

## COMPARISON OF STRATIFIED-ALGEBRAIC AND TOPOLOGICAL K-THEORY

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ABSTRACT. Stratified-algebraic vector bundles on real algebraic varieties have many desirable features of algebraic vector bundles but are more flexible. We give a characterization of the compact real algebraic varieties  $X$  having the following property: There exists a positive integer  $r$  such that for any constant rank topological vector bundle  $\xi$  on  $X$ , the direct sum of  $r$  copies of  $\xi$  is isomorphic to a stratified-algebraic vector bundle. In particular, each compact real algebraic variety of dimension at most 8 has this property. Our results are expressed in terms of K-theory.

### 1. INTRODUCTION AND MAIN RESULTS

In the recent paper [30], we introduced and investigated stratified-algebraic vector bundles on real algebraic varieties. They occupy an intermediate position between algebraic and topological vector bundles. Here we continue the line of research undertaken in [30, 28] and look for new relationships between stratified-algebraic and topological vector bundles. In a broader context, the present paper is also closely related to [5, 16, 23, 24, 26, 27, 29]. All results announced in this section are proved in Section 2.

Throughout this paper the term *real algebraic variety* designates a locally ringed space isomorphic to an algebraic subset of  $\mathbb{R}^N$ , for some  $N$ , endowed with the Zariski topology and the sheaf of real-valued regular functions (such an object is called an affine real algebraic variety in [7]). The class of real algebraic varieties is identical with the class of quasi-projective real varieties, cf. [7, Proposition 3.2.10, Theorem 3.4.4]. Morphisms of real algebraic varieties are called *regular maps*. Each real algebraic variety carries also the Euclidean topology, which is induced by the usual metric on  $\mathbb{R}$ . Unless explicitly stated otherwise, all topological notions relating to real algebraic varieties refer to the Euclidean topology.

Let  $\mathbb{F}$  stand for  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$  (the quaternions). All  $\mathbb{F}$ -vector spaces will be left  $\mathbb{F}$ -vector spaces. When convenient,  $\mathbb{F}$  will be identified with  $\mathbb{R}^{d(\mathbb{F})}$ , where  $d(\mathbb{F}) = \dim_{\mathbb{R}} \mathbb{F}$ .

Let  $X$  be a real algebraic variety. For any nonnegative integer  $n$ , let  $\varepsilon_X^n(\mathbb{F})$  denote the standard trivial  $\mathbb{F}$ -vector bundle on  $X$  with total space  $X \times \mathbb{F}^n$ , where  $X \times \mathbb{F}^n$  is regarded as a real algebraic variety. An algebraic  $\mathbb{F}$ -vector bundle on  $X$  is an algebraic  $\mathbb{F}$ -vector subbundle of  $\varepsilon_X^n(\mathbb{F})$  for some  $n$  (cf. [7, Chapters 12 and 13] for various characterizations of algebraic  $\mathbb{F}$ -vector bundles).

We now recall the fundamental notion introduced in [30]. By a *stratification* of  $X$  we mean a finite collection  $\mathcal{S}$  of pairwise disjoint Zariski locally closed subvarieties whose union is  $X$ . Each subvariety in  $\mathcal{S}$  is called a stratum of  $\mathcal{S}$ . A *stratified-algebraic  $\mathbb{F}$ -vector bundle on  $X$*  is a

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topological  $\mathbb{F}$ -vector subbundle  $\xi$  of  $\varepsilon_X^n(\mathbb{F})$ , for some  $n$ , such that for some stratification  $\mathcal{S}$  of  $X$ , the restriction  $\xi|_S$  of  $\xi$  to each stratum  $S$  of  $\mathcal{S}$  is an algebraic  $\mathbb{F}$ -vector subbundle of  $\varepsilon_S^n(\mathbb{F})$ .

A topological  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$  is said to *admit an algebraic structure* if it is isomorphic to an algebraic  $\mathbb{F}$ -vector bundle on  $X$ . Similarly,  $\xi$  is said to *admit a stratified-algebraic structure* if it is isomorphic to a stratified-algebraic  $\mathbb{F}$ -vector bundle on  $X$ . These two types of  $\mathbb{F}$ -vector bundles have been extensively investigated in [3, 4, 6, 7, 9, 10, 11, 13] and [30, 28], respectively. In general, their behaviors are quite different, cf. [30, Example 1.11]. Here we further develop the direction of research initiated in [30, 28]. It is convenient to bring into play Grothendieck groups.

Denote by  $K_{\mathbb{F}}(X)$  the Grothendieck group of topological  $\mathbb{F}$ -vector bundles on  $X$ . For any topological  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$ , let  $[\xi]$  denote its class in  $K_{\mathbb{F}}(X)$ . Since  $X$  has the homotopy type of a compact polyhedron [7, pp. 217, 225], it follows that the abelian group  $K_{\mathbb{F}}(X)$  is finitely generated (cf. [21, Exercise III.7.5] or the spectral sequence in [2, 15]). Let  $K_{\mathbb{F}\text{-str}}(X)$  be the subgroup of  $K_{\mathbb{F}}(X)$  generated by the classes of all  $\mathbb{F}$ -vector bundles admitting a stratified-algebraic structure.

If the variety  $X$  is compact, then the group  $K_{\mathbb{F}\text{-str}}(X)$  contains complete information on  $\mathbb{F}$ -vector bundles on  $X$  admitting a stratified-algebraic structure. More precisely, we have the following.

**Theorem 1.1** ([30, Corollary 3.14]). *Let  $X$  be a compact real algebraic variety. A topological  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$  admits a stratified-algebraic structure if and only if the class  $[\xi]$  is in  $K_{\mathbb{F}\text{-str}}(X)$ .*

In other words, with notation as in Theorem 1.1,  $\xi$  admits a stratified-algebraic structure if and only if there exists a stratified-algebraic  $\mathbb{F}$ -vector bundle  $\eta$  on  $X$  such that the direct sum  $\xi \oplus \eta$  admits a stratified-algebraic structure.

For our purposes it is convenient to distinguish some vector bundles by imposing a suitable condition on their rank. For any topological  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$ , we regard  $\text{rank } \xi$  (the rank of  $\xi$ ) as a function  $\text{rank } \xi: X \rightarrow \mathbb{Z}$ , which assigns to every point  $x$  in  $X$  the dimension of the fiber of  $\xi$  over  $x$ . Clearly,  $\text{rank } \xi$  is a constant function on each connected component of  $X$ .

We say that  $\xi$  has *property (rk)* if for every integer  $d$ , the set  $\{x \in X \mid (\text{rank } \xi)(x) = d\}$  is algebraically constructible. Recall that a subset of  $X$  is said to be algebraically constructible if it belongs to the Boolean algebra generated by the Zariski closed subsets of  $X$ . It readily follows that each stratified-algebraic  $\mathbb{F}$ -vector bundle on  $X$  has property (rk). Thus property (rk) is a necessary condition for  $\xi$  to admit a stratified-algebraic structure.

We now give a simple example to illustrate the role of property (rk). The real algebraic curve

$$C = \{(x, y) \in \mathbb{R}^2 : x^2(x^2 - 1)(x^2 - 4) + y^2 = 0\}$$

is irreducible with singular locus  $\{(0,0)\}$ . It has three connected components, the singleton  $\{(0,0)\}$  and two ovals. Clearly, every algebraic  $\mathbb{F}$ -vector bundle on  $C$  has constant rank, while the rank function of a topological  $\mathbb{F}$ -vector bundle on  $C$  may take three distinct values. On the other hand, the rank function of a stratified-algebraic vector bundle on  $C$  need not be constant, but must be constant on  $C \setminus \{(0,0)\}$ .

Returning to the general case, denote by  $K_{\mathbb{F}}^{(\text{rk})}(X)$  the subgroup of  $K_{\mathbb{F}}(X)$  generated by the classes of all topological  $\mathbb{F}$ -vector bundles having property (rk). By construction,

$$K_{\mathbb{F}\text{-str}}(X) \subseteq K_{\mathbb{F}}^{(\text{rk})}(X).$$

Since the group  $K_{\mathbb{F}}(X)$  is finitely generated, so is the quotient group

$$\Gamma_{\mathbb{F}}(X) := K_{\mathbb{F}}^{(\text{rk})}(X)/K_{\mathbb{F}\text{-str}}(X).$$

Thus the group  $\Gamma_{\mathbb{F}}(X)$  is finite if and only if  $r\Gamma_{\mathbb{F}}(X) = 0$  for some positive integer  $r$ . In the present paper the group  $\Gamma_{\mathbb{F}}(X)$  is the main object of investigation.

For any  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$  and any positive integer  $r$ , we denote by

$$\xi(r) = \xi \oplus \cdots \oplus \xi$$

the  $r$ -fold direct sum. The following preliminary result shows that our approach here is consistent with that of [28].

**Proposition 1.2.** *Let  $X$  be a compact real algebraic variety. For a positive integer  $r$ , the following conditions are equivalent:*

- (a) *The group  $\Gamma_{\mathbb{F}}(X)$  is finite and  $r\Gamma_{\mathbb{F}}(X) = 0$ .*
- (b) *For each topological  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$  having property (rk), the  $\mathbb{F}$ -vector bundle  $\xi(r)$  admits a stratified-algebraic structure.*
- (c) *For each topological  $\mathbb{F}$ -vector bundle  $\eta$  on  $X$  having constant rank, the  $\mathbb{F}$ -vector bundle  $\eta(r)$  admits a stratified-algebraic structure.*

In [28, Conjecture C], it is suggested that the group  $\Gamma_{\mathbb{F}}(X)$  is always finite (for  $X$  compact). We show here that the finiteness of the group  $\Gamma_{\mathbb{F}}(X)$  is equivalent to a certain condition involving cohomology classes of a special kind. For any nonnegative integer  $k$ , we defined in [30] a subgroup  $H_{\mathbb{C}\text{-str}}^{2k}(X; \mathbb{Z})$  of the cohomology group  $H^{2k}(X; \mathbb{Z})$ . For the convenience of the reader, the definition and basic properties of  $H_{\mathbb{C}\text{-str}}^{2k}(X; \mathbb{Z})$  are recalled in Section 2.

**Theorem 1.3.** *For any compact real algebraic variety  $X$ , the following conditions are equivalent:*

- (a) *The group  $\Gamma_{\mathbb{F}}(X)$  is finite.*
- (b) *The quotient group  $H^{4k}(X; \mathbb{Z})/H_{\mathbb{C}\text{-str}}^{4k}(X; \mathbb{Z})$  is finite for every positive integer  $k$  satisfying  $8k - 2 < \dim X$ .*

Since the groups  $H_{\mathbb{C}\text{-str}}^{2k}(-; \mathbb{Z})$  are hard to compute, it is worthwhile to give a simple topological criterion for the finiteness of the group  $\Gamma_{\mathbb{F}}(X)$ . To this end some preparation is required.

For any positive integer  $d$ , let  $\mathbb{S}^d$  denote the unit  $d$ -sphere

$$\mathbb{S}^d = \{(u_0, \dots, u_d) \in \mathbb{R}^{d+1} \mid u_0^2 + \cdots + u_d^2 = 1\}.$$

Let  $s_d$  be a generator of the cohomology group  $H^d(\mathbb{S}^d; \mathbb{Z}) \cong \mathbb{Z}$ . A cohomology class  $u$  in  $H^d(\Omega; \mathbb{Z})$ , where  $\Omega$  is an arbitrary topological space, is said to be *spherical* if  $u = h^*(s_d)$  for some continuous map  $h: \Omega \rightarrow \mathbb{S}^d$ . Denote by  $H_{\text{sph}}^d(\Omega; \mathbb{Z})$  the subgroup of  $H^d(\Omega; \mathbb{Z})$  generated by all spherical cohomology classes. In general a cohomology class in  $H_{\text{sph}}^d(\Omega; \mathbb{Z})$  need not be spherical.

**Theorem 1.4.** *Let  $X$  be a compact real algebraic variety. If the quotient group*

$$H^{4k}(X; \mathbb{Z})/H_{\text{sph}}^{4k}(X; \mathbb{Z})$$

*is finite for every positive integer  $k$  satisfying  $8k - 2 < \dim X$ , then the group  $\Gamma_{\mathbb{F}}(X)$  is finite.*

As a consequence we obtain the following.

**Corollary 1.5.** *Let  $X$  be a compact real algebraic variety. If each connected component of  $X$  is homotopically equivalent to  $\mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_n}$  for some positive integers  $d_1, \dots, d_n$ , then the group  $\Gamma_{\mathbb{F}}(X)$  is finite.*

*Proof.* Since  $H_{\text{sph}}^l(X; \mathbb{Z}) = H^l(X; \mathbb{Z})$  for every positive integer  $l$ , it suffices to make use of Theorem 1.4. □

It is interesting to compare Corollary 1.5 with related, previously known, results. If

$$X = X_1 \times \cdots \times X_n,$$

where each  $X_i$  is a compact real algebraic variety homotopically equivalent to  $\mathbb{S}^{d_i}$  for  $1 \leq i \leq n$ , then  $\Gamma_{\mathbb{F}}(X) = 0$  for  $\mathbb{F} = \mathbb{C}$  and  $\mathbb{F} = \mathbb{H}$ , and  $2\Gamma_{\mathbb{R}}(X) = 0$ , cf. [30, Theorem 1.10]. On the other hand, there exists a nonsingular real algebraic variety  $X$  diffeomorphic to the  $n$ -fold product  $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ ,  $n > d(\mathbb{F})$ , such that  $\Gamma_{\mathbb{F}}(X) \neq 0$ , cf. [30, Example 7.10].

For any compact real algebraic variety  $X$ , the equality  $H^l(X; \mathbb{Z}) = 0$  holds if  $l > \dim X$ , cf. [7, p. 217]. Hence, in view of either Theorem 1.3 or Theorem 1.4, the group  $\Gamma_{\mathbb{F}}(X)$  is finite for  $\dim X \leq 6$ . This is extended below to  $\dim X \leq 8$ . Actually, we obtain a result containing additional information.

Denote by  $e(\mathbb{F})$  the integer satisfying  $d(\mathbb{F}) = 2^{e(\mathbb{F})}$ , that is,

$$e(\mathbb{F}) = \begin{cases} 0 & \text{if } \mathbb{F} = \mathbb{R} \\ 1 & \text{if } \mathbb{F} = \mathbb{C} \\ 2 & \text{if } \mathbb{F} = \mathbb{H}. \end{cases}$$

Given a nonnegative integer  $n$ , set

$$\begin{aligned} a(n) &= \min\{l \in \mathbb{Z} \mid l \geq 0, 2^l \geq n\}, \\ a(n, \mathbb{F}) &= \max\{0, a(n) - e(\mathbb{F})\}. \end{aligned}$$

It is conjectured in [28] that  $2^{a(\dim X, \mathbb{F})} \Gamma_{\mathbb{F}}(X) = 0$  for every compact real algebraic variety  $X$ . This conjecture is confirmed in [28] for varieties of dimension not exceeding 5. Using different methods, we get the following.

**Theorem 1.6.** *For any compact real algebraic variety  $X$  of dimension at most 8, the group  $\Gamma_{\mathbb{F}}(X)$  is finite and*

$$2^{a(\dim X, \mathbb{F}) + a(X)} \Gamma_{\mathbb{F}}(X) = 0,$$

where  $a(X) = 0$  if  $\dim X \leq 7$  and  $a(X) = 2$  if  $\dim X = 8$ .

We are not able to decide whether Theorem 1.6 holds with  $a(X) = 0$  for  $\dim X = 8$ .

In Section 2 we establish relationships between the groups  $H_{\text{sph}}^{2k}(-; \mathbb{Z})$  and  $H_{\text{C-str}}^{2k}(-; \mathbb{Z})$  for  $k \geq 1$ . This leads to the proofs of Theorems 1.3 and 1.4. Along the way we obtain closely related results, Theorems 2.14, 2.15 and 2.16, which are of independent interest. Noteworthy is also Theorem 2.13, which plays a key role in the proof of Theorem 1.6. In Section 3 we investigate topological  $\mathbb{C}$ -line bundles admitting a stratified-algebraic structure.

**Notation.** Given two  $\mathbb{F}$ -vector bundles  $\xi$  and  $\eta$  on the same topological space, we will write  $\xi \cong \eta$  to indicate that they are isomorphic.

## 2. STRATIFIED-ALGEBRAIC VERSUS TOPOLOGICAL VECTOR BUNDLES

To begin with we establish a connection between vector bundles having property (rk) and those of constant rank.

**Lemma 2.1.** *Let  $X$  be a real algebraic variety and let  $\xi$  be a topological  $\mathbb{F}$ -vector bundle on  $X$ . If  $\xi$  has property (rk), then there exists a stratified-algebraic  $\mathbb{F}$ -vector bundle  $\eta$  on  $X$  such that the direct sum  $\xi \oplus \eta$  is of constant rank.*

*Proof.* Since  $X$  has the homotopy type of a compact polyhedron [7, pp. 217, 225], we may assume that  $\xi$  is a topological  $\mathbb{F}$ -vector subbundle of  $\varepsilon_X^n(\mathbb{F})$  for some positive integer  $n$ . Assume that  $\xi$  has property (rk). By definition, for each integer  $d$  satisfying  $0 \leq d \leq n$ , the set

$$R(d) = \{x \in X \mid (\text{rank } \xi)(x) = d\}$$

is algebraically constructible. Thus  $R(d)$  is the union of a finite collection of pairwise disjoint Zariski locally closed subvarieties of  $X$ . In particular, there exists a stratification  $\mathcal{S}$  of  $X$  such that each set  $R(d)$  is the union of some strata of  $\mathcal{S}$ . Furthermore, each nonempty set  $R(d)$  is the union of some connected components of  $X$ . It follows that we can find a topological  $\mathbb{F}$ -vector subbundle  $\eta$  of  $\varepsilon_X^n(\mathbb{F})$  whose restriction  $\eta|_{R(d)}$  is the trivial  $\mathbb{F}$ -vector subbundle of  $\varepsilon_{R(d)}^n(\mathbb{F})$  with total space  $R(d) \times (\mathbb{F}^{n-d} \times \{0\})$ , where  $\mathbb{F}^{n-d} \times \{0\} \subseteq \mathbb{F}^n$ . By construction,  $\eta$  is a stratified-algebraic  $\mathbb{F}$ -vector bundle and the direct sum  $\xi \oplus \eta$  is of rank  $n$ .  $\square$

In particular, if  $K_{\mathbb{F}}^{(\text{crk})}(X)$  is the subgroup of  $K_{\mathbb{F}}(X)$  generated by the classes of all topological  $\mathbb{F}$ -vector bundles of constant rank, then

$$K_{\mathbb{F}\text{-str}}(X) + K_{\mathbb{F}}^{(\text{crk})}(X) = K_{\mathbb{F}}^{(\text{rk})}(X).$$

Hence the group  $\Gamma_{\mathbb{F}}(X)$  is isomorphic to the quotient group

$$K_{\mathbb{F}}^{(\text{crk})}(X)/K_{\mathbb{F}\text{-str}}(X) \cap K_{\mathbb{F}}^{(\text{crk})}(X).$$

*Proof of Proposition 1.2.* Obviously, (b) implies (a). According to Theorem 1.1, (a) implies (b). Hence, in view of Lemma 2.1, (a) and (c) are equivalent.  $\square$

Let  $X$  be a real algebraic variety. Let  $\mathbb{K}$  be a subfield of  $\mathbb{F}$ , where  $\mathbb{K}$  (as  $\mathbb{F}$ ) stands for  $\mathbb{R}$ ,  $\mathbb{C}$  or  $\mathbb{H}$ . Any  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$  can be regarded as a  $\mathbb{K}$ -vector bundle, which is indicated by  $\xi_{\mathbb{K}}$ . In particular,  $\xi_{\mathbb{K}} = \xi$  if  $\mathbb{K} = \mathbb{F}$ . Furthermore,  $\xi_{\mathbb{R}} = (\xi_{\mathbb{K}})_{\mathbb{R}}$ . If the  $\mathbb{F}$ -vector bundle  $\xi$  admits a stratified-algebraic structure, then so does the  $\mathbb{K}$ -vector bundle  $\xi_{\mathbb{K}}$ .

The following result will be frequently referred to.

**Theorem 2.2.** *Let  $X$  be a compact real algebraic variety. A topological  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$  admits a stratified-algebraic structure if and only if the  $\mathbb{K}$ -vector bundle  $\xi_{\mathbb{K}}$  admits a stratified-algebraic structure.*

*Proof.* The proof for  $\mathbb{K} = \mathbb{R}$ , rather involved, is given in [30, Theorem 1.7]. The general case follows since  $\xi_{\mathbb{R}} = (\xi_{\mathbb{K}})_{\mathbb{R}}$ .  $\square$

We will also make use of the extension of scalars construction. Let  $X$  be a real algebraic variety. Any  $\mathbb{K}$ -vector bundle  $\xi$  on  $X$  gives rise to the  $\mathbb{F}$ -vector bundle  $\mathbb{F} \otimes \xi$  on  $X$ . Here  $\mathbb{F} \otimes \xi = \xi$  if  $\mathbb{K} = \mathbb{F}$ ,  $\mathbb{C} \otimes \xi$  is the complexification of  $\xi$  if  $\mathbb{K} = \mathbb{R}$ , and  $\mathbb{H} \otimes \xi$  is the quaternionization of  $\xi$  if  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ . If the  $\mathbb{K}$ -vector bundle  $\xi$  admits a stratified-algebraic structure, then so does the  $\mathbb{F}$ -vector bundle  $\mathbb{F} \otimes \xi$ .

For any  $\mathbb{C}$ -vector bundle  $\xi$ , let  $\bar{\xi}$  denote the conjugate bundle, cf. [31]. Note that  $\bar{\xi}_{\mathbb{R}} \cong \xi_{\mathbb{R}}$ . Furthermore, for the  $\mathbb{H}$ -vector bundle  $\mathbb{H} \otimes \xi$ , we have

$$(\mathbb{H} \otimes \xi)_{\mathbb{C}} \cong \xi \oplus \bar{\xi}.$$

**Lemma 2.3.** *Let  $X$  be a compact real algebraic variety and let  $\xi$  be a topological  $\mathbb{C}$ -vector bundle on  $X$ . For any positive integer  $q$ , the  $\mathbb{H}$ -vector bundle  $(\mathbb{H} \otimes \xi)(q)$  admits a stratified-algebraic structure if and only if so does the  $\mathbb{C}$ -vector bundle  $\xi(2q)$ .*

*Proof.* Since

$$((\mathbb{H} \otimes \xi)(q))_{\mathbb{C}} \cong (\mathbb{H} \otimes \xi)_{\mathbb{C}}(q) \cong (\xi \oplus \bar{\xi})(q)$$

and

$$((\xi \oplus \bar{\xi})(q))_{\mathbb{R}} \cong (\xi_{\mathbb{R}} \oplus \bar{\xi}_{\mathbb{R}})(q) \cong (\xi_{\mathbb{R}} \oplus \xi_{\mathbb{R}})(q) \cong (\xi(2q))_{\mathbb{R}},$$

we get

$$((\mathbb{H} \otimes \xi)(q))_{\mathbb{R}} \cong \xi(2q)_{\mathbb{R}}.$$

The proof is complete in view of Theorem 2.2. □

For any  $\mathbb{R}$ -vector bundle  $\xi$ , we have  $(\mathbb{C} \otimes \xi)_{\mathbb{R}} \cong \xi \oplus \xi$ .

**Lemma 2.4.** *Let  $X$  be a compact real algebraic variety and let  $\xi$  be a topological  $\mathbb{R}$ -vector bundle on  $X$ . For any positive integer  $q$ , the  $\mathbb{C}$ -vector bundle  $(\mathbb{C} \otimes \xi)(q)$  admits a stratified-algebraic structure if and only if so does the  $\mathbb{R}$ -vector bundle  $\xi(2q)$ .*

*Proof.* Since

$$((\mathbb{C} \otimes \xi)(q))_{\mathbb{R}} \cong (\mathbb{C} \otimes \xi)_{\mathbb{R}}(q) \cong (\xi \oplus \xi)(q) \cong \xi(2q),$$

the proof is complete in view of Theorem 2.2. □

For the convenience of the reader we recall the definition and basic properties of stratified- $\mathbb{C}$ -algebraic cohomology classes, introduced and investigated in [30].

Let  $V$  be a compact nonsingular real algebraic variety. A *nonsingular projective complexification* of  $V$  is a pair  $(\mathbb{V}, \iota)$ , where  $\mathbb{V}$  is a nonsingular projective scheme over  $\mathbb{R}$  and  $\iota: V \rightarrow \mathbb{V}(\mathbb{C})$  is an injective map such that  $\mathbb{V}(\mathbb{R})$  is Zariski dense in  $\mathbb{V}$ ,  $\iota(V) = \mathbb{V}(\mathbb{R})$  and  $\iota$  induces a biregular isomorphism between  $V$  and  $\mathbb{V}(\mathbb{R})$ . Here the set  $\mathbb{V}(\mathbb{R})$  of real points of  $\mathbb{V}$  is regarded as a subset of the set  $\mathbb{V}(\mathbb{C})$  of complex points of  $\mathbb{V}$ . The existence of  $(\mathbb{V}, \iota)$  follows from Hironaka's theorem on resolution of singularities [19] (cf. also [22] for a very readable exposition). We identify  $\mathbb{V}(\mathbb{C})$  with the set of complex points of the scheme  $\mathbb{V}_{\mathbb{C}} := \mathbb{V} \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C}$  over  $\mathbb{C}$ . For any nonnegative integer  $k$ , denote by  $H_{\text{alg}}^{2k}(\mathbb{V}(\mathbb{C}); \mathbb{Z})$  the subgroup of  $H^{2k}(\mathbb{V}(\mathbb{C}); \mathbb{Z})$  that consists of the cohomology classes corresponding to algebraic cycles (defined over  $\mathbb{C}$ ) on  $\mathbb{V}_{\mathbb{C}}$  of codimension  $k$ , cf. [14] or [17, Chapter 19]. The subgroup

$$H_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z}) := \iota^*(H_{\text{alg}}^{2k}(\mathbb{V}(\mathbb{C}); \mathbb{Z}))$$

of  $H^{2k}(V; \mathbb{Z})$  does not depend on the choice of  $(\mathbb{V}, \iota)$ , cf. [6]. Cohomology classes in  $H_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z})$  are called  *$\mathbb{C}$ -algebraic*. The groups  $H_{\mathbb{C}\text{-alg}}^{2k}(-; \mathbb{Z})$  are subtle invariants with numerous applications, cf. [6, 8, 11, 13, 25].

Let  $X$  and  $Y$  be real algebraic varieties. A map  $f: X \rightarrow Y$  is said to be *stratified-regular* if it is continuous and for some stratification  $\mathcal{S}$  of  $X$ , the restriction  $f|_S: S \rightarrow Y$  of  $f$  to each stratum  $S$  of  $\mathcal{S}$  is a regular map. A cohomology class  $u$  in  $H^{2k}(X; \mathbb{Z})$  is said to be *stratified- $\mathbb{C}$ -algebraic* if there exists a stratified-regular map  $\varphi: X \rightarrow V$ , into a compact nonsingular real algebraic variety  $V$ , such that  $u = \varphi^*(v)$  for some cohomology class  $v$  in  $H_{\mathbb{C}\text{-alg}}^{2k}(V; \mathbb{Z})$ . The set  $H_{\mathbb{C}\text{-str}}^{2k}(X; \mathbb{Z})$  of all stratified- $\mathbb{C}$ -algebraic cohomology classes in  $H^{2k}(X; \mathbb{Z})$  forms a subgroup. The direct sum

$$H_{\mathbb{C}\text{-str}}^{\text{even}}(X; \mathbb{Z}) := \bigoplus_{k \geq 0} H_{\mathbb{C}\text{-str}}^{2k}(X; \mathbb{Z})$$

is a subring of the ring

$$H^{\text{even}}(X; \mathbb{Z}) := \bigoplus_{k \geq 0} H^{2k}(X; \mathbb{Z}).$$

If  $\xi$  is a stratified-algebraic  $\mathbb{C}$ -vector bundle on  $X$ , then the  $k$ th Chern class  $c_k(\xi)$  of  $\xi$  is in  $H_{\mathbb{C}\text{-str}}^{2k}(X; \mathbb{Z})$  for every nonnegative integer  $k$ . The reader can find proofs of these facts in [30].

For any topological  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$ , one can interpret  $\text{rank } \xi$  as an element of  $H^0(X; \mathbb{Z})$ . Then the following holds.

**Lemma 2.5.** *Let  $X$  be a real algebraic variety and let  $\xi$  be a topological  $\mathbb{F}$ -vector bundle on  $X$ . If  $\xi$  has property (rk), then  $\text{rank } \xi$  is in  $H_{\mathbb{C}\text{-str}}^0(X; \mathbb{Z})$ .*

*Proof.* Assume that the  $\mathbb{F}$ -vector bundle  $\xi$  has property (rk). We make use of the notation introduced in the proof of Lemma 2.1. Furthermore, we regard  $V = \{0, \dots, n\}$  as a real algebraic variety and  $\text{rank } \xi$  as a map  $\text{rank } \xi: X \rightarrow V$ . Then  $\text{rank } \xi$  is a stratified-regular map. Note that  $\text{rank } \xi$  interpreted as a cohomology class in  $H^0(X; \mathbb{Z})$  coincides with  $(\text{rank } \xi)^*(v)$ , where  $v$  is the cohomology class in  $H^0(V; \mathbb{Z})$  whose restriction to the singleton  $\{i\}$  is equal to 1 in  $H^0(\{i\}; \mathbb{Z})$  for every  $i$  in  $V$ . Since  $H_{\mathbb{C}\text{-alg}}^0(V; \mathbb{Z}) = H^0(V; \mathbb{Z})$ , the cohomology class  $(\text{rank } \xi)^*(v)$  is in  $H_{\mathbb{C}\text{-str}}^0(X; \mathbb{Z})$ , as required.  $\square$

The following observation will prove to be useful.

**Proposition 2.6.** *Let  $X$  be a compact real algebraic variety. For a topological  $\mathbb{C}$ -vector bundle  $\xi$  on  $X$ , the following conditions are equivalent:*

- (a) *There exists a positive integer  $r$  such that the  $\mathbb{C}$ -vector bundle  $\xi(r)$  admits a stratified-algebraic structure.*
- (b) *The  $\mathbb{C}$ -vector bundle  $\xi$  has property (rk) and for every positive integer  $j$ , there exists a positive integer  $b_j$  such that the cohomology class  $b_j c_j(\xi)$  is in  $H_{\mathbb{C}\text{-str}}^{2j}(X; \mathbb{Z})$ .*

*Proof.* Assume that condition (a) is satisfied. Then  $\xi(r)$  has property (rk) and hence  $\xi$  has it as well. Furthermore, the total Chern class  $c(\xi(r))$  is in  $H_{\mathbb{C}\text{-str}}^{\text{even}}(X; \mathbb{Z})$ . We have

$$c(\xi(r)) = c(\xi) \smile \dots \smile c(\xi),$$

where the right-hand-side is the  $r$ -fold cup product. In particular,  $c_1(\xi(r)) = r c_1(\xi)$  is in  $H_{\mathbb{C}\text{-str}}^2(X; \mathbb{Z})$ . By induction, for every positive integer  $j$ , we can find a positive integer  $b_j$  such that the cohomology class  $b_j c_j(\xi)$  is in  $H_{\mathbb{C}\text{-str}}^{2j}(X; \mathbb{Z})$ . Thus (a) implies (b).

Now assume that condition (b) is satisfied. Since  $\xi$  has property (rk), by Lemma 2.5,  $\text{rank } \xi$  is in  $H_{\mathbb{C}\text{-str}}^0(X; \mathbb{Z})$ . Hence (b) implies that the Chern character  $\text{ch}(\xi)$  is in  $H_{\mathbb{C}\text{-str}}^{\text{even}}(X; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ . Consequently, for some positive integer  $r$ , the class  $r[\xi] = [\xi(r)]$  is in  $K_{\mathbb{C}\text{-str}}(X)$ , cf. [30, Proposition 8.9]. According to Theorem 1.1, the  $\mathbb{C}$ -vector bundle  $\xi(r)$  admits a stratified-algebraic structure. Thus (b) implies (a), which completes the proof.  $\square$

We now collect some results on spherical cohomology classes. Every compact real algebraic variety is triangulable [7, p. 217] and hence a result due to Serre can be stated as follows.

**Proposition 2.7** ([32, p. 289, Propoposition 2']). *Let  $X$  be a compact real algebraic variety. Then there exists a positive integer  $a$  such that for every positive integer  $d$  satisfying*

$$\dim X \leq 2d - 2$$

*and every cohomology class  $u$  in  $H^d(X; \mathbb{Z})$ , the cohomology class  $au$  is spherical. In particular, the inclusion*

$$aH^d(X; \mathbb{Z}) \subseteq H_{\text{sph}}^d(X; \mathbb{Z})$$

*holds for such  $a$  and  $d$ .*

Let  $X$  and  $Y$  be real algebraic varieties. A map  $f: X \rightarrow Y$  is said to be *continuous rational* if it is continuous and its restriction to some Zariski open and dense subvariety of  $X$  is a regular map. Assuming that the variety  $X$  is nonsingular, the map  $f$  is continuous rational if and only if it is stratified-regular, cf. [23, Proposition 8] and [30, Remark 2.3].

**Lemma 2.8.** *Let  $X$  be a compact nonsingular real algebraic variety and let  $d$  be a positive integer. For any continuous map  $h: X \rightarrow \mathbb{S}^d$  and any continuous map  $\varphi: \mathbb{S}^d \rightarrow \mathbb{S}^d$  of (topological) degree 2, the composite map  $\varphi \circ h: X \rightarrow \mathbb{S}^d$  is homotopic to a stratified-regular map.*

*Proof.* We may assume without loss of generality that  $h$  is a  $C^\infty$  map. By Sard's theorem,  $h$  is transverse to each point in some open subset  $U$  of  $\mathbb{S}^d$  diffeomorphic to  $\mathbb{R}^d$ . Let  $y$  and  $z$  be distinct points in  $U$ , and let  $A$  be a  $C^\infty$  arc in  $U$  joining  $y$  and  $z$ . Then

$$M := h^{-1}(y) \cup h^{-1}(z)$$

is a compact  $C^\infty$  submanifold of  $X$ . Furthermore,  $B := h^{-1}(A)$  is a compact  $C^\infty$  manifold with boundary  $\partial B = M$ , embedded in  $X$  with trivial normal bundle.

Hence, according to [12, Theorem 1.12], there exists a  $C^\infty$  map  $F: X \rightarrow \mathbb{R}^d$  transverse to 0 in  $\mathbb{R}^d$  and such that  $M = F^{-1}(0)$ . By the Weierstrass approximation theorem, the  $C^\infty$  map  $F$  can be approximated, in the  $C^\infty$  topology, by a regular map  $G: X \rightarrow \mathbb{R}^d$ . If  $G$  is sufficiently close to  $F$ , then  $G$  is transverse to 0 and  $V := G^{-1}(0)$  is a nonsingular Zariski closed subvariety of  $X$ . Furthermore,  $V$  is isotopic to  $M$  in  $X$ , cf. [1, Theorem 20.2].

We can choose a  $C^\infty$  map  $\psi: \mathbb{S}^d \rightarrow \mathbb{S}^d$  of degree 2 that is transverse to  $y$  and satisfies  $\psi^{-1}(y) = \{y, z\}$ . By Hopf's theorem,  $\psi$  is homotopic to  $\varphi$ . Consequently, the maps  $\varphi \circ h$  and  $\psi \circ h$  are homotopic. It suffices to prove that  $\psi \circ h$  is homotopic to a stratified-regular map. By construction, the map  $\psi \circ h$  is transverse to  $y$  and

$$(\psi \circ h)^{-1}(y) = h^{-1}(\psi^{-1}(y)) = h^{-1}(y) \cup h^{-1}(z) = M.$$

Since  $M$  is isotopic to  $V$ , according to [24, Theorem 2.4], the map  $\psi \circ h$  is homotopic to a continuous rational map  $f: X \rightarrow \mathbb{S}^d$ . The map  $f$  is stratified-regular, the variety  $X$  being nonsingular.  $\square$

As a consequence, we obtain the following observation.

**Remark 2.9.** For any compact nonsingular real algebraic variety  $X$ , the inclusion

$$2H_{\text{sph}}^{2k}(X; \mathbb{Z}) \subseteq H_{\text{C-str}}^{2k}(X; \mathbb{Z})$$

holds for every positive integer  $k$ . Indeed, it suffices to prove that for any spherical cohomology class  $u$  in  $H^{2k}(X; \mathbb{Z})$ , the cohomology class  $2u$  is in  $H_{\text{C-str}}^{2k}(X; \mathbb{Z})$ . To this end, let  $h: X \rightarrow \mathbb{S}^{2k}$  be a continuous map with  $h^*(s_{2k}) = u$  and let  $\varphi: \mathbb{S}^{2k} \rightarrow \mathbb{S}^{2k}$  be a continuous map of degree 2. Then

$$(\varphi \circ h)^*(s_{2k}) = h^*(\varphi^*(s_{2k})) = h^*(2s_{2k}) = 2u.$$

Recall that  $H_{\text{C-alg}}^{2k}(\mathbb{S}^{2k}; \mathbb{Z}) = H^{2k}(\mathbb{S}^{2k}; \mathbb{Z})$ , cf. [6, Proposition 4.8]. Since, according to Lemma 2.8, the map  $\varphi \circ h$  is homotopic to a stratified-regular map, it follows that the cohomology class  $2u$  is in  $H_{\text{C-str}}^{2k}(X; \mathbb{Z})$ .

It would be interesting to decide whether the nonsingularity of  $X$  in Remark 2.9 is essential. Dropping the nonsingularity assumption, we obtain below a weaker but useful result, Lemma 2.12. First some preparation is necessary.

By a *multiblowup* of a real algebraic variety  $X$  we mean a regular map  $\pi: X' \rightarrow X$  which is the composition of a finite collection of blowups with nonsingular centers. If  $C$  is a Zariski closed subvariety of  $X$  and the restriction  $\pi_C: X' \setminus \pi^{-1}(C) \rightarrow X \setminus C$  of  $\pi$  is a biregular isomorphism, then we say that the multiblowup  $\pi$  is *over*  $C$ .

A *filtration* of  $X$  is a finite sequence  $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$  of Zariski closed subvarieties satisfying

$$\emptyset = X_{-1} \subseteq X_0 \subseteq \dots \subseteq X_m = X.$$

We will make use of the following result.



**Theorem 2.10** ([30, Theorem 5.4]). *Let  $X$  be a compact real algebraic variety. For a topological  $\mathbb{F}$ -vector bundle  $\xi$  on  $X$ , the following conditions are equivalent:*

- (a) *The  $\mathbb{F}$ -vector bundle  $\xi$  admits a stratified-algebraic structure.*
- (b) *There exists a filtration  $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$  of  $X$ , and for each  $i = 0, \dots, m$ , there exists a multiblowup  $\pi_i: X'_i \rightarrow X_i$  over  $X_{i-1}$  such that the pullback  $\mathbb{F}$ -vector bundle  $\pi_i^*(\xi|_{X_i})$  on  $X'_i$  admits a stratified-algebraic structure.*

We now derive the following.

**Lemma 2.11.** *Let  $X$  be a compact real algebraic variety. Let  $d$  be a positive integer and let  $\theta$  be a topological  $\mathbb{F}$ -vector bundle on  $\mathbb{S}^d$ . For any continuous map  $h: X \rightarrow \mathbb{S}^d$  and any continuous map  $\varphi: \mathbb{S}^d \rightarrow \mathbb{S}^d$  of degree 2, the pullback  $\mathbb{F}$ -vector bundle  $(\varphi \circ h)^*\theta$  on  $X$  admits a stratified-algebraic structure.*

*Proof.* Let  $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$  be a filtration of  $X$  such that the variety  $X_i \setminus X_{i-1}$  is nonsingular for  $0 \leq i \leq m$ . According to Hironaka’s theorem on resolution of singularities [19, 22], for each  $i = 0, \dots, m$ , there exists a multiblowup  $\pi_i: X'_i \rightarrow X_i$  over  $X_{i-1}$  with  $X'_i$  nonsingular. In view of Theorem 2.10, the  $\mathbb{F}$ -vector bundle  $\xi := (\varphi \circ h)^*\theta$  on  $X$  admits a stratified-algebraic structure if and only if the  $\mathbb{F}$ -vector bundle  $\xi_i := \pi_i^*(\xi|_{X_i})$  on  $X'_i$  admits a stratified-algebraic structure for  $0 \leq i \leq m$ . If  $e_i: X_i \hookrightarrow X$  is the inclusion map, then

$$\xi_i = \pi_i^*(e_i^*\xi) = \pi_i^*(e_i^*((\varphi \circ h)^*\theta)) = (\varphi \circ h \circ e_i \circ \pi_i)^*\theta.$$

Since the variety  $X'_i$  is nonsingular and the map  $h \circ e_i \circ \pi_i: X'_i \rightarrow \mathbb{S}^d$  is continuous, according to Lemma 2.8, the map  $\varphi \circ h \circ e_i \circ \pi_i$  is homotopic to a stratified-regular map  $f_i: X'_i \rightarrow \mathbb{S}^d$ . In particular,  $\xi_i \cong f_i^*\theta$ . We may assume that the  $\mathbb{F}$ -vector bundle  $\theta$  is algebraic since each topological  $\mathbb{F}$ -vector bundle on  $\mathbb{S}^d$  admits an algebraic structure, cf. [34, Theorem 11.1] and [7, Proposition 12.1.12; pp. 325, 326, 352]. Thus  $f_i^*\theta$  is a stratified-algebraic  $\mathbb{F}$ -vector bundle on  $X'_i$ . Consequently, the  $\mathbb{F}$ -vector bundle  $\xi_i$  admits a stratified-algebraic structure, as required.  $\square$

Here is the result we have already alluded to in the comment following Remark 2.9.

**Lemma 2.12.** *For any compact real algebraic variety  $X$ , the inclusion*

$$2(k-1)!H_{\text{sph}}^{2k}(X; \mathbb{Z}) \subseteq H_{\mathbb{C}\text{-str}}^{2k}(X; \mathbb{Z}).$$

*holds for every positive integer  $k$ .*

*Proof.* Let  $k$  be a positive integer. It suffices to prove that for every spherical cohomology class  $u$  in  $H^{2k}(X; \mathbb{Z})$ , the cohomology class  $2(k-1)!u$  is in  $H_{\mathbb{C}\text{-str}}^{2k}(X; \mathbb{Z})$ . To this end, let  $h: X \rightarrow \mathbb{S}^{2k}$  be a continuous map with  $h^*(s_{2k}) = u$  and let  $\varphi: \mathbb{S}^{2k} \rightarrow \mathbb{S}^{2k}$  be a continuous map of degree 2. Then

$$(\varphi \circ h)^*(s_{2k}) = h^*(\varphi^*(s_{2k})) = h^*(2s_{2k}) = 2u.$$

Now we choose a topological  $\mathbb{C}$ -vector bundle  $\theta$  on  $\mathbb{S}^{2k}$  with  $c_k(\theta) = (k-1)!s_{2k}$ , cf. [2, p. 19] or [18, p. 155]. Then

$$c_k((\varphi \circ h)^*\theta) = (\varphi \circ h)^*(c_k(\theta)) = (\varphi \circ h)^*((k-1)!s_{2k}) = 2(k-1)!u.$$

According to Lemma 2.11, the  $\mathbb{C}$ -vector bundle  $(\varphi \circ h)^*\theta$  on  $X$  admits a stratified-algebraic structure, and hence the cohomology class  $2(k-1)!u$  is in  $H_{\mathbb{C}\text{-str}}^{2k}(X; \mathbb{Z})$ . The proof is complete.  $\square$

The following result will be used in the proof of Theorem 1.6 and is also of independent interest.

**Theorem 2.13.** *Let  $X$  be a compact real algebraic variety. Let  $k$  be a positive integer and let  $\theta$  be a topological  $\mathbb{F}$ -vector bundle on  $\mathbb{S}^{2k}$ , where  $\mathbb{F} = \mathbb{C}$  or  $\mathbb{F} = \mathbb{H}$ . For any continuous map  $h: X \rightarrow \mathbb{S}^{2k}$ , the  $\mathbb{F}$ -vector bundle  $h^*\theta \oplus h^*\theta$  on  $X$  admits a stratified-algebraic structure.*

*Proof.* Let  $\varphi: \mathbb{S}^{2k} \rightarrow \mathbb{S}^{2k}$  be a continuous map of degree 2. Then

$$c_k(\varphi^*\theta_{\mathbb{C}}) = \varphi^*(c_k(\theta_{\mathbb{C}})) = 2c_k(\theta_{\mathbb{C}}) = c_k(\theta_{\mathbb{C}} \oplus \theta_{\mathbb{C}}),$$

and hence the  $\mathbb{C}$ -vector bundles  $\varphi^*\theta_{\mathbb{C}}$  and  $\theta_{\mathbb{C}} \oplus \theta_{\mathbb{C}}$  on  $\mathbb{S}^{2k}$  are stably equivalent, cf. [2, p. 19] or [18, p. 155]. Consequently, the  $\mathbb{C}$ -vector bundles

$$h^*(\varphi^*\theta_{\mathbb{C}}) = (\varphi \circ h)^*\theta_{\mathbb{C}} \text{ and } h^*(\theta_{\mathbb{C}} \oplus \theta_{\mathbb{C}}) = (h^*\theta \oplus h^*\theta)_{\mathbb{C}}$$

on  $X$  are stably equivalent as well. By Lemma 2.11, the  $\mathbb{C}$ -vector bundle  $(\varphi \circ h)^*\theta_{\mathbb{C}}$  admits a stratified-algebraic structure. Hence, according to Theorem 1.1, the  $\mathbb{C}$ -vector bundle  $(h^*\theta \oplus h^*\theta)_{\mathbb{C}}$  admits a stratified-algebraic structure. Now the proof is complete in view of Theorem 2.2.  $\square$

The next three theorems are crucial for the proof of Theorem 1.3. We first consider  $\mathbb{H}$ -vector bundles. Note that for any  $\mathbb{H}$ -vector bundle  $\xi$ , we have  $c_l(\xi_{\mathbb{C}}) = 0$  for every odd positive integer  $l$ .

**Theorem 2.14.** *Let  $X$  be a compact real algebraic variety. For a topological  $\mathbb{H}$ -vector bundle  $\xi$  on  $X$ , the following conditions are equivalent:*

- (a) *There exists a positive integer  $r$  such that the  $\mathbb{H}$ -vector bundle  $\xi(r)$  admits a stratified-algebraic structure.*
- (b) *The  $\mathbb{H}$ -vector bundle  $\xi$  has property (rk) and there exists a positive integer  $a$  such that the cohomology class  $ac_{2k}(\xi_{\mathbb{C}})$  is in  $H_{\mathbb{C}\text{-str}}^{4k}(X; \mathbb{Z})$  for every positive integer  $k$  satisfying  $8k - 2 < \dim X$ .*

*Proof.* If condition (a) is satisfied, then the  $\mathbb{C}$ -vector bundle  $\xi_{\mathbb{C}}(r)$  admits a stratified-algebraic structure, being isomorphic to  $(\xi(r))_{\mathbb{C}}$ . Thus condition (b) holds in view of Proposition 2.6.

Now assume that condition (b) is satisfied. By Proposition 2.7 and Lemma 2.12, there exists a positive integer  $b$  such that the cohomology class  $bc_{2k}(\xi_{\mathbb{C}})$  is in  $H_{\mathbb{C}\text{-str}}^{4k}(X; \mathbb{Z})$  for every positive integer  $k$ . Furthermore,  $c_l(\xi_{\mathbb{C}}) = 0$  for every odd positive integer  $l$ . Hence, according to Proposition 2.6, there exists a positive integer  $r$  such that the  $\mathbb{C}$ -vector bundle  $\xi_{\mathbb{C}}(r)$  admits a stratified-algebraic structure. Since the  $\mathbb{C}$ -vector bundles  $\xi_{\mathbb{C}}(r)$  and  $(\xi(r))_{\mathbb{C}}$  are isomorphic, by Theorem 2.2, the  $\mathbb{H}$ -vector bundle  $\xi(r)$  admits a stratified-algebraic structure. Thus (b) implies (a). The proof is complete.  $\square$

Recall that for any topological  $\mathbb{C}$ -vector bundle  $\xi$ , the equality  $c_k(\bar{\xi}) = (-1)^k c_k(\xi)$  holds for every nonnegative integer  $k$ , cf. [31, p. 168].

**Theorem 2.15.** *Let  $X$  be a compact real algebraic variety. For a topological  $\mathbb{C}$ -vector bundle  $\xi$  on  $X$ , the following conditions are equivalent:*

- (a) *There exists a positive integer  $r$  such that the  $\mathbb{C}$ -vector bundle  $\xi(r)$  admits a stratified-algebraic structure.*
- (b) *The  $\mathbb{C}$ -vector bundle  $\xi$  has property (rk) and there exists a positive integer  $a$  such that the cohomology class  $ac_{2k}(\xi \oplus \bar{\xi})$  is in  $H_{\mathbb{C}\text{-str}}^{4k}(X; \mathbb{Z})$  for every positive integer  $k$  satisfying  $8k - 2 < \dim X$ .*

*Proof.* Since  $(\mathbb{H} \otimes \xi)_{\mathbb{C}} \cong \xi \oplus \bar{\xi}$ , the equality  $c_l((\mathbb{H} \otimes \xi)_{\mathbb{C}}) = c_l(\xi \oplus \bar{\xi})$  holds for every nonnegative integer  $l$ . Furthermore, the  $\mathbb{C}$ -vector bundle  $\xi$  has property (rk) if and only if the  $\mathbb{H}$ -vector bundle  $\mathbb{H} \otimes \xi$  has it. Hence the proof is complete in view of Lemma 2.3 and Theorem 2.14.  $\square$

Let  $\xi$  be an  $\mathbb{R}$ -vector bundle. Recall that for any nonnegative integer  $k$ , the  $k$ th Pontryagin class of  $\xi$  is defined by  $p_k(\xi) = (-1)^k c_{2k}(\mathbb{C} \otimes \xi)$ .

**Theorem 2.16.** *Let  $X$  be a compact real algebraic variety. For a topological  $\mathbb{R}$ -vector bundle  $\xi$  on  $X$ , the following conditions are equivalent:*

- (a) *There exists a positive integer  $r$  such that the  $\mathbb{R}$ -vector bundle  $\xi(r)$  admits a stratified-algebraic structure.*
- (b) *The  $\mathbb{R}$ -vector bundle  $\xi$  has property (rk) and there exists a positive integer  $a$  such that the cohomology class  $ap_k(\xi)$  is in  $H_{\mathbb{C}\text{-str}}^{4k}(X; \mathbb{Z})$  for every positive integer  $k$  satisfying  $8k - 2 < \dim X$ .*

*Proof.* Assume that condition (a) is satisfied. Then the  $\mathbb{R}$ -vector bundle  $\xi(r)$  has property (rk) and hence  $\xi$  has it as well. Furthermore, the  $\mathbb{C}$ -vector bundle  $(\mathbb{C} \otimes \xi)(r)$  admits a stratified-algebraic structure, being isomorphic to  $\mathbb{C} \otimes \xi(r)$ . According to Proposition 2.6, for every positive integer  $j$ , there exists a positive integer  $b_j$  such that the cohomology class  $b_j c_j(\mathbb{C} \otimes \xi)$  is in  $H_{\mathbb{C}\text{-str}}^{2j}(X; \mathbb{Z})$ . In particular, (a) implies (b) in view of the definition of  $p_k(\xi)$ .

Now assume that condition (b) is satisfied. By Proposition 2.7 and Lemma 2.12, there exists a positive integer  $b$  such that the cohomology class  $bc_{2k}(\mathbb{C} \otimes \xi)$  is in  $H_{\mathbb{C}\text{-str}}^{4k}(X; \mathbb{Z})$  for every positive integer  $k$ . Recall that  $2c_l(\mathbb{C} \otimes \xi) = 0$  for every odd positive integer  $l$ , cf. [31, p. 174]. Hence, according to Proposition 2.6, the  $\mathbb{C}$ -vector bundle  $(\mathbb{C} \otimes \xi)(q)$  admits a stratified-algebraic structure for some positive integer  $q$ . In view of Lemma 2.4, the  $\mathbb{R}$ -vector bundle  $\xi(2q)$  admits a stratified-algebraic structure. Thus (b) implies (a). The proof is complete.  $\square$

We need one more technical result.

**Lemma 2.17.** *Let  $X$  be a compact real algebraic variety. If the group  $\Gamma_{\mathbb{C}}(X)$  is finite, then the quotient group  $H^{2j}(X; \mathbb{Z})/H_{\mathbb{C}\text{-str}}^{2j}(X; \mathbb{Z})$  is finite for every positive integer  $j$ . If the group  $\Gamma_{\mathbb{F}}(X)$  is finite, where  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{F} = \mathbb{H}$ , then the quotient group  $H^{4k}(X; \mathbb{Z})/H_{\mathbb{C}\text{-str}}^{4k}(X; \mathbb{Z})$  is finite for every positive integer  $k$ .*

*Proof.* Recall that the cohomology group  $H^*(X; \mathbb{Z})$  is finitely generated, the variety  $X$  being triangulable.

There exists a positive integer  $b$  such that for every positive integer  $j$  and every cohomology class  $u$  in  $H^{2j}(X; \mathbb{Z})$ , one can find a topological  $\mathbb{C}$ -vector bundle  $\xi$  on  $X$  with

$$c_i(\xi) = 0 \text{ for } 1 \leq i \leq j - 1 \text{ and } c_j(\xi) = bu,$$

cf. [2, p. 19] or [18, p. 155, Theorem A]. We can choose such a  $\mathbb{C}$ -vector bundle  $\xi$  of constant rank.

Assume that the group  $\Gamma_{\mathbb{C}}(X)$  is finite and  $r\Gamma_{\mathbb{C}}(X) = 0$  for some positive integer  $r$ . Then the  $\mathbb{C}$ -vector bundle  $\xi(r)$  admits a stratified-algebraic structure, and hence the cohomology class

$$c_j(\xi(r)) = rc_j(\xi) = rbu$$

is in  $H_{\mathbb{C}\text{-str}}^{2j}(X; \mathbb{Z})$ . Thus the quotient group  $H^{2j}(X; \mathbb{Z})/H_{\mathbb{C}\text{-str}}^{2j}(X; \mathbb{Z})$  is finite, as asserted.

Note that the complexification  $\mathbb{C} \otimes \xi_{\mathbb{R}}$  of the  $\mathbb{R}$ -vector bundle  $\xi_{\mathbb{R}}$  satisfies

$$\mathbb{C} \otimes \xi_{\mathbb{R}} \cong \xi \oplus \bar{\xi}.$$

Similarly, for the quaternionization  $\mathbb{H} \otimes \xi$  of the  $\mathbb{C}$ -vector bundle  $\xi$ , we have

$$(\mathbb{H} \otimes \xi)_{\mathbb{C}} \cong \xi \oplus \bar{\xi}.$$

If the group  $\Gamma_{\mathbb{R}}(X)$  is finite and  $q\Gamma_{\mathbb{R}}(X) = 0$  for some positive integer  $q$ , then the  $\mathbb{R}$ -vector bundle  $\xi_{\mathbb{R}}(q)$  admits a stratified-algebraic structure, and hence so do the  $\mathbb{C}$ -vector bundles

$$\mathbb{C} \otimes \xi_{\mathbb{R}}(q) \cong (\mathbb{C} \otimes \xi_{\mathbb{R}})(q) \cong (\xi \oplus \bar{\xi})(q).$$

If the group  $\Gamma_{\mathbb{H}}(X)$  is finite and  $q\Gamma_{\mathbb{H}}(X) = 0$ , then the  $\mathbb{H}$ -vector bundle  $(\mathbb{H} \otimes \xi)(q)$  admits a stratified-algebraic structure, and hence so do the  $\mathbb{C}$ -vector bundles

$$((\mathbb{H} \otimes \xi)(q))_{\mathbb{C}} \cong (\mathbb{H} \otimes \xi)_{\mathbb{C}}(q) \cong (\xi \oplus \bar{\xi})(q).$$

Consequently, if  $q\Gamma_{\mathbb{F}}(X) = 0$ , where  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{F} = \mathbb{H}$ , then the Chern class  $c_j((\xi \oplus \bar{\xi})(q))$  is in  $H_{\mathbb{C}\text{-str}}^{2j}(X; \mathbb{Z})$ . Now suppose that  $j = 2k$ , where  $k$  is a positive integer. Then

$$c_i(\xi \oplus \bar{\xi}) = 0 \text{ for } 1 \leq i \leq 2k - 1 \text{ and } c_{2k}(\xi \oplus \bar{\xi}) = 2c_{2k}(\xi) = 2bu,$$

which implies the equality

$$c_{2k}((\xi \oplus \bar{\xi})(q)) = qc_{2k}(\xi \oplus \bar{\xi}) = 2qbu.$$

Thus the cohomology class  $2qbu$  is in  $H_{\mathbb{C}\text{-str}}^{4k}(X; \mathbb{Z})$ . In conclusion, the quotient group

$$H^{4k}(X; \mathbb{Z})/H_{\mathbb{C}\text{-str}}^{4k}(X; \mathbb{Z})$$

is finite. The proof is complete. □

We are now ready to prove the theorems announced in Section 1.

*Proof of Theorem 1.3.* In view of Lemma 2.17, condition (a) implies (b). By combining Theorems 2.14, 2.15 and 2.16, we conclude that (b) implies (a). □

*Proof of Theorem 1.4.* It suffices to make use of Theorem 1.3 and Lemma 2.12. □

*Proof of Theorem 1.6.* Let  $n = \dim X$ . According to Proposition 1.2, it suffices to prove that for any topological  $\mathbb{F}$ -vector bundle  $\xi$  of constant positive rank on  $X$ , the  $\mathbb{F}$ -vector bundle  $\xi(r)$  admits a stratified-algebraic structure, where

$$r = \begin{cases} 2^{a(n, \mathbb{F})} & \text{if } n \leq 7 \\ 2^{a(n, \mathbb{F})+2} & \text{if } n = 8. \end{cases}$$

If  $n \leq d(\mathbb{F})$ , then  $a(n, \mathbb{F}) = 1$  and the  $\mathbb{F}$ -vector bundle  $\xi(1) = \xi$  admits a stratified-algebraic structure, cf. [30, Corollary 3.6].

Henceforth we assume that  $n \geq d(\mathbb{F}) + 1$ .

The rest of the proof is divided into three steps.

*Case 1.* Suppose that  $\mathbb{F} = \mathbb{H}$ .

The 4-sphere  $\mathbb{S}^4$  can be identified (as a topological space) with the quaternionic projective line  $\mathbb{P}^1(\mathbb{H})$ . Let  $\theta$  be the  $\mathbb{H}$ -line bundle on  $\mathbb{S}^4$  corresponding to the tautological  $\mathbb{H}$ -line bundle on  $\mathbb{P}^1(\mathbb{H})$ . Since  $5 \leq n \leq 8$ , we have  $a(n, \mathbb{H}) = 1$ .

First suppose that  $5 \leq n \leq 7$ . Then  $\xi$  can be expressed as  $\xi = \lambda \oplus \varepsilon$ , where  $\lambda$  and  $\varepsilon$  are topological  $\mathbb{H}$ -vector bundles,  $\text{rank } \lambda = 1$  and  $\varepsilon$  is trivial, cf. [20, p. 99]. For the same reason, the  $\mathbb{H}$ -vector bundle  $\lambda \oplus \lambda$  has a nowhere vanishing continuous section. Thus the  $\mathbb{H}$ -line bundle  $\lambda$  is generated by two continuous sections. It follows that we can find a continuous map  $h: X \rightarrow \mathbb{S}^4$  with  $\lambda \cong h^*\theta$ . According to Theorem 2.13, the  $\mathbb{H}$ -vector bundle  $\lambda \oplus \lambda = \lambda(2)$  admits a stratified-algebraic structure. Since  $\xi(2) \cong \lambda(2) \oplus \varepsilon(2)$ , the  $\mathbb{H}$ -vector bundle  $\xi(2)$  admits a stratified-algebraic structure, as required.

Now suppose that  $n = 8$ . It remains to prove that the  $\mathbb{H}$ -vector bundle  $\xi(8)$  admits a stratified-algebraic structure. This can be done as follows. Let  $\mathcal{F} = (X_{-1}, X_0, \dots, X_m)$  be a filtration of  $X$  such that the variety  $X_i \setminus X_{i-1}$  is nonsingular of pure dimension for  $0 \leq i \leq m$ . According to Hironaka's theorem on resolution of singularities [19, 22], for each  $i = 0, \dots, m$ , there exists a multiblowup  $\pi_i: X'_i \rightarrow X_i$  over  $X_{i-1}$  with  $X'_i$  nonsingular of pure dimension. Consider the

pullback  $\mathbb{H}$ -vector bundle  $\xi_i := \pi_i^*(\xi|_{X_i})$  on  $X'_i$ . According to Theorem 2.10, it suffices to prove that the  $\mathbb{H}$ -vector bundle  $\xi_i(8)$  admits a stratified-algebraic structure. If  $\dim X'_i \leq 7$ , we already established a stronger result, namely,  $\xi_i(2)$  admits a stratified-algebraic structure. If  $\dim X'_i = 8$ , we choose a finite subset  $A_i$  of  $X'_i$  whose intersection with each connected component of  $X'_i$  consists of one point. Let  $\sigma_i: X''_i \rightarrow X'_i$  be the blowup of  $X'_i$  with center  $A_i$ . We can replace  $\pi_i: X'_i \rightarrow X_i$  by the composite map  $\sigma_i \circ \pi_i: X''_i \rightarrow X_i$  and replace the  $\mathbb{H}$ -vector bundle  $\xi_i$  on  $X'_i$  by the  $\mathbb{H}$ -vector bundle  $(\sigma_i \circ \pi_i)^*(\xi|_{X_i})$  on  $X''_i$ . Note that  $X''_i$  is a compact nonsingular real algebraic variety of pure dimension 8, and each connected component of  $X''_i$  is nonorientable as a  $\mathcal{C}^\infty$  manifold. Thus in order to simplify notation we may assume that the variety  $X$  is nonsingular of pure dimension 8, and each connected component of  $X$  is nonorientable as a  $\mathcal{C}^\infty$  manifold. The last condition implies the equality  $2H^8(X; \mathbb{Z}) = 0$ . Since  $c_l(\xi_{\mathbb{C}}) = 0$  for every odd positive integer  $l$ , we get

$$c_4((\xi(4))_{\mathbb{C}}) = c_4(\xi_{\mathbb{C}}(4)) = 4c_4(\xi_{\mathbb{C}}) + 6c_2(\xi_{\mathbb{C}}) \smile c_2(\xi_{\mathbb{C}}) = 0$$

in  $H^8(X; \mathbb{Z})$ . The  $\mathbb{H}$ -vector bundle  $\xi(4)$  can be expressed as the direct sum of a topological  $\mathbb{H}$ -vector bundle  $\eta$  of rank 2 and a trivial  $\mathbb{H}$ -vector bundle, cf. [20, p. 99]. Then

$$c_4(\eta_{\mathbb{C}}) = c_4((\xi(4))_{\mathbb{C}}) = 0.$$

Recall that  $c_4(\eta_{\mathbb{C}})$  is the Euler class  $e(\eta_{\mathbb{R}})$  of the oriented  $\mathbb{R}$ -vector bundle  $\eta_{\mathbb{R}} = (\eta_{\mathbb{C}})_{\mathbb{R}}$ , cf. [31, p. 159]. Interpreting  $e(\eta_{\mathbb{R}})$  as an obstruction, we conclude that the  $\mathbb{H}$ -vector bundle  $\eta$  has a nowhere vanishing continuous section, cf. [31, pp. 139, 140, 147] and [33]. Consequently, the  $\mathbb{H}$ -vector bundle  $\xi(4)$  can be expressed as  $\xi(4) = \mu \oplus \delta$ , where  $\mu$  and  $\delta$  are topological  $\mathbb{H}$ -vector bundles,  $\text{rank } \mu = 1$  and  $\delta$  is trivial.

Since  $\xi(8) \cong \mu(2) \oplus \delta(2)$ , it suffices to prove that the  $\mathbb{H}$ -vector bundle  $\mu(2)$  admits a stratified-algebraic structure. Note that

$$c_4((\mu(2))_{\mathbb{C}}) = c_4((\xi(8))_{\mathbb{C}}) = 8c_4(\xi_{\mathbb{C}}) + 28c_2(\xi_{\mathbb{C}}) \smile c_2(\xi_{\mathbb{C}}) = 0$$

in  $H^8(X; \mathbb{Z})$ . Now, interpreting  $c_4(\mu(2)) = e((\mu(2))_{\mathbb{R}})$  as an obstruction, we get a nowhere vanishing continuous section of  $\mu(2)$ . In other words, the  $\mathbb{H}$ -line bundle  $\mu$  is generated by two continuous sections. It follows that we can find a continuous map  $g: X \rightarrow \mathbb{S}^4$  with  $\mu \cong g^*\theta$ . According to Theorem 2.13, the  $\mathbb{H}$ -vector bundle  $\mu \oplus \mu = \mu(2)$  admits a stratified-algebraic structure. The proof of Case 1 is complete.

*Case 2.* Suppose that  $\mathbb{F} = \mathbb{C}$ .

Since  $n \geq 3$ , we have  $a(n, \mathbb{C}) = a(n, \mathbb{H}) + 1$ . Hence it suffices to apply Case 1 and Lemma 2.3 to the  $\mathbb{H}$ -vector bundle  $\mathbb{H} \otimes \xi$ .

*Case 3.* Suppose that  $\mathbb{F} = \mathbb{R}$ .

Since  $n \geq 2$ , we have  $a(n, \mathbb{R}) = a(n, \mathbb{C}) + 1$ . Hence it suffices to apply Case 2 and Lemma 2.4 to the  $\mathbb{C}$ -vector bundle  $\mathbb{C} \otimes \xi$ .

The proof is complete. □

### 3. LINE BUNDLES

In this short section we concentrate our attention on  $\mathbb{C}$ -line bundles. For any real algebraic variety  $X$ , let  $\text{VB}_{\mathbb{C}}^1(X)$  denote the group of isomorphism classes of topological  $\mathbb{C}$ -line bundles on  $X$  (with operation induced by tensor product). Let  $\text{VB}_{\mathbb{C}\text{-str}}^1(X)$  be the subgroup of  $\text{VB}_{\mathbb{C}}^1(X)$  consisting of the isomorphism classes of all  $\mathbb{C}$ -line bundles admitting a stratified-algebraic structure. Since  $X$  has the homotopy type of a compact polyhedron [7, pp. 217, 225], the group  $\text{VB}_{\mathbb{C}}^1(X)$

is finitely generated, being isomorphic to the cohomology group  $H^2(X; \mathbb{Z})$ . In particular, the quotient group

$$\Gamma_{\mathbb{C}}^1(X) := \text{VB}_{\mathbb{C}}^1(X) / \text{VB}_{\mathbb{C}\text{-str}}^1(X)$$

is finitely generated. Thus the group  $\Gamma_{\mathbb{C}}^1(X)$  is finite if and only if  $r\Gamma_{\mathbb{C}}^1(X) = 0$  for some positive integer  $r$ . Furthermore, the latter condition holds if and only if for every topological  $\mathbb{C}$ -line bundle  $\lambda$  on  $X$  its  $r$ th tensor power  $\lambda^{\otimes r}$  admits a stratified-algebraic structure.

**Proposition 3.1.** *Let  $X$  be a real algebraic variety. For any topological  $\mathbb{C}$ -line bundle  $\lambda$  on  $X$  and positive integer  $r$ , if  $\lambda(r)$  admits a stratified-algebraic structure, then so does  $\lambda^{\otimes r}$ .*

*Proof.* If the  $\mathbb{C}$ -vector bundle  $\lambda(r)$  admits a stratified-algebraic structure, then so does the  $\mathbb{C}$ -line bundle  $\det \lambda(r)$ , cf. [30, Proposition 3.15]. Here  $\det \lambda(r)$  stands for the  $r$ th exterior power of  $\lambda(r)$ . The proof is complete since the  $\mathbb{C}$ -line bundles  $\det \lambda(r)$  and  $\lambda^{\otimes r}$  are isomorphic.  $\square$

As a consequence, we obtain the following.

**Corollary 3.2.** *Let  $X$  be a compact real algebraic variety. If  $r$  is a positive integer and  $r\Gamma_{\mathbb{C}}^1(X) = 0$ , then  $r\Gamma_{\mathbb{C}}^1(X) = 0$ .*

*Proof.* It suffices to make use of Propositions 1.2 and 3.1.  $\square$

**Corollary 3.3.** *For any compact real algebraic variety  $X$  of dimension at most 8, the group  $\Gamma_{\mathbb{C}}^1(X)$  is finite and*

$$2^{a(\dim X, \mathbb{C}) + a(X)} \Gamma_{\mathbb{C}}^1(X) = 0,$$

where  $a(X) = 0$  if  $\dim X \leq 7$  and  $a(X) = 2$  if  $\dim X = 8$ .

*Proof.* This follows from Theorem 1.6 and Corollary 3.2.  $\square$

A different proof of Corollary 3.3 for varieties of dimension at most 5 is given in [28]. It is plausible that  $2\Gamma_{\mathbb{C}}^1(X) = 0$  for every compact real algebraic variety  $X$ , cf. [28, Conjecture B, Proposition 1.5]. This is confirmed by Corollary 3.3 for  $\dim X \leq 4$ . Without restrictions on the dimension of  $X$  we have the following.

**Theorem 3.4.** *Let  $X$  be a compact real algebraic variety with  $H_{\text{sph}}^2(X; \mathbb{Z}) = H^2(X; \mathbb{Z})$ . Then the group  $\Gamma_{\mathbb{C}}^1(X)$  is finite and  $2\Gamma_{\mathbb{C}}^1(X) = 0$ .*

*Proof.* According to Lemma 2.12,  $2H^2(X; \mathbb{Z}) \subseteq H_{\mathbb{C}\text{-str}}^2(X; \mathbb{Z})$ . Hence for any topological  $\mathbb{C}$ -line bundle  $\lambda$  on  $X$ , the Chern class  $c_1(\lambda^{\otimes 2}) = 2c_1(\lambda)$  is in  $H_{\mathbb{C}\text{-str}}^2(X; \mathbb{Z})$ .

In view of [30, Proposition 8.6], the  $\mathbb{C}$ -line bundle  $\lambda^{\otimes 2}$  admits a stratified-algebraic structure. Thus  $2\Gamma_{\mathbb{C}}^1(X) = 0$ , as asserted.  $\square$

The following special case is of interest.

**Corollary 3.5.** *Let  $X$  be a compact real algebraic variety. If each connected component of  $X$  is homotopically equivalent to  $\mathbb{S}^{d_1} \times \cdots \times \mathbb{S}^{d_n}$  for some positive integers  $d_1, \dots, d_n$ , then the group  $\Gamma_{\mathbb{C}}^1(X)$  is finite and  $2\Gamma_{\mathbb{C}}^1(X) = 0$ .*

*Proof.* Since  $H_{\text{sph}}^2(X; \mathbb{Z}) = H^2(X; \mathbb{Z})$ , it suffices to apply Theorem 3.4.  $\square$

According to [30, Example 7.10], there exists a nonsingular real algebraic variety  $X$  diffeomorphic to the  $n$ -fold product  $\mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ ,  $n \geq 3$ , with  $\Gamma_{\mathbb{C}}^1(X) \neq 0$ .

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