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A GEOMETRIC DESCRIPTION OF THE MONODROMY OF BRIESKORN-PHAM POLYNOMIALS

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Abstract. We give an explicit construction of Lê's vanishing polyhedra for a Brieskorn-Pham polynomial f. Then we use it to give a geometric description of the monodromy associated to f. It allows us to write the matrix that determines the induced algebraic monodromy. In particular, this provides another proof for the Brieskorn-Pham theorem, which says that the characteristic polynomial associated to the monodromy of f is given by $\Delta(t) = \Pi(t - \omega_1\omega_2 \dots \omega_n)$, where each ω_j ranges over all a_j-th roots of unity other than 1.

1. Introduction

Let $f : \mathbb{C}^n \to \mathbb{C}$ be the polynomial map given by

$$f(z_1, \dots, z_n) = z_1^{a_1} + \dots + z_n^{a_n}\,,$$

with $a_j \in \mathbb{N}$ and $a_j \geq 2$, for $j = 1, \dots, n$.

Pham [8] constructed a polyhedron $\mathcal{P}$ in the Milnor fiber F_f of f which is a deformation retract of F_f. Moreover, he showed that $\mathcal{P}$ (and hence F_f) has the homotopy type of a wedge of $\mu(f)$-many spheres $\mathbb{S}^{n-1}$, with

$$\mu(f) = (a_1 - 1)(a_2 - 1)\dots(a_n - 1)\,.$$

Afterwards, Brieskorn [2] studied the topology of the complex variety $f^{-1}(0)$, so now the polynomials above are known as *Brieskorn-Pham polynomials*.

They also studied the algebraic monodromy

$$h^* : H_{n-1}(F_f; \mathbb{C}) \to H_{n-1}(F_f; \mathbb{C})$$

associated to the Milnor fibration of f. They showed that the characteristic roots of the linear transformation h^* are the products $\omega_1\omega_2 \dots \omega_n$, where each ω_j ranges over all the a_j-th roots of unity other than 1. So the characteristic polynomial of h^* is given by

$$\Delta(t) = \Pi(t - \omega_1\omega_2 \dots \omega_n)\,.$$

Later, many other mathematicians have studied the monodromy associated to singularities. See [3] for a survey on this subject.

In this paper, we use Lê's construction ([4] and [5]) of the vanishing polyhedron $\mathcal{P}$ in F_f to give a geometric description of the induced monodromy $h : \mathcal{P} \to \mathcal{P}$. It allows us to explicitly construct the matrix defined by the induced geometric monodromy h^* with respect to a given basis for $H_{n-1}(\mathcal{P})$ (compare to [7], page 75). In particular, it provides another proof for the Brieskorn-Pham theorem.

The approach suggested by this paper could be useful to study the monodromy associated to real analytic map-germs with an isolated critical point.

On the other hand, the explicit construction of a Lê's vanishing polyhedron for this family of complex functions is a quite interesting example illustrating Lê's construction in a concrete case.

There is another way of describing the geometric monodromy of certain classes of singularities, which have recently been developed by A'Campo. In the last section of his very interesting preprint [1] he explains the so-called *tête-à-tête monodromy* for Brieskorn-Pham polynomials in three variables.

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2. Lê's vanishing polyhedron

In [4] D.T. Lê sketched a proof of the following theorem, whose complete proof was given later in [5] by the author and himself.

Theorem 2.1. *Let $X \subset \mathbb{C}^N$ be a reduced equidimensional complex analytic space and let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification of X. Let $f : (X, x) \to (\mathbb{C}, 0)$ be a germ of complex analytic function at a point $x \in X$. If f has an isolated singularity at x relatively to $\mathcal{S}$ and if ϵ and η are sufficiently small positive real numbers as above, then for each $t \in \mathbb{D}_\eta^*$ there exist:*

- (i) *a polyhedron P_t of real dimension $\dim_{\mathbb{C}} X_t$ in the Milnor fiber X_t, compatible with the Whitney stratification $\mathcal{S}$, and a continuous simplicial map:*

$$\tilde{\xi}_t : \partial X_t \to P_t$$

  compatible with $\mathcal{S}$, such that X_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$;
- (ii) *a continuous map $\psi_t : X_t \to X_0$ that sends P_t to $\{0\}$ and that restricts to a homeomorphism $X_t \backslash P_t \to X_0 \backslash \{0\}$.*

In this section, we review the general lines of Lê's construction of such a vanishing polyhedron in the case of a complex function-germ $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with $n \geq 2$ and with isolated critical point.

Let $\ell : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a linear form and consider the map-germ

$$\phi_\ell : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$$

defined by $\phi_\ell(z) := (\ell(z), f(z))$.

For a generic choice of ℓ the critical set of ϕ_ℓ is either empty or a smooth reduced complex curve, whose closure Γ has image by ϕ_ℓ a complex curve Δ in $\mathbb{C}^2$ (Lemma 21 of [5]). We say that Γ is the *polar curve of f relatively to ℓ* and that Δ *is the polar discriminant of f relatively to ℓ.*

Then the map ϕ_ℓ induces a locally trivial fibration

$$\phi_| : \phi_\ell^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta) \cap \mathbb{B}_\epsilon \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \setminus \Delta \,,$$

where η_1 and η_2 are small enough real numbers, with $0 < \eta_2 \ll \eta_1 \ll \epsilon$ (Proposition 22 of [5]). The Milnor fiber $f^{-1}(t) \cap \mathbb{B}_\epsilon$ of f is homeomorphic to the set $F_t := \phi_\ell^{-1}(D_t) \cap \mathbb{B}_\epsilon$ (see Theorem 2.3.1 of [6]) for $t \in \mathbb{D}_{\eta_2} \setminus \{0\}$, where

$$D_t := \mathbb{D}_{\eta_1} \times \{t\} \,.$$

Notice that for each $t \in \mathbb{D}_{\eta_2} \setminus \{0\}$ fixed, the restriction of ϕ_ℓ induces a locally trivial fibration

$$\ell_t : (F_t \setminus \{y_1(t), \dots, y_k(t)\}) \cap \mathbb{B}_\epsilon \to D_t \setminus \{y_1(t), \dots, y_k(t)\} \,,$$

where

$$\{y_1(t), \dots, y_k(t)\} := \Delta \cap D_t \,.$$

We can suppose that $\lambda_t := (0,t)$ is in $D_t \setminus \{y_1(t), \dots, y_k(t)\}$. For each $j = 1, \dots, k$, let $\delta(y_j(t))$ be a simple path in D_t starting at λ_t and ending at $y_j(t)$. We can choose λ_t in such a way that these paths are disjoint away from λ_t. Finally, set

$$Q_t := \bigcup_{j=1}^{k} \delta(y_j(t)) \,.$$

With this notation, we can now construct the Lê's vanishing polyhedron. This is done by induction on n.

For $n = 2$ we just set

$$P_t := \ell_t^{-1}(Q_t)$$

and the lifting of a suitable vector field on D_t that deformation retracts it onto Q_t gives a deformation retraction of F_t onto P_t (see Lemma 25 and Proposition 27 of [5]).

Actually, the constructions above can be made simultaneously for every t in a simple path γ in $\mathbb{D}_{\eta_2}$ joining 0 and some $t_0 \in \partial\mathbb{D}_{\eta_2}$. The resulting polyhedron P_γ is called a *collapsing cone along* γ.

Now suppose $n > 2$. By the induction hypothesis we have a vanishing polyhedron P'_t in the local Milnor fiber F'_t of the hyperplane section

$$f' : \mathbb{C}^n \cap \{\ell = 0\} \to \mathbb{C} \,.$$

For each point $y_j(t) \in \Delta \cap D_t$ let $x_j(t)$ be a point in the intersection of the polar curve Γ with $\ell_t^{-1}\big(y_j(t)\big)$. Without losing generality, we can assume that $x_j(t)$ is the only point in such intersection. Also by the induction hypothesis, there is a collapsing cone P_j for the restriction of the map ℓ_t to a small neighborhood of $x_j(t)$. The "basis" of a such cone is the polyhedron $P_j(a_j) := P_j \cap \ell_t^{-1}(a_j)$, where a_j is a point in $\delta(y_j(t))\backslash y_j(t)$ close to $y_j(t)$.

Since ℓ_t is a locally trivial fiber bundle over $\delta(y_j(t))\backslash y_j(t)$, we can "extend" the cone P_j until it reaches the "central" polyhedron P'_t. This gives a polyhedron C_j. The union of all the polyhedra C_j together with P'_t gives our vanishing polyhedron P_t.

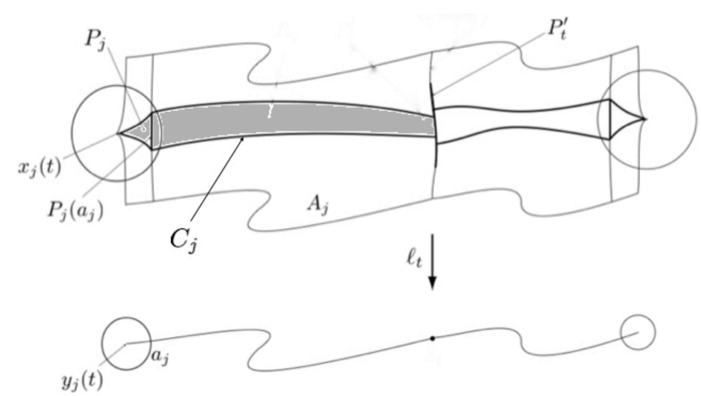

FIGURE 1.

3. Vanishing polyhedron for Brieskorn-Pham polynomials

In this section, we will follow the steps pointed in Section 2 above to construct a Lê's vanishing polyhedron for a Brieskorn-Pham polynomial

$$f(z_1, \dots, z_n) = z_1^{a_1} + \cdots + z_n^{a_n} \,,$$

with $a_j \in \mathbb{N}$ and $a_j \geq 2$, for $j = 1, \dots, n$.

3.1. **The two-dimensional case.** Since the construction of a Lê's vanishing polyhedron is made by induction on the dimension of the domain of the complex function f, we start with the two-dimensional case. That is, we consider a Brieskorn-Pham polynomial $f : \mathbb{C}^2 \to \mathbb{C}$ given by

$$f(x, y) = x^a + y^b ,$$

with $a, b \in \mathbb{N}$ and $a, b \geq 2$.

Define the linear form $\ell(x, y) = x$ and consider $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\phi := (\ell, f)$, that is

$$\phi(x, y) = (x, x^a + y^b) .$$

Its critical set is the curve $\Gamma = \{y = 0\}$, which we call the polar curve of f relatively to the form ℓ. We say that its image $\Delta = f(\Gamma)$ is the polar discriminant of f relatively to ℓ. It is the complex curve in $\mathbb{C}^2$ given by

$$\Delta = \{(u, v) \in \mathbb{C}^2;\ u^a - v = 0\} .$$

One can consider small real numbers $0 < \eta_2 \ll \eta_1 \ll \epsilon \ll 1$ such that the restriction

$$\phi_| : \phi^{-1}\big((\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \setminus \Delta\big) \cap \mathbb{B}_\epsilon \to (\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \setminus \Delta$$

is a topological locally trivial fibration (see Proposition 22 of [5]).

For any $t \in \mathbb{D}_{\eta_2}$ set

$$D_t := \mathbb{D}_\eta \times \{t\} .$$

If $t \neq 0$, the local Milnor fiber $f^{-1}(t) \cap \mathbb{B}_\epsilon$ of f at $0 \in \mathbb{C}^2$ is homeomorphic to

$$F_t := f^{-1}(t) \cap \ell^{-1}(\mathbb{D}_\eta) \cap \mathbb{B}_\epsilon$$

(see Theorem 2.3.1 of [6]).

Now, for any $t \in \mathbb{D}_{\eta_2}$ the map ϕ induces a map

$$\ell_t : F_t \to D_t$$

which is a locally trivial fibration over $D_t \setminus (\Delta \cap D_t)$.

Notice that

$$\Delta \cap D_t = \{(t^{\frac{1}{a}} \omega_a^\alpha, t) \in \mathbb{C}^2;\ 0 \leq \alpha \leq a - 1\} ,$$

where $\omega_a := \exp(\frac{2\pi i}{a})$. Moreover, notice that for each $\alpha = 0, \dots, a-1$ one has that

$$(\ell_t)^{-1}\big((t^{\frac{1}{a}} \omega_a^\alpha, t)\big) = \{(t^{\frac{1}{a}} \omega_a^\alpha, 0)\} .$$

Now, for each $\alpha = 0, \dots, a-1$ fixed, consider the path $\delta_{t,\alpha}$ in D_t given by

$$\delta_{t,\alpha}(r) := (r t^{\frac{1}{a}} \omega_a^\alpha, t) \ ;\ 0 \leq r \leq 1$$

Notice that

$$(\ell_t)^{-1}\big((r t^{\frac{1}{a}} \omega_a^\alpha, t)\big) = \{(r t^{\frac{1}{a}} \omega_a^\alpha, (1 - r^a)^{\frac{1}{b}} t^{\frac{1}{b}} \omega_b^\beta) \in \mathbb{C}^2;\ 0 \leq \beta \leq b - 1\} .$$

Hence $(\ell_t)^{-1}(\delta_{t,\alpha})$ is the union of the b-many paths $p_{\alpha,\beta}$ in F_t given by

$$p_{\alpha,\beta}(r) := (r t^{\frac{1}{a}} \omega_a^\alpha, (1 - r^a)^{\frac{1}{b}} t^{\frac{1}{b}} \omega_b^\beta) \ ;\ 0 \leq r \leq 1$$

with $\beta = 0, \dots, b-1$. Each path $p_{\alpha,\beta}$ start at the corresponding point $(0, t^{\frac{1}{b}} \omega_b^\beta) \in (\ell_t)^{-1}\big((0, t)\big)$. All the paths $p_{\alpha,\beta}$ end at the point $(t^{\frac{1}{a}} \omega_a^\alpha, 0) = (\ell_t)^{-1}\big((t^{\frac{1}{a}} \omega_a^\alpha, t)\big)$.

So the vanishing polyhedron P_t is given by

$$P_t := \bigcup_{\substack{0 \leq \alpha \leq a-1 \\ 0 \leq \beta \leq b-1}} tr(p_{\alpha,\beta})$$

where $tr(p_{\alpha,\beta})$ denotes the trace of the path $p_{\alpha,\beta}(r)$, with $0 \leq r \leq 1$.

Following [5] we have that P_t is a deformation retract of F_t. It is easy to see that P_t is homeomorphic to the join of $(\ell_t)^{-1}\big((0,t)\big)$ and $(\ell_t)^{-1}\big(\Delta \cap D_t\big)$. The first one is a set of b-many points and the second one is a set of a-many points. Hence the Milnor number of f is given by $\mu(f) = (a-1)(b-1)$.

3.2. **The general case.** Now, given $n > 2$, consider a Brieskorn-Pham polynomial

$$f(z_1, \ldots, z_n) = z_1^{a_1} + \cdots + z_n^{a_n},$$

with $a_j \in \mathbb{N}$ and $a_j \geq 2$, for $j = 1, \ldots, n$.

Define the linear form $\ell(z_1, \ldots, z_n) = z_n$ and consider $\phi : \mathbb{C}^n \to \mathbb{C}^2$ given by $\phi := (\ell, f)$. Its critical set is the polar curve

$$\Gamma = \{z_1 = \cdots = z_{n-1} = 0\},$$

and its image

$$\Delta = \{(u,v) \in \mathbb{C}^2;\ u^{a_n} - v = 0\}.$$

is the polar discriminant of f relatively to ℓ.

As before, one can consider small real numbers $0 < \eta_2 \ll \eta_1 \ll \epsilon \ll 1$ such that the restriction

$$\phi_| : \phi^{-1}\big((\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \setminus \Delta\big) \cap \mathbb{B}_\epsilon \to (\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \setminus \Delta$$

is a topological locally trivial fibration, so that for any $t \in \mathbb{D}_{\eta_2}$ the map ϕ induces a map

$$\ell_t : F_t \to D_t$$

which is a locally trivial fibration over $D_t \setminus (\Delta \cap D_t)$, where $D_t := \mathbb{D}_{\eta_1} \times \{t\}$ and

$$F_t := f^{-1}(t) \cap \ell^{-1}(\mathbb{D}_\eta) \cap \mathbb{B}_\epsilon$$

is homeomorphic to the local Milnor fiber of f at $0 \in \mathbb{C}^n$.

Notice that

$$\Delta \cap D_t = \{(t^{1/a_n}\omega_{a_n}^{\alpha_n}, t) \in \mathbb{C}^2;\ 0 \leq \alpha_n \leq a_n - 1\},$$

where $\omega_{a_n} := \exp(\frac{2\pi i}{a_n})$.

Let f' be the restriction of f to $\ell^{-1}(0)$. That is

$$f'(z_1, \ldots, z_{n-1}, 0) := z_1^{a_1} + \cdots + z_{n-1}^{a_{n-1}}.$$

By induction on n, we have a Lê's polyhedron P'_t in $F'_t := F_t \cap \{z_n = 0\}$ such that

$$P'_t = \bigcup_{\substack{0 \leq \alpha_j \leq a_j - 1 \\ 1 \leq j \leq n-1}} tr(p_{\alpha_1, \ldots, \alpha_{n-1}})$$

where each $p_{\alpha_1,\ldots,\alpha_{n-1}} : \big([0,1]\big)^{n-2} \to F'_t$ is a parametrized space.

Example 3.1. For $n = 3$ we have

$$p_{\alpha_1,\alpha_2}(r) = (rt^{\frac{1}{a_1}}\omega_{a_1}^{\alpha_1}, (1-r^{a_1})^{\frac{1}{a_2}}t^{\frac{1}{a_2}}\omega_{a_2}^{\alpha_2})\ ;\ 0 \leq r \leq 1.$$

□

Now, for each point $y_{\alpha_n} := (t^{1/a_n}\omega_{a_n}^{\alpha_n}, t)$ in $(\Delta \cap D_t)$, with $0 \leq \alpha_n \leq a_n - 1$, set

$$x_{\alpha_n} := (\ell_t)^{-1}(y_{\alpha_n}) \cap \Gamma = (0, \ldots, 0, t^{1/a_n}\omega_{a_n}^{\alpha_n}).$$

Then consider the map-germ

$$\tilde{\ell}_{\alpha_n} : (F_t, x_{\alpha_n}) \to (\mathbb{C}, y_{\alpha_n})$$

given by the restriction of ℓ to F_t. As in Section 2 above, we can use the induction hypothesis to construct a collapsing cone P_{α_n} of $\tilde{\ell}_{\alpha_n}$, for each $\alpha_n = 0, \ldots, a_n - 1$ fixed, so that:

(i) Each P_{α_n} is the union of parametrized spaces

$$q^{\alpha_n}_{\alpha_1,\dots,\alpha_{n-1}} : \big([0,1]\big)^{n-1} \to F_t \ ; \text{ and}$$

(ii) Any two of them intersect exactly at P'_t.

So

$$P_t = \bigcup_{\substack{0\le\alpha_j\le a_j-1\\1\le j\le n}} tr(q^{\alpha_n}_{\alpha_1,\dots,\alpha_{n-1}})\,.$$

Example 3.2. In the case $n = 3$ we have the map-germ

$$\tilde{\phi}_{\alpha_3} : (F_t, x_{\alpha_3}) \to (\mathbb{C}^2, \tilde{y}_{\alpha_3})$$

given by $\tilde{\phi}_{\alpha_3}(z_1,z_2,z_3) := (z_1,z_3)$, where $\tilde{y}_{\alpha_3} := (0, t^{1/a_3}\omega^{\alpha_3}_{a_3})$. Its critical points are the points in F_t at which

$$\det\begin{pmatrix} \frac{\partial f}{\partial z_1} & \frac{\partial f}{\partial z_2} & \frac{\partial f}{\partial z_3}\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} = 0$$

Hence the relative polar curve of $\tilde{\ell}_{\alpha_3}$ is the curve

$$\tilde{\Gamma}_{\alpha_3} := F_t \cap \{z_2 = 0\}$$

and its polar discriminant is the curve

$$\tilde{\Delta}_{\alpha_3} = \{u^{a_1} + v^{a_3} = t\}\,.$$

Setting $D_\tau := \mathbb{D}_{\tilde{\eta}_1} \times \{\tau\}$ for $\tilde{\eta}_1$ sufficiently small, we have that

$$\tilde{\Delta}_{\alpha_3} \cap D_\tau = \{((t-\tau^{a_3})^{1/a_1}\omega^{\alpha_1}_{a_1}, \tau) \in \mathbb{C}^2;\ 0 \le \alpha_1 \le a_1 - 1\}\,.$$

So for each $\alpha_1 = 0,\dots,a_1-1$ fixed, consider the path $\delta^{\alpha_3}_{\tau,\alpha_1}$ in D_τ given by

$$\delta^{\alpha_3}_{\tau,\alpha_1}(r) := (r(t-\tau^{\alpha_3})^{1/a_1}\omega^{\alpha_1}_{a_1}, \tau) \ ;\ 0 \le r \le 1\,.$$

Then $(\tilde{\phi}_{\alpha_3})^{-1}\big(\delta^{\alpha_3}_{\tau,\alpha_1}(r)\big)$ is the set of points $(z_1,z_2,\tau) \in \mathbb{C}^3$ such that

$$z_1^{a_1} + z_2^{a_2} + \tau^{a_3} = t \ \text{ and } \ z_1 = r(t-\tau^{\alpha_3})^{1/a_1}\omega^{\alpha_1}_{a_1}\,.$$

Since

$$r^{a_1}(t-\tau^{a_3}) + z_2^{a_2} + \tau^{a_3} = t \Leftrightarrow z_2 = (t-\tau^{a_3})^{\frac{1}{a_2}}(1-r^{a_1})^{\frac{1}{a_2}}\omega^{\alpha_2}_{a_2}\,,$$

with $\alpha_2 = 0,\dots,a_2-1$, it follows that $(\tilde{\phi}_{\alpha_3})^{-1}\big(\delta^{\alpha_3}_{\tau,\alpha_1}\big)$ is the union of the a_2-many paths

$$q^{\alpha_3}_{\alpha_1,\alpha_2}(r) := \big(r(t-\tau^{a_3})^{\frac{1}{a_1}}\omega^{\alpha_1}_{a_1}, (1-r^{a_1})^{\frac{1}{a_2}}(t-\tau^{a_3})^{\frac{1}{a_2}}\omega^{\alpha_2}_{a_2}, \tau\big) \ ;\ 0 \le r \le 1\,.$$

Now make τ move along the semi-line that passes through $t^{\frac{1}{a_3}}\omega^{\alpha_3}_{a_3}$, that is, consider:

$$\tau_{\alpha_3}(k) := (1-k)t^{\frac{1}{a_3}}\omega^{\alpha_3}_{a_3} \ ;\ 0 \le k \le 1\,.$$

Then the collapsing cone P_{α_3} of $\tilde{\ell}_{\alpha_3}$, for each $\alpha_3 = 0,\dots,a_3-1$, is given by

$$P_{\alpha_3} := \bigcup_{\substack{0\le\alpha_1\le a_1-1\\0\le\alpha_2\le a_2-1}} q^{\alpha_3}_{\alpha_1,\alpha_2}\big([0,1]\times[0,1]\big)\,,$$

where $q^{\alpha_3}_{\alpha_1,\alpha_2}$ is the parametrized surface in P_t given by

$$q^{\alpha_3}_{\alpha_1,\alpha_2}(r,k) = \big(rt^{\frac{1}{a_1}}(1-(1-k)^{a_3})^{\frac{1}{a_1}}\omega^{\alpha_1}_{a_1}, (1-r^{a_1})^{\frac{1}{a_2}}t^{\frac{1}{a_2}}(1-(1-k)^{a_3})^{\frac{1}{a_2}}\omega^{\alpha_2}_{a_2}, (1-k)t^{\frac{1}{a_3}}\omega^{\alpha_3}_{a_3}\big)\,.$$

Notice that $q^{\alpha_3}_{\alpha_1,\alpha_2}(r,1) = p_{\alpha_1,\alpha_2}(r)$, so any two collapsing cones of the type P_{α_3} as above intersect at P'_t.

So we finally have that

$$P_t = \bigcup_{0 \le \alpha_3 \le a_3 - 1} P_{\alpha_3}$$

and hence

$$P_t = \bigcup_{\substack{0 \le \alpha_1 \le a_1 - 1 \\ 0 \le \alpha_2 \le a_2 - 1 \\ 0 \le \alpha_3 \le a_3 - 1}} q^{\alpha_3}_{\alpha_1, \alpha_2}\big([0,1] \times [0,1]\big)$$

is the Lê's vanishing polyhedron for f.

Since P'_t has the homotopy type of a wedge of $(a_1 - 1)(a_2 - 1)$-many circles, it follows that P_t has the homotopy type of a wedge of $(a_1 - 1)(a_2 - 1)(a_3 - 1)$-many spheres $\mathbb{S}^2$. $\square$

4. The monodromy of the Brieskorn-Pham polynomial

Consider the characteristic homeomorphism $h_t : F_t \to F_t$ given by

$$h_t(z_1, \dots, z_n) := (e^{2\pi i/a_1} z_1, \dots, e^{2\pi i/a_n} z_n)\,.$$

Identifying $a_i \sim 0$ for each $i = 1, \dots, n$ one can check that the characteristic homeomorphism h_t takes each $q^{\alpha_n}_{\alpha_1, \dots, \alpha_{n-1}}$ onto $q^{\alpha_n + 1}_{\alpha_1 + 1, \dots, \alpha_{n-1} + 1}$. This gives a geometric view of the monodromy of f (see the examples below).

Notice that the homology group $H_{n-1}(P_t)$ is generated by $(n-1)$-cycles $\sigma(\alpha_1, \dots, \alpha_n)$, where each one of them is a sum (with signals) of 2^n-many parametrized spaces $q^{\alpha_n}_{\alpha_1, \dots, \alpha_{n-1}}$.

Moreover, one can check that

$$h_t\big(\sigma(\alpha_1, \dots, \alpha_n)\big) = \sigma(\alpha_1 + 1, \dots, \alpha_n + 1)$$

if $0 \le \alpha_i \le a_i - 3$ for any $i = 1, \dots, n$; and that $h_t\big(\sigma(\alpha_1, \dots, \alpha_n)\big)$ equals

$$(-1)^k \sum_{i_1 = 0}^{a_{i_1} - 2 + 1} \cdots \sum_{i_k = 0}^{a_{i_k} - 2} \sigma(\alpha_1 + 1, \dots, i_1, \dots, i_2, \dots, i_k, \dots, \alpha_n + 1)$$

if $\alpha_{i_j} = a_{i_j} - 2$ for $j = 1, \dots, k$ and $\alpha_i < a_i - 2$ for $i \notin \{i_1, \dots, i_k\}$. This gives a homological view of the monodromy of f.

Next we consider the two and the three dimensional cases, so the reader can actually see this geometric description of the monodromy of a Brieskorn-Pham polynomial.

4.1. Two-dimensional case.

Consider $f(x, y) = x^a + y^b$ and let $h_t : F_t \to F_t$ be the characteristic homeomorphism, given by

$$h_t(x, y) := (e^{2\pi i/a} x, e^{2\pi i/b} y)\,.$$

Notice that

$$h_t\big(p_{\alpha,\beta}(r)\big) = h_t\big((r t^{\frac{1}{a}} \omega_a^{\alpha}, (1 - r^a)^{\frac{1}{b}} t^{\frac{1}{b}} \omega_b^{\beta})\big) = (r t^{\frac{1}{a}} \omega_a^{\alpha+1}, (1 - r^a)^{\frac{1}{b}} t^{\frac{1}{b}} \omega_b^{\beta+1})$$

for any $0 \le r \le 1$, $0 \le \alpha \le a - 1$ and $0 \le \beta \le b - 1$. So if we identify $a \sim 0$ and $b \sim 0$ we have that

$$h_t\big(p_{\alpha,\beta}(r)\big) = p_{\alpha+1,\beta+1}(r)\,.$$

In particular, $h_t(P_t) = P_t$.

Now observe that the homology group $H_1(P_t)$ is generated by the cycles

$$\sigma(\alpha, \beta) := p_{\alpha,\beta} - p_{\alpha,\beta+1} - p_{\alpha+1,\beta} + p_{\alpha+1,\beta+1}$$

with $0 \le \alpha \le a-2$ and $0 \le \beta \le b-2$. So we have that

$$h_t\big(\sigma(\alpha,\beta)\big) = p_{\alpha+1,\beta+1} - p_{\alpha+1,\beta+2} - p_{\alpha+2,\beta+1} + p_{\alpha+2,\beta+2}\,.$$

We have some cases:

(i) If $0 \le \alpha \le a-3$ and $0 \le \beta \le b-3$ then one clearly has

$$h_t\big(\sigma(\alpha,\beta)\big) = \sigma(\alpha+1,\beta+1)\,.$$

(ii) If $0 \le \alpha \le a-3$ and $\beta = b-2$ then

$$\begin{array}{rcl} h_t\big(\sigma(\alpha,b-2)\big) & = & -p_{\alpha+1,0} + p_{\alpha+1,b-1} + p_{\alpha+2,0} - p_{\alpha+2,b-1} \\ & = & -\sigma(\alpha+1,0) - \sigma(\alpha+1,1) - \cdots - \sigma(\alpha+1,b-1)\,. \end{array}$$

(iii) Analogously, if $\alpha = a-2$ and $0 \le \beta \le b-3$ we have that

$$h_t\big(\sigma(a-2,\beta)\big) = -\sigma(0,\beta+1) - \sigma(1,\beta+1) - \cdots - \sigma(a-1,\beta+1)\,.$$

(iv) If $\alpha = a-2$ and $\beta = b-2$ then

$$\begin{array}{rcl} h_t\big(\sigma(a-2,b-2)\big) & = & p_{0,0} - p_{0,b-1} - p_{a-1,0} + p_{a-1,b-1} \\ & = & \sum_{i=0}^{a-2}\sum_{j=0}^{b-2}\sigma(i,j)\,. \end{array}$$

So we have showed that

$$h_t\big(\sigma(\alpha,\beta)\big) = \begin{cases} \sigma(\alpha+1,\beta+1) & \text{if } 0 \le \alpha \le a-3 \text{ and } 0 \le \beta \le b-3 \\ -\sum_{j=0}^{b-2}\sigma(\alpha+1,j) & \text{if } 0 \le \alpha \le a-3 \text{ and } \beta = b-2 \\ -\sum_{i=0}^{a-2}\sigma(i,\beta+1) & \text{if } \alpha = a-2 \text{ and } 0 \le \beta \le b-3 \\ \sum_{i=0}^{a-2}\sum_{j=0}^{b-2}\sigma(i,j) & \text{if } \alpha = a-2 \text{ and } \beta = b-2 \end{cases}$$

Notice that since $H_1(P_t)$ has a finite basis, then h_t^* has finite order. So, by a theorem from Linear Algebra, we know that the minimal polynomial of h_t^* is a product of distinct cyclotomic polynomials. In particular, the roots of the characteristic polynomial of h_t are products of roots of the unity $\omega_a^k\omega_b^l$.

Example 4.1. Consider $f(x,y) = x^3 + y^3$. Then $a = b = 3$ and we have the following basis for $H_1(P_t)$:

$$B = \{\sigma(0,0), \sigma(0,1), \sigma(1,0), \sigma(1,1)\}\,.$$

So the matrix of the homomorphism $h_t^* : H_1(P_t) \to H_1(P_t)$ in the basis B is given by:

$$[h_t^*]_B^B = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

A simple calculation shows that the characteristic polynomial is

$$p(\lambda) = (\lambda - 1)(\lambda^3 + 1)\,.$$

Example 4.2. Consider $f(x,y) = x^3 + y^4$ and consider the following basis for $H_1(P_t)$:

$$B = \{\sigma(0,0), \sigma(0,1), \sigma(0,2), \sigma(1,0), \sigma(1,1), \sigma(1,2)\}\,.$$

So the matrix of the homomorphism $h_t^*: H_1(P_t) \to H_1(P_t)$ in the basis B is given by:

$$[h_t^*]_B^B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 & -1 & 1 \end{pmatrix}.$$

4.2. **Three-dimensional case.** Consider

$$f(z_1, z_2, z_3) = z_1^{a_1} + z_2^{a_2} + z_3^{a_3}.$$

The characteristic homeomorphism $h_t : F_t \to F_t$ is given by

$$h_t(z_1, z_2, z_3) := (e^{2\pi i/a_1} z_1, e^{2\pi i/a_2} z_2, e^{2\pi i/a_3} z_3).$$

So if we identify $a_i \sim 0$, $i = 1, 2, 3$, we have that

$$h_t\big(q^{\alpha_3}_{\alpha_1,\alpha_2}(r,k)\big) = q^{\alpha_3+1}_{\alpha_1+1,\alpha_2+1}(r,k),$$

for any $(r, k) \in [0, 1] \times [0, 1]$. In particular, $h_t(P_t) = P_t$.

Now observe that the homology group $H_2(P_t)$ is generated by the 2-cycles given by

$$\begin{aligned} \sigma(\alpha_1,\alpha_2,\alpha_3) := {} & q^{\alpha_3}_{\alpha_1,\alpha_2} - q^{\alpha_3}_{\alpha_1+1,\alpha_2} - q^{\alpha_3}_{\alpha_1,\alpha_2+1} - q^{\alpha_3+1}_{\alpha_1,\alpha_2} \\ & + q^{\alpha_3}_{\alpha_1+1,\alpha_2+1} + q^{\alpha_3+1}_{\alpha_1+1,\alpha_2} + q^{\alpha_3+1}_{\alpha_1,\alpha_2+1} - q^{\alpha_3+1}_{\alpha_1+1,\alpha_2+1} \end{aligned}$$

with $0 \le \alpha_i \le a_i - 2$ for $i = 1, 2, 3$.

Then some calculations as before give that $h_t\big(\sigma(\alpha_1,\alpha_2,\alpha_3)\big)$ equals to:

$$\begin{cases} \sigma(\alpha_1+1,\alpha_2+1,\alpha_3+1) & \text{if } 0 \le \alpha_i \le a_i - 3, \text{for } i = 1,2,3 \\ -\sum_{i=0}^{a_1-2} \sigma(i,\alpha_2+1,\alpha_3+1) & \text{if } \alpha_1 = a_1 - 2,\ 0 \le \alpha_2 \le a_2 - 3 \text{ and } 0 \le \alpha_3 \le a_3 - 3 \\ -\sum_{j=0}^{a_2-2} \sigma(\alpha_1+1,j,\alpha_3+1) & \text{if } 0 \le \alpha_1 \le a_1 - 3,\ \alpha_2 = a_2 - 2 \text{ and } 0 \le \alpha_3 \le a_3 - 3 \\ -\sum_{k=0}^{a_3-2} \sigma(\alpha_1+1,\alpha_2+1,k) & \text{if } 0 \le \alpha_1 \le a_1 - 3,\ 0 \le \alpha_2 \le a_2 - 3 \text{ and } \alpha_3 = a_3 - 2 \\ \sum_{i=0}^{a_1-2}\sum_{j=0}^{a_2-2} \sigma(i,j,\alpha_3+1) & \text{if } \alpha_1 = a_1 - 2,\ \alpha_2 = a_2 - 2 \text{ and } 0 \le \alpha_3 \le a_3 - 3 \\ \sum_{i=0}^{a_1-2}\sum_{k=0}^{a_3-2} \sigma(i,\alpha_2+1,k) & \text{if } \alpha_1 = a_1 - 2,\ 0 \le \alpha_2 \le a_2 - 3 \text{ and } \alpha_3 = a_3 - 2 \\ \sum_{j=0}^{a_2-2}\sum_{k=0}^{a_3-2} \sigma(\alpha_1+1,j,k) & \text{if } 0 \le \alpha_1 \le a_1 - 3,\ \alpha_2 = a_2 - 2 \text{ and } \alpha_3 = a_3 - 2 \\ -\sum_{i=0}^{a_1-2}\sum_{j=0}^{a_2-2}\sum_{k=0}^{a_3-2} \sigma(i,j,k) & \text{if } \alpha_1 = a_1 - 2,\ \alpha_2 = a_2 - 2 \text{ and } \alpha_3 = a_3 - 2 \end{cases}$$

Example 4.3. Consider $f(x, y) = z_1^2 + z_2^3 + z_3^3$ and consider the following basis for $H_2(P_t)$:

$$B = \{\sigma(0,0,0), \sigma(0,0,1), \sigma(0,1,0), \sigma(0,1,1)\}.$$

So the matrix of the homomorphism $h_t^*: H_2(P_t) \to H_2(P_t)$ in the basis B is given by:

$$[h_t^*]_B^B = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}.$$

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