Journal *of* Singularities
Volume 23 (2021), 15-18

received: 3 March 2020
in revised form: 30 November 2020
DOI: 10.5427/jsing.2021.23b

ERRATUM FOR "THE SHEAF $\alpha_X^\bullet$"

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1. Erratum for "the sheaf $\alpha_X^\bullet$"

The aim of this erratum is to correct several mistakes in [3]. The main mistake is in Theorem 4.1.1 of [3] which is wrong in the very general setting in which it is stated.

So we give here a much more modest version of the "pull-back theorem" for these sheaves which has a rather simple proof.

Recall that on a reduced complex space X the sheaf $\alpha_X^\bullet$ is the integral closure in the sheaf $\omega_X^\bullet$ of the sheaf $\Omega_X^\bullet\big/torsion$, where $\Omega_X^\bullet$ is the sheaf of Kähler differential forms and where the sheaf $\omega_X^\bullet$ is the sheaf of $(\bullet, 0)-\bar\partial-$closed currents on X modulo its torsion sub-sheaf (see [1]).

Theorem 1.0.1. *Let $f : X \to Y$ be a holomorphic map between reduced complex spaces and assume that $f^{-1}(S(Y))$ has empty interior in X, where $S(Y)$ is the singular set of Y. Then there exists a natural "pull-back map"*

$$\hat{f}^* : f^*(\alpha_Y^\bullet) \to \alpha_X^\bullet$$

which extends the usual pull-back of the graduate sheaf of holomorphic differential forms

$$f^* : f^*(\Omega_Y^\bullet\big/torsion) \to \Omega_X^\bullet\big/torsion.$$

For any holomorphic maps $f : X \to Y$ and $g : Y \to Z$ between reduced complex spaces such that $f^{-1}\big(S(Y) \cup g^{-1}(S(Z))\big)$ has empty interior in X and $g^{-1}(S(Z))$ has empty interior in Y we have

$$\hat{f}^*(\hat{g}^*(\sigma)) = \widehat{f \circ g}^*(\sigma) \quad \forall \sigma \in \alpha_Z^\bullet. \tag{1}$$

Proof. The problem is local. Let σ be a section of the sheaf $\alpha_Y^\bullet$ on an open set V in Y. Let V' be the set of regular points in V and let U'' the set of regular points in the open set $U' := f^{-1}(V')$. This is a Zariski dense open set in $U := f^{-1}(V)$ and, as σ is a holomorphic form on V', $f^*(\sigma)$ is a well defined holomorphic form on U'' which is Zariski open and dense in U. Take a point x in U; by definition (see Proposition 2.2.4 in [3]) there exists an open neighborhood W of $y := f(x)$ in V and a monic polynomial

$$P(z) = z^k + \sum_{h=1}^{k} S_h.z^{k-h}$$

such that S_h is a section on W of the symmetric algebra of degree h, $S^h(\Omega_Y^\bullet\big/torsion)$, of the sheaf $\Omega_Y^\bullet\big/torsion$, which satisfies $P(\sigma) = 0$ in $\Gamma(W, S^k(\Omega_Y^\bullet\big/torsion))$. Then the pull-back $f^*(P)$ of P by f is well defined on $f^{-1}(W)$ and is a monic polynomial whose coefficients are sections on $f^{-1}(W)$ of the symmetric algebra of $\Omega_X^\bullet\big/torsion$. On the open set $U'' \cap f^{-1}(W)$ the holomorphic

2010 *Mathematics Subject Classification.* 32S05, 32S10, 32S20.
Key words and phrases. singular complex spaces, singularity invariants, meromorphic differential forms.
I thank the referee for several careful readings and for many relevant remarks.

form $f^*(\sigma)$ is a root of $f^*(P)$ and so the meromorphic[1] form $f^*(\sigma)$ on $U \cap f^{-1}(W)$ is integrally dependent on the sheaf $\Omega^\bullet_X\big/torsion$. So it defines a unique section on U of the sheaf $\alpha^\bullet_X$.
As the equality (1) holds generically on X the conclusion follows from the fact that the sheaf $\alpha^\bullet_X$ has no torsion. ■

The second mistake (which is a consequence of the previous one) is that, in Definition 5.1.5 of [3], it is necessary to ask that the $p-$dimensional irreducible analytic subset Y is not contained in the singular set of X in order to define the integral on Y of a form of the type $\rho.\alpha \wedge \bar{\beta}$, where α, β are sections of the sheaf α^p_X in X.

To be clear we give here the correct statements for Definition 5.1.5, Lemma 5.1.6 and for Theorem 5.1.7. The statement of such a result makes sense only assuming that the pull-back for the sheaf $\alpha^\bullet$ is defined. This is consequence of the hypothesis that Y is not contained in $S(X)$ which allows one to apply Theorem 1.0.1 above.

Definition 1.0.2. *Let X be a reduced complex space and let $Y \subset X$ be a closed irreducible $p-$dimensional analytic subset in X; assume that Y is not contained in the singular set $S(X)$ of X. We shall note $j : Y \to X$ the inclusion map. Let ρ be a continuous function with compact support in X and let α, β be sections on X of the sheaf α^p_X. We define the number $\int_Y \ \rho.\alpha \wedge \bar{\beta}$ as the integral*

$$\int_Y \ j^*(\rho).\hat{j}^*(\alpha) \wedge \overline{\hat{j}^*(\beta)}$$

which is well-defined by Theorem 1.0.1 above.

This definition extends by additivity to any $p-$cycle Y in X such that its support has no irreducible component contained in $S(X)$.

Remark. The definition above makes sense, more generally, still assuming that Y is not contained in $S(X)$, when α and β are sections of the sheaf L^p_X of meromorphic forms which become holomorphic on any desingularisation of X because we see that the improper integral on $Y \setminus S(X)$ converges by looking at the strict transform of Y by the desingularisation map.

Lemma 1.0.3. *Let $f : X \to Y$ be a holomorphic map between reduced complex spaces such that $f^{-1}(S(Y))$ has empty interior in X. Let Z be a closed $p-$dimensional irreducible analytic subset in X such that Z is not contained in the singular set $S(X)$ of X, the restriction of f to Z is proper and $f(Z)$ is not contained in the singular set of Y. Let α, β be sections on Y of the sheaf α^p_Y and let ρ be a continuous function with compact support in Y. Then we have the equality*

$$\int_Z \ f^*(\rho).\hat{f}^*(\alpha) \wedge \overline{\hat{f}^*(\beta)} = \int_{f_*(Z)} \ \rho.\hat{j}^*(\alpha) \wedge \overline{\hat{j}^*(\beta)}$$

where $f_*(Z)$ is the direct image cycle of Z by f and $j : |f_*(Z)| \to Y$ the inclusion in Y of the support of the cycle $f_*(Z)$.

Moreover if the set $f(Z)$ is contained in $S(Y)$ the singular set of Y and has dimension at most $p-1$ (so that $f_*(Z)$ is the empty $p-$cycle) we have

$$\int_Z \ f^*(\rho).\hat{f}^*(\alpha) \wedge \overline{\hat{f}^*(\beta)} = 0.$$

Of course, when $f(Z) \not\subset S(Y)$ and satisfies $f_*(Z) = 0$ as a $p-$cycle in Y, the first part of the lemma gives also the vanishing of $\int_Z \ f^*(\rho).\hat{f}^*(\alpha) \wedge \overline{\hat{f}^*(\beta)} = 0$.

[1]Remember that σ is a meromorphic form on V with poles in $S(Y) \cap V$.

Proof. The first assertion is an easy consequence of the same result when α, β are holomorphic forms (see [2] Ch.IV Prop. 2.3.1, or Prop. 4.2.17 in the English translation), by considering a modification of X where it is the case, using for instance, a desingularisation of X (see [5]).

When $f(Z) \subset S(Y)$ and $f_*(Z) = 0$ the restriction of f to Z has generic rank at most $p-1$, so the pull-back of any holomorphic $p-$form on Y to Z is torsion. Then the monic polynomial giving an integral dependence relation of α (or of β) reduces to $z^k = 0$ on $f(Z)$ and so α (and β) vanishes on Z. ■

We give now a correct version of Theorem 5.1.7 in [3].

Theorem 1.0.4. *Let X be a reduced complex space and $(Y_t)_{t\in T}$ be a proper analytic family of compact $p-$cycles in X parametrized by a reduced complex space T (see [2] Section IV.3). Assume that for t in a dense open subset T' in T no component of the cycle Y_t is contained in $S(X)$, the singular set of X. Let ρ be a continuous function with support in the compact set K in X and let α, β be two sections of the sheaf α_X^p. Define the function*

$$\varphi : T' \to \mathbb{C} \quad \text{by} \quad \varphi(t) := \int_{Y_t} \rho.\hat{j}_t^*(\alpha) \wedge \overline{\hat{j}_t^*(\beta)}$$

where $j_t : |Y_t| \to X$ is the inclusion in X of the support of the cycle Y_t.

Then φ is continuous on T' and locally bounded near each point in T.

For any continuous hermitian metric h on X, there exists a constant $C > 0$ (depending on K, α, β, h but not of the choice of ρ with support in K) such that for each $t \in T'$ we have:

$$|\varphi(t)| \leq C.\int_{Y_t} |\rho|.h^{\wedge p} \leq C.||\rho||_\infty.\int_{Y_t\cap K} h^{\wedge p}. \tag{2}$$

Proof. Let $\tau : \tilde{X} \to X$ be a desingularisation of X; so $\hat{\tau}^*(\alpha)$ and $\hat{\tau}^*(\beta)$ are holomorphic $p-$forms on $\tilde{X}$. Using Corollary IV 9.1.3 in [2] we may lift the analytic family $(Y_t)_{t\in T}$ to an analytic family $(\tilde{Y})_{\tilde{t}\in\tilde{T}}$ where $\theta : \tilde{T} \to T$ is a (proper) modification such that for each $\tilde{t} \in \theta^{-1}(T')$ we have the equality of cycles in X

$$\tau_*(\tilde{Y}_{\tilde{t}}) = Y_{\theta(\tilde{t})}. \tag{S}$$

Then Proposition IV 2.3.1 in [2] gives the continuity of the function $\tilde{\varphi} : \tilde{T} \to \mathbb{C}$ defined by

$$\tilde{\varphi}(\tilde{t}) = \int_{\tilde{Y}_{\tilde{t}}} \tau^*(\rho).\hat{\tau}^*(\alpha) \wedge \overline{\hat{\tau}^*(\beta)}.$$

The point is now to show that for $\tilde{t} \in \theta^{-1}(T')$ we have $\tilde{\varphi}(\tilde{t}) = \varphi(\theta(\tilde{t}))$. Thanks to Corollary IV 2.5.5 in [2] this is clear using the formula (S) if we can prove that for $\theta(\tilde{t}) \in T'$ the contribution to the integral $\tilde{\varphi}(\tilde{t})$ of an irreducible component Z of $\tilde{Y}_{\tilde{t}}$ satisfying $\tau_*(Z) = 0$ as a $p-$cycle in X vanishes, because this implies the equality $\varphi(\theta(\tilde{t})) = \tilde{\varphi}(\tilde{t})$. But this is precisely the content of the second part of Lemma 1.0.3. This gives the continuity of φ on T'.

As $\tilde{\varphi}$ is continuous on $\tilde{T}$, the function φ is locally bounded near each point in T.

The estimate (2) is a direct consequence of Corollary 5.1.2 in [3]. ■

Remarks.

(1) Assuming only that α and β are sections of the sheaf L_X^p, it is not clear that φ is continuous on T' because in order to lift the family of cycles $(Y_t)_{t\in T}$ in a continuous family of cycles on a desingularisation of X it may be necessary to add exceptional components to the strict transform of Y_t for some values of $t \in T'$ and the argument used above to show that these components do not contribute to the integral upstairs

does not works for sections in L^p_X. Moreover, the estimate (2) is not true in general for sections in L^p_X (see Remark 2 following Corollary 5.1.2 in [3]).

(2) For any analytic family of compact cycles $(Y_t)_{t\in T}$ in X, the subset of points $t \in T$ where the cycle Y_t has at least one irreducible component contained in $S(X)$ is a closed analytic subset in T by a general result on analytic families of compact cycles (see the exercise following Theorem IV 3.3.1 in [2]). So, assuming that T is irreducible, if there exists a point t such that Y_t has no irreducible component contained in $S(X)$, there exists a Zariski open and dense subset T' of T which satisfies the hypothesis in the previous theorem.

(3) The previous theorem is in fact a local result on X and T, but we consider here only the case of a proper analytic family of compact cycles in X to have a simple argument to lift the analytic family of cycles in X to an analytic family of cycles in $\tilde{X}$ such that (S) is satisfied.

The last mistake is Lemma 6.2.2 which is wrong for $k \geq 4$. The correct computation of $\alpha^2_{S_k}$ is given in Paragraph 2.3 in [4].

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