preserving homeomorphism

$$\alpha_* \colon (\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_\xi).$$

For simplicity, we write τ instead of $\tau_{\mathbb{P}^2}$. We have an orientation-reversing homeomorphism

$$\tau \colon (\mathbb{P}^2, Z_\xi) \xrightarrow{\sim} (\mathbb{P}^2, \tau[Z_\xi])$$

obtained by the complex conjugation. Since

$$\tau[Z_\xi] = Q_r + \bar{l}'_1 + \cdots + \bar{l}'_m + \tau[\bar{c}'_1] + \cdots + \tau[\bar{c}'_n],$$

and, for $j = 1, \ldots, n$, the 4-tangent conic $\tau[\bar{c}'_j]$ of $\tau[Q_r] = Q_r$ belongs to the same connected family as $\bar{c}'_j$ by Lemma 5.11, we see that $\mathcal{Q}^{(m,n)}$-curves $\tau[Z_\xi]$ and Z_ξ have the same deformation type, and we have an orientation-preserving homeomorphism

$$\beta_* \colon (\mathbb{P}^2, \tau[Z_\xi]) \xrightarrow{\sim} (\mathbb{P}^2, Z_\xi).$$

Composing α_*, τ, β_* and α_*^{-1}, we obtain an orientation-reversing self-homeomorphism of $(\mathbb{P}^2, Z_\zeta)$. □

5.4. **Homeomorphism types.** Let ζ be a point of $\mathcal{Z}^{(m,n)}$ such that Z_ζ has the decomposition

$$Z = Q + \bar{l}_1 + \cdots + \bar{l}_m + \bar{c}_1 + \cdots + \bar{c}_n.$$

We consider another point $\zeta' \in \mathcal{Z}^{(m,n)}$ with the decomposition

$$Z_{\zeta'} = Q_{u'} + \bar{l}'_1 + \cdots + \bar{l}'_m + \bar{c}'_1 + \cdots + \bar{c}'_n.$$

Recall that the involution ι_u of Y_u acts on the orthogonal complement Σ_u of $h_u \in SY_u$ as -1. Let $g\colon SY_u \xrightarrow{\sim} SY_{u'}$ be an isometryof lattices. Suppose that g maps h_u to $h_{u'}$. Then we have $g \circ \iota_u = \iota_{u'} \circ g$. Moreover, by definitions (3.3) and (4.1), the isometry g maps L_u to $L_{u'}$ and F_u to $F_{u'}$. Therefore g induces a bijection $P_u^{(m,n)} \xrightarrow{\sim} P_{u'}^{(m,n)}$.

Theorem 1.3 is an immediate consequence of the following:

Theorem 5.12. *The following are equivalent:*

(i) *ζ and ζ' belong to the same connected component of $\mathcal{Z}^{(m,n)}$,*

(ii) *$\theta(\zeta)$ and $\theta(\zeta')$ belong to the same connected component of $\mathcal{P}^{(m,n)}$,*

(iii) *there exists an isometry $g\colon SY_u \xrightarrow{\sim} SY_{u'}$ of lattices that maps h_u to $h_{u'}$ and such that the induced bijection $P_u^{(m,n)} \xrightarrow{\sim} P_{u'}^{(m,n)}$ maps $\theta(\zeta)$ to $\theta(\zeta')$, and*

(iv) *there exists a homeomorphism $(\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_{\zeta'})$.*

Proof. By Proposition 5.8, condition (iv) is equivalent to the following:

(iv)$'$ there exists an orientation-preserving homeomorphism $(\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_{\zeta'})$.

We will show that (i), (ii), (iii) and (iv)$'$ are equivalent. The implication (i) $\iff$ (ii) follows from Lemma 5.1, and (i) $\Longrightarrow$ (iv)$'$ follows from Example 5.7.

We show (iv)$'$ $\Longrightarrow$ (iii). Suppose that $\sigma\colon (\mathbb{P}^2, Z_\zeta) \xrightarrow{\sim} (\mathbb{P}^2, Z_{\zeta'})$ is an orientation-preserving homeomorphism. We can assume, after renumbering the curves, that σ induces homeomorphisms $\bar{l}_i \xrightarrow{\sim} \bar{l}'_i$ and $\bar{c}_j \xrightarrow{\sim} \bar{c}'_j$ that preserve the orientation. We have a homeomorphism $\sigma_Y\colon Y_u \xrightarrow{\sim} Y_{u'}$ that covers σ, and σ_Y induces an isometry

$$\sigma_{SY}\colon SY_u \xrightarrow{\sim} SY_{u'},$$

which maps h_u to $h_{u'}$. If $l_i \subset Y_u$ is a Y-lift of a bitangent $\bar{l}_i \subset Z_\zeta$, there exists a Y-lift $l'_i \subset Y_{u'}$ of the bitangent $\bar{l}'_i \subset Z_{\zeta'}$ such that σ_Y induces a homeomorphism $l_i \xrightarrow{\sim} l'_i$ preserving the orientation. In particular, we have $\sigma_{SY}([l_i]) = [l'_i]$. The same holds for a Y-lift $c_j \subset Y_u$ of a 4-tangent conic $\bar{c}_j \subset Z_\zeta$. Hence the bijection $P_u^{(m,n)} \xrightarrow{\sim} P_{u'}^{(m,n)}$ induced by the isometry σ_{SY} maps $\theta(\zeta)$ to $\theta(\zeta')$. Thus (iii) holds.

We show (iii) $\Longrightarrow$ (ii). Suppose that (iii) holds. We choose a path $\beta\colon I \to \mathcal{U}$ from u to the base-point b and a path $\beta'\colon I \to \mathcal{U}$ from u' to b, and consider the isometries

$$\beta_*\colon SY_u \xrightarrow{\sim} SY_b, \quad \beta'_*\colon SY_{u'} \xrightarrow{\sim} SY_b$$

obtained by the parallel transports along β and β'. Note that $\beta_*(h_u) = h_b$ and $\beta'_*(h_{u'}) = h_b$. Hence $\beta'_* \circ g \circ \beta_*^{-1}$ is an element of $\mathrm{O}(SY_b, h_b)$. By Theorem 3.1, there exists a loop $\alpha\colon I \to \mathcal{U}$ with the base point b such that

$$\alpha_* = \beta'_* \circ g \circ \beta_*^{-1}.$$

Therefore the isometry $g\colon SY_u \xrightarrow{\sim} SY_{u'}$ is equal to the parallel transport γ_* along the path $\gamma := \beta'^{-1}\alpha\beta$ from u to u'. Let

$$\tilde{\gamma}\colon I \to \mathcal{P}^{(m,n)}$$

be the lift of γ such that $\tilde{\gamma}(0) = \theta(\zeta)$. Since $g = \gamma_*$ maps $\theta(\zeta)$ to $\theta(\zeta')$, we see that $\tilde{\gamma}(1) = \theta(\zeta')$. Therefore $\theta(\zeta)$ and $\theta(\zeta')$ are in the same connected component of $\mathcal{P}^{(m,n)}$. $\square$

6. Geometry of the $K3$ surface X_u

We investigate the connected families of 4-tangent conics more closely for a *general* point $u \in \mathcal{U}$. Our main result of this section is as follows.

Theorem 6.1. *Suppose that $u \in \mathcal{U}$ is general. Then each connected family of 4-tangent conics $\bar{c}$ of Q_u is parameterized by a rational curve minus $12+6$ points. A member $\bar{c}$ of this family becomes a 3-tangent conic at each of 12 punctured points, and $\bar{c}$ degenerates into a union of two distinct bitangents at each of the remaining 6 punctured points.*

For the proof, we add the following easy result to Proposition 2.4.

Proposition 6.2. *Let $\bar{c}$ be a 3-tangent conic of Q_u. Then $\gamma_u^*(\bar{c})$ is a union of two one-nodal rational curves.* □

Recall that the double covering $\eta_u \colon X_u \to Y_u$ induces a primitive embedding of lattices $\eta_u^* \colon SY_u(2) \hookrightarrow SX_u$.

Proposition 6.3. *If $u \in \mathcal{U}$ is general, then η_u^* is an isomorphism.*

Proof. Kondo [10] studied the moduli of genus-3 curves by considering the periods of $K3$ surfaces X that are cyclic covers of $\mathbb{P}^2$ of degree 4 branched along quartic curves $Q \subset \mathbb{P}^2$. Let j denote the generator of $\mathrm{Gal}(X/\mathbb{P}^2) \cong \mu_4$ that acts on $H^{2,0}(X)$ as $\sqrt{-1}$. Kondo exhibits an action of the cyclic group μ_4 on the $K3$ lattice

$$\mathbf{L} := E_8^{\oplus 2} \oplus U^{\oplus 3}$$

that is obtained by a marking $H^2(X,\mathbb{Z}) \cong \mathbf{L}$. Let $\mathbf{L}_S$ and $\mathbf{L}_T$ be the kernel of $j^{*2}-1$ and of $j^{*2}+1$ on $\mathbf{L}$, respectively. Then $\mathbf{L}_S$ is of rank 8, and, via the marking, equal to the image of the pull-back of $H^2(Y,\mathbb{Z})(2)$ by the double covering $X \to Y := X/\langle j^2 \rangle$. The period $H^{2,0}(X)$ is a point of $\mathbb{P}_*(V_{\sqrt{-1}})$, where $V_{\sqrt{-1}}$ is the kernel of $j^* - \sqrt{-1}$ on $\mathbf{L}_T \otimes \mathbb{C}$. We have $\dim \mathbb{P}_*(V_{\sqrt{-1}}) = 6$. The result of [10] implies that, when Q varies, the point $H^{2,0}(X)$ of $\mathbb{P}_*(V_{\sqrt{-1}})$ sweeps an open subset of $\mathbb{P}_*(V_{\sqrt{-1}})$.

We fix a marking $H^2(X_u,\mathbb{Z}) \cong \mathbf{L}$. Since $u \in \mathcal{U}$ is general, the period $H^{2,0}(X_u)$ is general in $\mathbb{P}_*(V_{\sqrt{-1}})$. Since $\mathbf{L}_T \otimes \mathbb{C} = V_{\sqrt{-1}} \oplus \overline{V_{\sqrt{-1}}}$, the minimal $\mathbb{Z}$-submodule M of $\mathbf{L}$ such that $M \otimes \mathbb{C}$ contains $H^{2,0}(X_u)$ is equal to $\mathbf{L}_T$, and hence its orthogonal complement $M^\perp = SX_u$ is equal to $\mathbf{L}_S = \eta_u^*(SY_u(2))$. □

Let $Rats(X_u)$ denote the set of rational curves on X_u, and $Ells(X_u)$ the set of elliptic fibrations on X_u.

Proposition 6.4. *Suppose that $u \in \mathcal{U}$ is general. Then $Rats(X_u)$ is equal to the set*

$$\widetilde{L}_u := \{\, \eta_u^*(l) \mid l \in L_u \,\}$$

of 56 smooth rational curves on X_u.

This proposition is proved by Proposition 6.3 and [13, Proposition 98]. See also [13, Remark 99]. We give a proof, however, because the argument is also used in the proof of Proposition 6.5 below. Recall from the proof of Theorem 4.1 that, for $v \in F_u$, there exists an elliptic fibration $\phi_v \colon X_u \to \mathbb{P}^1$ such that the class of a fiber of ϕ_v is $\eta_u^*(v)$.

Proposition 6.5. *Suppose that $u \in \mathcal{U}$ is general. Then $v \mapsto \phi_v$ gives a bijection $F_u \cong Ells(X_u)$. Each fibration ϕ_v has no section. The singular fibers of ϕ_v consist of 6 fibers of type I_2 and 12 fibers of type I_1.*

Proof of Propositions 6.4 and 6.5. The space

$$\{\, v \in SX_u \otimes \mathbb{R} \mid \langle v, v \rangle_X > 0 \,\}$$

has two connected components. Let $\mathcal{P}_u$ be the connected component containing the ample class $\tilde{h}_u$. For a vector $v \in SX_u \otimes \mathbb{R}$ with $\langle v, v\rangle_X < 0$, let $[v]^\perp$ be the hyperplane of $SX_u \otimes \mathbb{R}$ defined by $\langle x, v\rangle_X = 0$, and we put $(v)^\perp := [v]^\perp \cap \mathcal{P}_u$. We then put

$$N_u := \{\, v \in \mathcal{P}_u \mid \langle v, \Gamma\rangle_X \geq 0 \ \text{ for all } \ \Gamma \subset X_u \,\}.$$

It is well known that N_u is equal to

$$\{\, v \in \mathcal{P}_u \mid \langle v, \Gamma\rangle_X \geq 0 \ \text{ for all } \ \Gamma \in \mathit{Rats}(X_u) \,\},$$

and that each $\Gamma \in \mathit{Rats}(X_u)$ defines a wall of the cone N_u, that is, $(\Gamma)^\perp \cap N_u$ contains a non-empty open subset of $(\Gamma)^\perp$. Let $\overline{N}_u$ be the closure of N_u in $SX_u \otimes \mathbb{R}$. For the proof of Proposition 6.4, it is enough to show that $\overline{N}_u$ is equal to

$$\overline{N}'_u := \{\, v \in SX_u \otimes \mathbb{R} \mid \langle v, \tilde{l}\rangle_X \geq 0 \ \text{ for all } \ \tilde{l} \in \widetilde{L}_u \,\}.$$

A *face* of the cone $\overline{N}'_u$ is a closed subset F of $\overline{N}'_u$ of the form $F = V \cap \overline{N}'_u$, where V is an intersection of some of the hyperplanes $[\tilde{l}]^\perp$ ($\tilde{l} \in \widetilde{L}_u$) such that F contains a non-empty open subset of V. We say that V is the *supporting linear subspace* of the face F, and put $\dim F := \dim V$. A *ray* is a 1-dimensional face. For the proof of $\overline{N}_u = \overline{N}'_u$, it is enough to show that all rays of $\overline{N}'_u$ are contained in $\overline{N}_u$. We can calculate all the faces F of $\overline{N}'_u$ by descending induction on $d := \dim F$ using linear programming method (see [17, Section 2.2]). The result is as follows. Suppose that $d \geq 2$. Then a linear subspace

$$V = [\tilde{l}_1]^\perp \cap \cdots \cap [\tilde{l}_k]^\perp \tag{6.1}$$

with $\tilde{l}_1, \dots, \tilde{l}_k \in \widetilde{L}_u$ is the supporting linear subspace of a face F with $\dim F = d$ if and only if $k = 8 - d$ and $\tilde{l}_1, \dots, \tilde{l}_k$ are disjoint from each other, that is, their dual graph is the Dynkin diagram of type $(8-d)A_1$. Suppose that $d = 1$. Then a linear subspace V as (6.1) is the supporting linear subspace of a ray F if and only if one of the following holds:

($7A_1$) $k = 7$ and the dual graph of $\tilde{l}_1, \dots, \tilde{l}_7$ is the Dynkin diagram of type $7A_1$. In this case, F is generated by a vector $v \in SX_u$ with $\langle \tilde{h}_u, v\rangle_X = 6$ and $\langle v, v\rangle_X = 2$. There exist exactly 576 rays of this type.

($6\widetilde{A}_1$) $k = 12$ and the dual graph of $\tilde{l}_1, \dots, \tilde{l}_{12}$ is the Dynkin diagram of type $6\widetilde{A}_1$, where $\widetilde{A}_1$ is $\circ\!\!=\!\!\circ$. In this case, F is generated by a primitive vector $\tilde{v}$ with $\langle \tilde{h}_u, \tilde{v}\rangle_X = 4$ and $\langle \tilde{v}, \tilde{v}\rangle_X = 0$. There exist exactly 126 rays of this type, and these generators $\tilde{v}$ are equal to $\eta_u^*(v)$ for some $v \in F_u$.

In Table 6.1, the numbers of faces of $\overline{N}'_u$ are given.

Suppose that there exists a ray F of $\overline{N}'_u$ not contained in $\overline{N}_u$. Then the generating class $v \in SX_u$ of F given above is effective but not nef. Let D be an effective divisor of X_u such that $[D] = v$. Then D contains a smooth rational curve Γ with $\langle \Gamma, v\rangle_X < 0$ as an irreducible component. Since $\tilde{h}_u$ is ample, the (-2)-vector $r = [\Gamma]$ satisfies $\langle \tilde{h}_u, r\rangle_X < \langle \tilde{h}_u, v\rangle_X \leq 6$. We make the set of all (-2)-vectors $r' \in SX_u$ with $\langle \tilde{h}_u, r'\rangle_X = 1, \dots, 5$, and confirm that this set is equal to the set of classes of $\widetilde{L}_u$. In particular, it contains no element r' satisfying $\langle r', v\rangle_X < 0$. This contradiction shows $\overline{N}'_u = \overline{N}_u$, and $\mathit{Rats}(X_u) = \widetilde{L}_u$ is proved.

It is well known that there exists a bijection between $\mathit{Ells}(X_u)$ and the set of rays contained in $\overline{N}_u \cap \partial\overline{\mathcal{P}}_u$. Hence we have $|\mathit{Ells}(X_u)| = 126$, and $v \mapsto \phi_v$ gives a bijection from F_u to $\mathit{Ells}(X_u)$. Therefore, as was shown in the proof of Theorem 4.1, every fiber f of any elliptic fibration ϕ_v is a double cover of a splitting conic of Q_u. The class of f is equal to $\eta_u^*(v)$. Since no element $\tilde{l} \in \mathit{Rats}(X_u)$ satisfies $\langle f, \tilde{l}\rangle_X = 1$, the fibration ϕ_v has no section. Since the dual graph of the set of $\tilde{l} \in \mathit{Rats}(X_u)$ with $\langle f, \tilde{l}\rangle_X = 0$ is of type $6\widetilde{A}_1$, the fibration ϕ_v has exactly 6 reducible fibers, each of which is either of type I_2 or of type III. If $\tilde{l}_i, \tilde{l}_j \in \mathit{Rats}(X_u)$ are in the same fiber of ϕ_v, then they satisfy $\langle \tilde{l}_i, \tilde{l}_j\rangle_X = 2$ and hence $\bar{l}_i := \gamma_u(\tilde{l}_i)$ and $\bar{l}_j := \gamma_u(\tilde{l}_j)$ are distinct bitangents of

$\dim F$	7	6	5	4	3	2	1
#	56	756	4032	10080	12096	6048	$576 + 126$

Table 6.1. Numbers of faces F

Q_u by Table 2.1. Since $u \in \mathcal{U}$ is general, the intersection point of $\bar{l}_i$ and $\bar{l}_j$ is not on Q_u. Hence every reducible fiber of ϕ_v is of type I_2. The irreducible singular fibers are either of type I_1 or of type II. By Lemma 2.2 and Proposition 6.2, we see that all irreducible singular fibers must be of type I_1. Calculating the Euler number, we conclude that the number of singular fibers of type I_1 is 12. □

Remark 6.6. The set of 576 rays of type $7A_1$ is in bijective correspondence with the set $L_u^{\{7\}}$ in the proof of Theorem 3.1. A ray F of type $7A_1$ corresponds to a 7-tuple $\{l_1, \dots, l_7\} \in L_u^{\{7\}}$ as follows. The generator v of F with $\langle v, v\rangle_X = 2$ is the class of the pull-back of a line of a plane $\mathbf{P}$ by the double covering $X_u \to Y_u \to \mathbf{P}$, where $Y_u \to \mathbf{P}$ is the blowing down of the (-1)-curves $l_1, \dots, l_7$.

Proof of Theorem 6.1. In fact, the proof was already given in the last paragraph of the proof of Proposition 6.5. □

7. Configurations of Y-lifts

Throughout this section, let u be a general point of $\mathcal{U}$.

7.1. **Lemmas on quartic polynomials.** Let $[d_1, \dots, d_m]$ be a list of positive integers satisfying $d_1 + \cdots + d_m = 4$. We put

$$\Gamma(d_1, \dots, d_m \colon 2) := \Gamma(d_1) \times \cdots \times \Gamma(d_m) \times \Gamma(2),$$

and denote by $\psi_{[d_1,\dots,d_m]} \colon \Gamma(d_1, \dots, d_m \colon 2) \to \Gamma(4)$ the morphism

$$(f_1, \cdots, f_m, q) \mapsto f_1 \cdots f_m + q^2.$$

Lemma 7.1. *The morphism $\psi_{[d_1,\dots,d_m]}$ is dominant.*

Proof. It is enough to show that $\psi_{[1,1,1,1]}$ is dominant, and then, it suffices to find a point P of $\Gamma(1, 1, 1, 1 \colon 2)$ at which the differential of $\psi := \psi_{[1,1,1,1]}$ is of rank $\dim \Gamma(4) = 15$. By choosing points P randomly and calculating the rank of $d_P\psi$, we can easily find such a point. □

Definition 7.2. For $[d_1, \dots, d_m]$ with $d_1 + \cdots + d_m = 4$, we have an open dense subset $\mathcal{V}_{[d_1,\dots,d_m]} \subset \Gamma(d_1, \dots, d_m \colon 2)$ and a dominant morphism

$$\Psi_{[d_1,\dots,d_m]} \colon \mathcal{V}_{[d_1,\dots,d_m]} \to \mathcal{U}$$

such that, for $p = (f_1, \dots, f_m, q) \in \mathcal{V}_{[d_1,\dots,d_m]}$, the quartic curve corresponding $\Psi_{[d_1,\dots,d_m]}(p) \in \mathcal{U}$ is defined by $f_1 \cdots f_m + q^2 = 0$.

Lemma 7.3. *If Q_u is defined by $f + q^2 = 0$ with $f \in \Gamma(4)$ and $q \in \Gamma(2)$, then Y_u has a divisor that is mapped isomorphically to the divisor $\{f = 0\}$ of $\mathbb{P}^2$.*

Proof. The surface Y_u is defined by $w^2 = f + q^2$, where w is a new variable, and hence contains a divisor defined by $f = w - q = 0$. It is obvious that π_u maps this divisor to the divisor $\{f = 0\}$ of $\mathbb{P}^2$ isomorphically. □

7.2. **Triangles of bitangents.** Recall that L_u is the set of Y-lifts l of bitangents $\bar{l} \in \overline{L}_u$ of Q_u.

Definition 7.4. A *triangle on* Y_u is a subset $\{l_1, l_2, l_3\}$ of L_u such that

$$\langle l_1, l_2\rangle_Y = \langle l_2, l_3\rangle_Y = \langle l_3, l_1\rangle_Y = 1.$$

A *liftable triangle* of bitangents of Q_u is a subset $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$ of $\overline{L}_u$ that is the image of a triangle on Y_u by π_u.

Let $\bar{l}_1, \bar{l}_2, \bar{l}_3$ be bitangents of Q_u. We choose Y-lifts $l_1, l_2, l_3 \in L_u$ of $\bar{l}_1, \bar{l}_2, \bar{l}_3$ in such a way that $\langle l_1, l_2\rangle_Y = \langle l_2, l_3\rangle_Y = 1$. Then $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$ is liftable if and only if $\langle l_3, l_1\rangle_Y = 1$.

Let T_u be the set of triangles on Y_u. We have calculated $L_u \subset SY_u$ explicitly. Using this data, we enumerate T_u, and see that $|T_u| = 2520$. Let

$$\overline{T}_u := T_u/\langle \iota_u\rangle$$

be the set of liftable triangles of bitangents of Q_u.

Corollary 7.5. *There exist exactly* $|\overline{T}_u| = 1260$ *liftable triangles.* □

By Theorem 3.1, we obtain the following:

Proposition 7.6. *By the monodromy,* $\pi_1(\mathcal{U}, b)$ *acts transitively on* T_b *and hence on* $\overline{T}_b$. □

Proposition 7.7. *Let* $\bar{l}_1, \bar{l}_2, \bar{l}_3$ *be bitangents of* Q_u. *Suppose that* $\bar{l}_i$ *is defined by* $f_i = 0$ *for* $i = 1, \dots, 3$, *where* $f_i \in \Gamma(1)$. *Then* $\{\bar{l}_1, \bar{l}_2, \bar{l}_3\}$ *is liftable if and only if there exist polynomials* $f_4 \in \Gamma(1)$ *and* $q \in \Gamma(2)$ *such that* Q_u *is defined by* $f_1 f_2 f_3 f_4 + q^2 = 0$.

Proof. The if-part follows from Lemma 7.3. Let $\bar{\tau}\colon \overline{\mathcal{T}} \to \mathcal{U}$ be the finite étale covering obtained from the family $\{\, \overline{T}_u \mid u \in \mathcal{U}\,\}$. Then $\overline{\mathcal{T}}$ is irreducible by Proposition 7.6. Let $p := (f'_1, \dots, f'_4, q')$ be a point of $\mathcal{V}_{[1,1,1,1]}$, and we put

$$u' := \Psi_{[1,1,1,1]}(p) \ \in \ \mathcal{U}$$

where $\mathcal{V}_{[1,1,1,1]}$ and $\Psi_{[1,1,1,1]}$ are given in Definition 7.2. Let $\bar{l}'_i \subset \mathbb{P}^2$ be the line $\{f'_i = 0\}$. By the if-part, we have $\{\bar{l}'_1, \bar{l}'_2, \bar{l}'_3\} \in \overline{T}_{u'}$. By $p \mapsto \{\bar{l}'_1, \bar{l}'_2, \bar{l}'_3\}$, we obtain a morphism $\Psi_{\overline{\mathcal{T}}}\colon \mathcal{V}_{[1,1,1,1]} \to \overline{\mathcal{T}}$. Since $\bar{\tau} \circ \Psi_{\overline{\mathcal{T}}} = \Psi_{[1,1,1,1]}$, $\bar{\tau}$ is étale, $\overline{\mathcal{T}}$ is irreducible, and $\Psi_{[1,1,1,1]}$ is dominant, we conclude that $\Psi_{\overline{\mathcal{T}}}$ is dominant. Since $u \in \mathcal{U}$ is general, we obtain the proof. □

Corollary 7.8. *There exists a set* $\overline{R}_u$ *consisting of* 315 *subsets* $\{\bar{l}_a, \bar{l}_b, \bar{l}_c, \bar{l}_d\} \subset \overline{L}_u$ *of size* 4 *with the following properties: a subset* $\{\bar{l}_i, \bar{l}_j, \bar{l}_k\} \subset \overline{L}_u$ *of size* 3 *is liftable if and only if there exists an element* $\{\bar{l}_a, \bar{l}_b, \bar{l}_c, \bar{l}_d\} \in \overline{R}_u$ *containing* $\{\bar{l}_i, \bar{l}_j, \bar{l}_k\}$. □

7.3. **Pairs of splitting conics.** Recall that $F_u \subset SY_u$ is the set of classes $[c]$ of Y-lifts c of 4-tangent conics $\bar{c}$ of Q_u, and that $\overline{F}_u = F_u/\langle \iota_u\rangle$ is regarded as the set of connected families of 4-tangent conics of Q_u, or equivalently as the set of connected families of splitting conics of Q_u. For a splitting conic $\bar{c}$, let $[\bar{c}] \in \overline{F}_u$ denote the connected family containing $\bar{c}$. By Theorem 3.1, we obtain the following:

Proposition 7.9. *By the monodromy,* $\pi_1(\mathcal{U}, b)$ *acts transitively on* F_b *and hence on* $\overline{F}_b$. □

Definition 7.10. Let $\bar{c}$ be a splitting conic of Q_u. We say that a decomposition $\pi_u^*(\bar{c}) = c + c'$ is *normal* if each of c and c' is a Y-lift of $\bar{c}$.

Note that, if $\bar{c}$ is smooth, then the decomposition $\pi_u^*(\bar{c}) = c + c'$ is normal, whereas if $\bar{c}$ is a sum of two bitangents $\bar{l} + \bar{l}'$, then $\pi_u^*(\bar{c}) = c + c'$ being normal means that $c = l + l'$ with $\langle l, l'\rangle_Y = 1$.

Definition 7.11. Let $\bar{c}_1$ and $\bar{c}_2$ be splitting conics of Q_u, and let $\pi_u^*(\bar{c}_1) = c_1 + c_1'$ and $\pi_u^*(\bar{c}_2) = c_2 + c_2'$ be the normal decompositions. We put

$$I([\bar{c}_1], [\bar{c}_2]) := \begin{bmatrix} \langle c_1, c_2\rangle_Y & \langle c_1, c_2'\rangle_Y \\ \langle c_1', c_2\rangle_Y & \langle c_1', c_2'\rangle_Y \end{bmatrix}.$$

Since we can make switchings $c_1 \leftrightarrow c_1'$ and $c_2 \leftrightarrow c_2'$, the matrix $I([\bar{c}_1], [\bar{c}_2])$ is well-defined only up to the transpositions of the two rows and of the two columns.

We have calculated $F_u \subset S_u$ explicitly. Using this data, we see that the matrix $I([\bar{c}_1], [\bar{c}_2])$ is one of the following:

$$\begin{aligned} I_A &:= \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, \\ I_B &:= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \\ I_C &:= \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \text{ or } \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}. \end{aligned}$$

Proposition 7.12. *Let $\bar{c}_1 = \{g_1 = 0\}$ and $\bar{c}_2 = \{g_2 = 0\}$ be splitting conics of Q_u. Consider the following conditions:*

(i) *$[\bar{c}_1] = [\bar{c}_2]$, that is, $\bar{c}_1$ and $\bar{c}_2$ belong to the same connected family.*
(ii) *The matrix $I([\bar{c}_1], [\bar{c}_2])$ is equal to I_A.*
(iii) *There exists a polynomial $q \in \Gamma(2)$ such that Q_u is defined by $g_1 g_2 + q^2 = 0$.*

Then we have (iii) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (i). *If* (i) *holds and $\bar{c}_1$ and $\bar{c}_2$ are general in the connected family $[\bar{c}_1] = [\bar{c}_2] \in \overline{F}_u$ of splitting conics, then* (iii) *holds.*

Proof. The implication (i) $\Longrightarrow$ (ii) follows immediately from Table 2.1, and the implication (iii) $\Longrightarrow$ (ii) follows from Lemma 7.3. Suppose that (ii) holds. Let $\pi_u^*(\bar{c}_1) = c_1 + c_1'$ and $\pi_u^*(\bar{c}_2) = c_2 + c_2'$ be the normal decompositions. Interchanging c_2 and c_2' if necessary, we can assume that $\langle c_1, c_2\rangle_Y = 0$. We put $f_1 := \eta_u^*(c_1)$ and $f_2 := \eta_u^*(c_2)$. Note that f_1 is a fiber of the elliptic fibration $\phi_1 \in \mathit{Ells}(X_u)$ corresponding to the class $[c_1] \in F_u$ of c_1 by $F_u \cong \mathit{Ells}(X_u)$. Since $\langle f_1, f_2\rangle_X = 2\langle c_1, c_2\rangle_Y = 0$, we conclude that f_2 is a fiber of ϕ_1, that is, the elliptic fibration corresponding to $[c_2] \in F_u \cong \mathit{Ells}(X_u)$ is equal to ϕ_1. Therefore $\bar{c}_1$ and $\bar{c}_2$ belong to the same connected family of splitting conics, and (i) holds. Thus (iii) $\Longrightarrow$ (ii) $\Longleftrightarrow$ (i) is proved.

Suppose that $[\bar{c}_1] = [\bar{c}_2]$. Let $\sigma \colon \overline{\mathcal{F}} \to \mathcal{U}$ be the finite étale covering defined by the family $\{\, \overline{F}_u \mid u \in \mathcal{U}\,\}$. By Proposition 7.9, we see that $\overline{\mathcal{F}}$ is irreducible. Let $p := (g_1', g_2', q')$ be a point of $\mathcal{V}_{[2,2]}$, and we put $u' := \Psi_{[2,2]}(p) \in \mathcal{U}$;

$$Q_{u'} = \{g_1' g_2' + {q'}^2 = 0\}.$$

Let $\bar{c}_i'$ be the splitting conic $\{g_i' = 0\}$ of $Q_{u'}$ for $i = 1, 2$. By the implication (iii) $\Longrightarrow$ (i), we have $[\bar{c}_1'] = [\bar{c}_2']$ in $\overline{F}_{u'}$. By $p \mapsto [\bar{c}_1']$, we obtain a morphism $\Psi_{\overline{\mathcal{F}}} \colon \mathcal{V}_{[2,2]} \to \overline{\mathcal{F}}$. By the same argument as in the proof of Proposition 7.7, we see that $\Psi_{\overline{\mathcal{F}}}$ is dominant. Since u is general in $\mathcal{U}$, the point $(u, [\bar{c}_1]) = (u, [\bar{c}_2])$ is general in $\overline{\mathcal{F}}$ and the fiber W of $\Psi_{\overline{\mathcal{F}}}$ over $(u, [\bar{c}_1])$ is of dimension

$$\dim \Gamma(2, 2\colon 2) - \dim \mathcal{U} = 18 - 14 = 4.$$

Let $S := \{\, \bar{c}(t) \mid t \in \mathbb{P}^1 \,\}$ be the connected family of splitting conics containing $\bar{c}_1$ and $\bar{c}_2$. If (g_1', g_2', q') is a point of the fiber W, then we have two members $\bar{c}_1' = \{g_1' = 0\}$ and $\bar{c}_2' = \{g_2' = 0\}$ of S, and thus we have a morphism $W \to \mathbb{P}^1 \times \mathbb{P}^1$, where $\mathbb{P}^1$ is the base curve of the family S. If two points (g_1', g_2', q') and (g_1'', g_2'', q'') of W are mapped to the same point of $\mathbb{P}^1 \times \mathbb{P}^1$, then there exist scalars $\lambda_1, \lambda_2 \in \mathbb{C}^\times$ such that $g_1'' = \lambda_1 g_1'$ and $g_2'' = \lambda_2 g_2'$. By the dimension reason, we see that $W \to \mathbb{P}^1 \times \mathbb{P}^1$ is dominant. Hence, if $\bar{c}_1 = \{g_1 = 0\}$ and $\bar{c}_2 = \{g_2 = 0\}$ are general members of the family S, there exists a polynomial $q \in \Gamma(2)$ such that $(g_1, g_2, q) \in W$, that is, Q_u is defined by $g_1 g_2 + q^2 = 0$. □

The following two propositions are confirmed by direct computation.

Proposition 7.13. *Among the* 1953 *non-ordered pairs* $\{[\bar{c}_1], [\bar{c}_2]\}$ *of distinct elements* $[\bar{c}_1], [\bar{c}_2]$ *of* $\overline{F}_u$*, exactly* 945 *pairs satisfy* $I([\bar{c}_1], [\bar{c}_2]) = I_B$*; the remaining* 1008 *pairs satisfy* $I([\bar{c}_1], [\bar{c}_2]) = I_C$*. When* $u = b$*, these two sets of pairs are the orbits of the monodromy action of* $\pi_1(\mathcal{U}, b)$ *on the set of non-ordered pairs of elements of* $\overline{F}_b$*.* □

Recall that each connected family $[c] \in F_u$ of Y-lifts of splitting conics contains exactly 6 reducible members, and the irreducible components l, l' of a reducible member satisfy $\langle l, l' \rangle_Y = 1$. We have a surjective map

$$\{\,\{l, l'\} \mid l, l' \in L_u,\ \langle l, l' \rangle_Y = 1\,\} \to F_u$$

defined by $\{l, l'\} \mapsto [l] + [l']$. Each fiber of size 6. The following gives how the cases

$$I([\bar{c}_1], [\bar{c}_2]) = I_B \qquad \text{and} \qquad I([\bar{c}_1], [\bar{c}_2]) = I_C$$

are distinguished.

Proposition 7.14. *Let* $[c_1]$ *and* $[c_2]$ *be elements of* F_u*, and let* $[\bar{c}_1]$ *and* $[\bar{c}_2]$ *be their images by* $F_u \to \overline{F}_u$*. Then* $I([\bar{c}_1], [\bar{c}_2]) = I_B$ *holds if and only if there exists a triangle* $\{l_1, l_2, l_3\}$ *on* Y_u *such that* $[c_1] = [l_1] + [l_3]$ *and* $[c_2] = [l_2] + [l_3]$*.* □

7.4. **Pairs of a bitangent and a splitting conic.** Let $\bar{l}$ be a bitangent of Q_u with $\pi_u^*(\bar{l}) = l + l'$, and let $\bar{c}$ be a splitting conic of Q_u with the normal decomposition $\pi_u^*(\bar{c}) = c + c'$. We put

$$J(\bar{l}, [\bar{c}]) := \begin{bmatrix} \langle l, c \rangle_Y & \langle l, c' \rangle_Y \\ \langle l', c \rangle_Y & \langle l', c' \rangle_Y \end{bmatrix}.$$

The matrix $J(\bar{l}, [\bar{c}])$ is one of the following:

$$\begin{aligned} J_\alpha &:= \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \text{or} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \\ J_\beta &:= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}. \end{aligned}$$

By direct computation, we confirm the following:

Proposition 7.15. *Let* $\bar{l}$ *be a bitangent of* Q_u*, and* $\bar{c}$ *a splitting conic of* Q_u*. Then* $J(\bar{l}, [\bar{c}])$ *is equal to* J_α *if and only if the connected family* $[\bar{c}] \in \overline{F}_u$ *of splitting conics has a singular member containing* $\bar{l}$ *as an irreducible component.*

When $u = b$*, the monodromy action of* $\pi_1(\mathcal{U}, b)$ *acts on the set of pairs* $(\bar{l}, [\bar{c}]) \in \overline{L}_b \times \overline{F}_b$ *with* $J(\bar{l}, [\bar{c}]) = J_\alpha$ *transitively, and the set of pairs* $(\bar{l}, [\bar{c}])$ *with* $J(\bar{l}, [\bar{c}]) = J_\beta$ *also transitively.* □

8. Intersection graph

Definition 8.1. An *intersection graph* is a pentad $(V_{\bar{l}}, V_{\bar{c}}, T, E_{\bar{c}\bar{c}}, E_{\bar{l}\bar{c}})$ such that

- $V_{\bar{l}}$ and $V_{\bar{c}}$ are finite sets,
- T is a subset of $S^3_0(V_{\bar{l}})$,
- $E_{\bar{c}\bar{c}}$ is a map $S^2(V_{\bar{c}}) \to \{A, B, C\}$, and
- $E_{\bar{l}\bar{c}}$ is a map $V_{\bar{l}} \times V_{\bar{c}} \to \{\alpha, \beta\}$.

Two intersection graphs $(V_{\bar{l}}, V_{\bar{c}}, T, E_{\bar{c}\bar{c}}, E_{\bar{l}\bar{c}})$ and $(V'_{\bar{l}}, V'_{\bar{c}}, T', E'_{\bar{c}\bar{c}}, E'_{\bar{l}\bar{c}})$ are *isomorphic* if there exists a pair of bijections $V_{\bar{l}} \cong V'_{\bar{l}}$ and $V_{\bar{c}} \cong V'_{\bar{c}}$ that induces $T \cong T'$, $E_{\bar{c}\bar{c}} \cong E'_{\bar{c}\bar{c}}$, and $E_{\bar{l}\bar{c}} \cong E'_{\bar{l}\bar{c}}$.

i	$\lvert o_i\rvert$	$\lvert T\rvert$	a_0	a_1	a_2
1	2016	0	0	0	0
2	1008	0	0	0	0
3	30240	4	0	0	6
4	60480	6	0	6	9
5	22680	8	2	10	16
6	181440	8	2	14	12
7	5040	8	4	12	12
8	12096	10	0	30	15
9	60480	10	2	24	19
10	1260	12	6	30	30

TABLE 9.1. The orbit decomposition for $(m, n) = (6, 0)$

Definition 8.2. For a $\mathcal{Q}^{(m,n)}$-curve Z as in (1.1), we define an intersection graph

$$g(Z) := (V_{\bar{l}}, V_{\bar{c}}, T, E_{\bar{c}\bar{c}}, E_{\bar{l}\bar{c}})$$

by the following:

- $V_{\bar{l}}$ is $\{\bar{l}_1, \dots, \bar{l}_m\}$ and $V_{\bar{c}}$ is $\{\bar{c}_1, \dots, \bar{c}_n\}$,
- T is the set of liftable triangles $\{\bar{l}_i, \bar{l}_j, \bar{l}_k\} \subset \{\bar{l}_1, \dots, \bar{l}_m\}$,
- $E_{\bar{c}\bar{c}}(\bar{c}_i, \bar{c}_j)$ is the type of the matrix $I([\bar{c}_i], [\bar{c}_j])$ defined in Section 7.3, and
- $E_{\bar{l}\bar{c}}(\bar{l}_i, \bar{c}_j)$ is the type of the matrix $J(\bar{l}_i, [\bar{c}_j])$ defined in Section 7.4.

Remark 8.3. By Proposition 7.12, the relation

$$\bar{c}_i \sim \bar{c}_j \iff E_{\bar{c}\bar{c}}(\bar{c}_i, \bar{c}_j) = A$$

is an equivalence relation on $V_{\bar{c}}$, and the functions $E_{\bar{c}\bar{c}}$ and $E_{\bar{l}\bar{c}}$ are compatible with this equivalence relation.

Remark 8.4. When $n = 0$, the intersection graph equal to the two-graph in [5].

It is obvious that, if ζ and ζ' are in the same connected component of $\mathcal{Z}^{(m,n)}$, the intersection graphs $g(Z_\zeta)$ and $g(Z_{\zeta'})$ are isomorphic. The converse is not true in general, as examples in the next section show.

9. Examples

9.1. **The case** $(m, n) = (6, 0)$**.** We have $|P_b^{(6,0)}| = 376740$. The action of $W(E_7)$ decomposes $P_b^{(6,0)}$ into orbits as in Table 9.1. For each orbit $o_i \subset P_b^{(6,0)}$, we choose a point $\zeta \in o_i$ and indicate the following data of the intersection graph $g(Z_\zeta)$ of $Z_\zeta = Q_u + \bar{l}_1 + \cdots + \bar{l}_6$: $|T| = k$ is the number of the liftable triangles $t_1, \dots, t_k$ in $\{\bar{l}_1, \dots, \bar{l}_6\}$, and a_ν is the number of pairs $\{t_i, t_j\}$ of liftable triangles such that $|t_i \cap t_j| = \nu$. The orbit o_1 and o_2 cannot be distinguished by the two-graph $(V_{\bar{l}}, T)$, but they belong to different $W(E_7)$-orbits, and hence the corresponding $\mathcal{Q}^{(6,0)}$-curves are of different homeomorphism types.

9.2. **The case** $n = 0$**.** We continue to consider the case where $n = 0$. From the two-graph $g = (V_{\bar{l}}, T)$, we can construct a graph $\tilde{g}$ whose set of vertices is T and whose edge connecting $t_\mu, t_\nu \in T$ has weight $|t_\mu \cap t_\nu|$. If the graphs $\tilde{g}$ and $\tilde{g}'$ are not isomorphic as graphs with weighted edges, then the two-graphs g and g' are not isomorphic. Using this method, we prove the following:

Proposition 9.1. *Except for the two orbits o_1 and o_2 in the case $m = 6$ described in Section 9.1, all $W(E_7)$-orbits of $P_b^{(m,0)}$ are distinguished by their two-graphs.* $\square$

i	edge labels	orbit sizes
1	AAA	63
2	ABB	1890
3	ACC	2016
4	BBB	$3780 + 315$
5	BBC	15120
6	BCC	15120
7	CCC	$5040 + 336$

TABLE 9.2. The orbit decomposition for $(m, n) = (0, 3)$

i	$E_{\bar{l}\bar{c}}$	$E_{\bar{c}\bar{c}}$	orbit sizes
1	$[[\alpha, \alpha], [\alpha, \alpha]]$	A	$3780 + 378$
2	$[[\alpha, \alpha], [\alpha, \alpha]]$	B	$3780 + 1890$
3	$[[\alpha, \alpha], [\alpha, \alpha]]$	C	15120
4	$[[\alpha, \alpha], [\alpha, \beta]]$	B	60480
5	$[[\alpha, \alpha], [\alpha, \beta]]$	C	$60480 + 12096$
6	$[[\alpha, \alpha], [\beta, \beta]]$	A	12096
7	$[[\alpha, \alpha], [\beta, \beta]]$	B	30240
8	$[[\alpha, \alpha], [\beta, \beta]]$	C	60480
9	$[[\alpha, \beta], [\alpha, \beta]]$	B	$45360 + 7560$
10	$[[\alpha, \beta], [\alpha, \beta]]$	C	30240
11	$[[\alpha, \beta], [\beta, \alpha]]$	B	60480
12	$[[\alpha, \beta], [\beta, \alpha]]$	C	$30240 + 6048$
13	$[[\alpha, \beta], [\beta, \beta]]$	B	120960
14	$[[\alpha, \beta], [\beta, \beta]]$	C	120960
15	$[[\beta, \beta], [\beta, \beta]]$	A	7560
16	$[[\beta, \beta], [\beta, \beta]]$	B	$22680 + 3780$
17	$[[\beta, \beta], [\beta, \beta]]$	C	45360

TABLE 9.3. The orbit decomposition for $(m, n) = (2, 2)$

Example 9.2. Let o'_1 and o'_2 be the orbits in $P_b^{(22,0)}$ containing 22-tuples obtained by taking the complement in $\overline{L}_b$ of 6-tuples in the orbits $o_1 \subset P_b^{(6,0)}$ and $o_2 \subset P^{(6,0)}$ above, respectively. Let g'_1 and g'_2 be the two-graphs of o'_1 and o'_2. We have $|T| = 600$ for both g'_1 and g'_2. The associated graphs $\tilde{g}'_1$ and $\tilde{g}'_2$ with weighted edges are not isomorphic. The graph $\tilde{g}'_1$ has exactly 8203640 triples $\{t_\lambda, t_\mu, t_\nu\}$ of liftable triangles with weight $|t_\lambda \cap t_\mu| = |t_\mu \cap t_\nu| = |t_\nu \cap t_\lambda| = 0$, whereas the number of such triples in $\tilde{g}'_2$ is 8203760.

9.3. **The case** $(m, n) = (0, 3)$**.** By Remark 8.3, the three edges of the graph $(V_{\bar{c}}, E_{\bar{c}\bar{c}})$ are labelled as in the second column of Table 9.2. The set $P_b^{(0,3)}$ of size 43680 is decomposed into nine $W(E_7)$-orbits with sizes given in the third column of Table 9.2.

9.4. **The case** $(m, n) = (2, 2)$**.** There exist 17 intersection graphs indicated in Table 9.3, where $E_{\bar{c}\bar{c}}$ is shown by the type of $I([\bar{c}_1], [\bar{c}_2])$, and

$$E_{\bar{l}\bar{c}} := [[J(\bar{l}_1, [\bar{c}_1]), J(\bar{l}_1, [\bar{c}_2])], [J(\bar{l}_2, [\bar{c}_1]), J(\bar{l}_2, [\bar{c}_2])]].$$

The set $P_b^{(2,2)}$ of size 762048 is decomposed into 23 orbits by the action of $W(E_7)$, and their sizes are given in the 4th column of Table 9.3.

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