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A QUICK TRIP THROUGH FIBRATION STRUCTURES

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Abstract. In this article we review the classical results about the existence of fibered structures for real and complex singularities in the local setting, commonly known in the literature as Milnor's fibration structures. After reviewing the classical studies, we describe some generalizations in two main directions, namely, the existence of open book structures on semi-algebraic manifolds, and the existence of the Milnor fibration in a stratified sense.

1. Introduction

The existence of a fibration near an isolated singularity is fundamental to the understanding of the local structure of the pair space-function.

In the famous Princeton notes of 1968 [Mi], J. Milnor established the foundations for studying fibration structures for germs of complex analytic functions $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ with $\dim \operatorname{Sing} f \geq 0$. In this setting, it was shown that given a representative $f : U \subset \mathbb{C}^{n+1} \to \mathbb{C}$ with U an open set in $\mathbb{C}^{n+1}$, $f(0) = 0$, there exists a small enough real number $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leq \varepsilon_0$,

$$\phi := \frac{f}{\|f\|} : S_\varepsilon^{2n+1} \setminus K_\varepsilon \to S^1 \tag{1}$$

is a locally trivial smooth fibration, where $K_\varepsilon = f^{-1}(0) \cap S_\varepsilon^{2n+1}$ is called the *link* of the singularity at the origin.

In chapters 5, 6 and 7 of [Mi], Milnor gave differentiable and topological descriptions of the link and the fibers $F_\theta = \phi^{-1}\left(e^{i\theta}\right)$, where $e^{i\theta} \in S^1$, showing that independent of the dimension of the singular locus, the fiber is a $(2n)$-dimensional smooth parallelizable manifold with the homotopy type of a k-dimensional CW-complex, with $k \leq n$.

In addition, whenever $\operatorname{Sing} f = \{0\}$, Milnor associated to the singular point of f a multiplicity denoted by $\mu(f)$, later named by several authors as *the Milnor number of the singularity*, given by the topological degree of the map

$$\varepsilon \frac{\nabla f}{\|\nabla f\|} : S_\varepsilon^{2n+1} \to S_\varepsilon^{2n+1}.$$

In this case it was also shown that the fiber F_θ has the same homotopy type of a bouquet of n-dimensional spheres $\bigvee_{i=1}^{\mu(f)} S_i^n$, with $\mu(f)$ spheres in the bouquet.

In 1976, Lê Dũng Tráng in his article [Le] proved the existence of a general fibration structure on a complex analytic set, as follows.

Let X be an analytic set in an open neighborhood U of the origin $0 \in \mathbb{C}^{n+1}$. Let $f : (X, 0) \to (\mathbb{C}, 0)$ be a germ of a holomorphic function.

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Theorem 1.1. [Le, Milnor-Lê Fibration] *For any small enough* $\varepsilon > 0$*, there exists* η*,* $0 < \eta \ll \varepsilon$*, such that*

$$f_| : B_\varepsilon^{2n+2} \cap X \cap f^{-1}(D_\eta \setminus \{0\}) \to D_\eta \setminus \{0\} \tag{2}$$

is a locally trivial topological fibration.

An important point to notice here is that this topological fibration structure becomes a smooth fibration if $X \setminus V_f$ is a non-singular analytic set in $\mathbb{C}^{n+1}$ (see details in [Ham, Le]).

As a particular case of the previous theorem, one can state:

Corollary 1.2. [Le, Existence of Milnor-Lê (tube) fibration] *Let* $f : \left(\mathbb{C}^{n+1}, 0\right) \to (\mathbb{C}, 0)$ *be a holomorphic function germ. Then there exists small enough* $\varepsilon > 0$*, such that for any* $0 < \delta \ll \varepsilon$*, the map*

$$f_| : \overline{B}_\varepsilon^{2n+2} \cap f^{-1}\left(D_\delta \setminus \{0\}\right) \to D_\delta \setminus \{0\} \tag{3}$$

is the projection of a locally trivial smooth fibration. In addition, for any small enough ε*, there exists* η*,* $0 < \eta \ll \varepsilon$*, such that*

$$f_| : B_\varepsilon^{2n+2} \cap f^{-1}\left(S_\eta^1\right) \to S_\eta^1 \tag{4}$$

is the projection of a locally trivial smooth fibration. Moreover, the fibrations (1) and (4) are equivalent[1]*.*

Milnor also explained how to extend the study to a real analytic map germ

$$G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0),\ m > p \geq 2,$$

with isolated singular point at the origin, i.e., $\operatorname{Sing} G = \{0\}$ as a germ of a set. In this case he observed that, for any small enough $\varepsilon > 0$, there exists a projection map

$$S_\varepsilon^{n-1} \setminus K_\varepsilon \to S_1^{p-1}$$

that is a smooth locally trivial fibration, induced by G, but which in general fails to be the canonical map $G/\|G\|$ like (1) (see section 2.2). However, one gets that G always induces a trivial fibration structure over a neighborhood of the *link* K_ε, and consequently an *open book structure* (or $NS-$pair) on S_ε^{n-1} for some extension of the projection $G/\|G\|$ (see Section 3).

More recently in [ACT1, AT1, AT2], the authors have defined and proved the existence of *singular higher open book structures* on spheres of small enough radius, which extends the real and complex fibrations results previously proved by Milnor.

In another direction, the authors in [DACA] have shown how it is possible to extend these results to the class of semi-algebraic maps, in such a way that it is possible to derive, as a particular case, the existence of fibration structures mentioned above. More precisely, let $G : \mathbb{R}^m \to \mathbb{R}^p$, $m > p \geq 2$, be a C^2 semi-algebraic map and $W \hookrightarrow \mathbb{R}^N$ an embedded compact and connected semi-algebraic manifold. The authors adapted some conditions used in [ACT1, ACT2, AT1, AT2, Ma] to ensure that the restriction map

$$\overline{G} = \frac{G}{\|G\|} : W \setminus V_G \to S^{p-1}$$

with $V_G := G^{-1}(0)$, gives a higher open book structure on W and consequently a locally trivial smooth fibration. In this case, the link of the structure is $V_W(G) = W \cap V_G$.

[1] Two locally trivial smooth fibrations $p : E \to B$ and $p' : E' \to B$ are said to be *equivalent* if there is a smooth diffeomorphism $h : E \to E'$ such that $p' \circ h = p$.

In the past few years the study of the existence of fibration structures in the real setting has concentrated on real maps with isolated singularities and on classes of singular maps with the property $\operatorname{Sing} G \subset V_G$, which in this work will be denoted by $\operatorname{Disc} G = \{0\}$ (cf [ACT1, AT1, AT2, C, CSS3, DA, Ma, Mi, PT, RSV]).

The complementary case, when $\operatorname{Disc} G$ is larger than $\{0\}$, has been studied, for instance, by Hamm in [Ham]. Hamm studied the case where the germs of holomorphic maps

$$G : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^p, 0)$$

are also an ICIS - *Isolated Complete Intersection Singularity* [2]. This means the map defines a local complete intersection germ V_G such that V_G has an isolated singularity at the origin, i.e., the ICIS condition amounts to the condition $\operatorname{Sing} G \cap V_G = \{0\}$. Hamm proved the following result.

Theorem 1.3. *Let $G := (G_1, \dots, G_p) : (\mathbb{C}^{n+p}, 0) \to (\mathbb{C}^p, 0)$, $p \geq 1$, be an ICIS at 0. Then,*

$$G_| : B_\varepsilon^{2(n+p)} \cap G^{-1}(B_\eta^{2p} \setminus \operatorname{Disc} G) \to B_\eta^{2p} \setminus \operatorname{Disc} G \tag{5}$$

is a locally trivial smooth fibration.

This fibration was also called *the Milnor fibration* and it generalizes the previous isolated singular case for holomorphic functions. The discriminant set $\operatorname{Disc} G$ is a complex hypersurface of $\mathbb{C}^p$. Hence, it does not disconnect the complement $B_\eta^{2p} \setminus \operatorname{Disc} G$ and the topological type of the fibers of (5) does not change. Moreover, the fiber F is a real $2n$-dimensional smooth manifold with the homotopy type of a bouquet of n-dimensional spheres $\bigvee_{i=1}^{\mu} S_i^n$, where now $\mu := \operatorname{rank} H_n(F, \mathbb{Z})$, the rank of the homology in the middle dimension of the fiber with integer coefficients.

For a real analytic map germ $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with positive dimensional discriminant set, i.e. $\dim \operatorname{Disc} G > 0$, the existence of fibration structures was pointed out theoretically in [ACT1, Theorem 1.3] and [MS], but no concrete families of examples have been studied. In [CGS], the authors presented a Milnor-Lê type result over the complement of the image $G(\operatorname{Sing} G)$, under assumptions of Thom regularity.

In [ART1] the authors have considered this general situation and have introduced two local fibrations structures. The first one was over the complement of the discriminant, which was called a *Milnor-Hamm tube fibration*. The second was a general notion of *stratified tube fibration* by considering in addition all singular fibers over the stratified discriminant. In the latter case, the tube fibration, which was called a *singular Milnor tube fibration*, is actually a collection of finitely many fibrations over path-connected subanalytic sets.

In [ART2], the authors considered again the setting $\dim \operatorname{Disc} G > 0$ and introduced the *Milnor-Hamm sphere fibration.* They gave natural sufficient conditions for which this fibration exists, and they presented several classes of maps which satisfies these conditions. Moreover, they have shown that the Milnor-Hamm tube and Milnor-Hamm sphere fibrations are extensions of the previous ones treated in [ACT1, AT1, AT2, CGS, CSS2, Ma, Mi].

In this work we present a brief survey about the results described above, as well as some comparisons between the main results found in the literature. This paper complements the nice survey paper [S2], recently published.

[2] One of the richest sources of information on ICIS is Looijenga's classical book [Lo2]. See also the reedited version [Lo3].

2. 0-DIMENSIONAL DISCRIMINANT SET

In this section we consider the fibration on the so-called *Milnor's tube*, and the fibration on a sphere of radius small enough for the case where the classical discriminant set is 0-dimensional. Classically, this case was studied in two approaches: isolated critical point and isolated critical value.

2.1. **Isolated critical point: tube fibration.** Given a representative of $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$, in the first part of the proof of [Mi, Theorem 11.2], Milnor proved that if G has an isolated critical point at the origin $0 \in \mathbb{R}^m$, then for any small enough $\varepsilon > 0$, there exists η, $0 < \eta \ll \varepsilon$, such that the restriction map

$$G_{|} : \overline{B}_{\varepsilon}^{m} \cap G^{-1}(S_{\eta}^{p-1}) \to S_{\eta}^{p-1} \tag{6}$$

is the projection of a locally trivial smooth fibration. More precisely, Milnor proved the following result:

Theorem 2.1. [Mi] *Let* $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ *be a real analytic map germ such that* $\operatorname{Sing} G = \{0\}$ *as a germ of an analytic set at the origin. Then there exists* $\varepsilon_0 > 0$ *such that, for each* ε, $0 < \varepsilon \leq \varepsilon_0$, *there exists* η, $0 < \eta \ll \varepsilon$, *such that* (6) *is a smooth fiber bundle.*

Geometrically, a standard picture for the total space $\overline{B}_{\varepsilon}^{m} \cap G^{-1}(S_{\eta}^{p-1})$ is as in the Figure 1 below[3]. The boundary manifold $\overline{B}_{\varepsilon}^{m} \cap G^{-1}(S_{\eta}^{p-1})$ looks like a "tube" surrounding the special fiber V_G. For this reason several authors called this space "the Milnor tube".

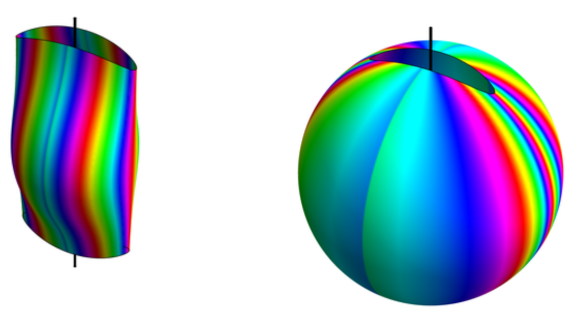


FIGURE 1. $G(x, y, z) = (x, y(x^2 + y^2 + z^2))$ Milnor tube and Milnor sphere fibrations.

REMARK 2.2. It is not hard to see that the structure of the fibration (6) does not change up to isotopy for any $\varepsilon > 0$ and $\eta > 0$ small enough. Consequently, we will denote the Milnor tube as M_G.

2.2. **Sphere fibration: Milnor's example.** Concerning the sphere fibration in this real setting, Milnor guaranteed the existence of a diffeomorphism between the Milnor tube M_G and the complement $S_{\varepsilon}^{m-1} \setminus \operatorname{int}(T)$ of an open tubular neighborhood $\operatorname{int}(T)$ of the link K_ε in S_{ε}^{m-1}, where $T := \{x \in S_{\varepsilon}^{m-1} \mid \|G(x)\| \leq \eta\}$. This diffeomorphism is the identity on the boundary of

[3] In the case the link $K_\varepsilon = V_G \cap S_{\varepsilon}^{m-1}$ is not empty for any small enough ε.

the tube, which allows one to extend it to an open book structure (see Section 3). This diffeomorphism and the locally trivial smooth fibration (6) guaranteed by Theorem 2.1, can be composed to get a map

$$\zeta : S_\varepsilon^{m-1} \setminus \mathrm{int}(T) \to S_\eta^{p-1}$$

which is a fibration, as stated in the following result:

Theorem 2.3. [Mi, Theorem 11.2, p. 97] *Let* $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p \geq 2$, *be a real analytic map germ such that* $\mathrm{Sing}\, G = \{0\}$ *as a germ of an analytic set at the origin. Then there exists* $\varepsilon_0 > 0$ *such that, for each* ε, $0 < \varepsilon \leq \varepsilon_0$, *there exists* η, $0 < \eta \ll \varepsilon$, *such that*

$$\zeta : S_\varepsilon^{m-1} \setminus int(T) \to S_\eta^{p-1} \tag{7}$$

is a smooth fiber bundle.

Moreover, Milnor showed that each fiber F_ζ of the fibration ζ is a smooth compact $(m-p)$-dimensional manifold bounded by a copy of K_ε. If the link K_ε is not empty for any small enough $\varepsilon > 0$, it is a $(m-p-1)$-dimensional closed smooth submanifold of the sphere and the fiber is $(p-2)$-connected. On the other hand, if the link K_ε is empty, then the manifold $\overline{B}_\varepsilon^m \cap G^{-1}(S_\eta^{p-1})$ is diffeomorphic to the sphere S_ε^{m-1}. Moreover, when $m > p$ the fibration (7) given in Theorem 2.3 becomes a *Hopf fibration*[4] $G_| : S^{2t-1} \to S^t$, with $t = 2, 4, 8$.

Next, Milnor presented the following remark without a proof [Mi, remark on p.99]:

"*with a little more effort one can prove that the entire complement* $S_\varepsilon^{m-1} \setminus K_\varepsilon$ *also fibers on* S_η^{p-1}".

In order to make this more precise, in [AT1, AT2] and [ACT1], the authors gave a complete proof for this remark.

Milnor also noted that in general the map projection of the fibration (7) fails to be the canonical map $G/\|G\|$, like it is for the above cited case of holomorphic function germs. In particular, in [Mi, p. 99], Milnor considered the mapping $G := (G_1, G_2) : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ given by $G(x, y) = (x, x^2 + y(x^2 + y^2))$ which satisfies $\mathrm{Sing}\, G = V_G = \{0\}$ and consequently has an isolated singular point at the origin. Theorem 2.3 gives the existence of the fibration in the sphere. However, the map $G/\|G\|$ cannot be the projection of a locally trivial smooth fibration on S_ε^1, because it is not a submersion for ε small enough.

In fact, considering $\mathbf{v} := (x, y)$ and the matrix

$$A(\mathbf{v}) = \begin{pmatrix} G_1(\mathbf{v})\nabla G_2(\mathbf{v}) - G_2(\mathbf{v})\nabla G_1(\mathbf{v}) \\ \mathbf{v} \end{pmatrix}$$

one can see that there exists a curve C (see Figure 2) of tangency points between the fibers of the map

$$G/\|G\| : B_\varepsilon^2 \setminus V_G \to S^1$$

and the small spheres [5]. The curve C contains the origin in its closure, hence the intersection $C \cap S_\varepsilon^1$ provides the critical locus of the map $G/\|G\| : S_\varepsilon^1 \to S^1$ for any small enough $\varepsilon > 0$.

As we will see in more details in the next section, the curve C represents the set of ρ-nonregular points of $G/\|G\|$ (see Lemma 2.10 and Remark 2.11). Consequently (c.f. Definition 2.9), the map $G/\|G\|$ is not ρ-regular and this is precisely the reason why the map $G/\|G\|$ fails to be the projection of a locally trivial smooth fibration.

[4] It is well known that this case is only possible for the pairs of dimensions $(m,p) \in \{(4,3),(8,5),(16,9)\}$, according to [CL, Lemma 1, p. 151], and $G : A \times A \to A \times \mathbb{R}$ is given by $G(x,y) = (2x\bar{y}, |y|^2 - |x|^2)$, where A denotes the complex numbers, the quaternions, or the Cayley numbers.

[5] It is also known as the polar curve.

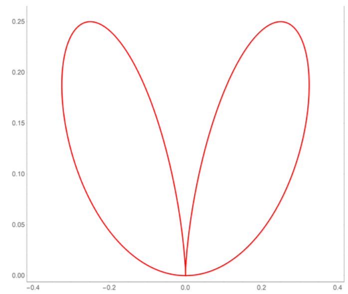


FIGURE 2. Curve of tangencies between the fibers of $G/\|G\|$ and spheres centered at the origin, for $G(x, y) = (x, x^2 + y(x^2 + y^2))$

REMARK 2.4. The phenomenon described above in the Milnor example can be reproduced in higher dimensions using the isolated singularity map $G : (\mathbb{R}^{m+2}, 0) \to (\mathbb{R}^2, 0)$ given by

$$G(x, y, z_1, \dots, z_m) = (x, x^2 + y(x^2 + y^2 + z_1^2 + \cdots + z_m^2)).$$

2.3. **Non-isolated singular case: tube fibration.** Both fibrations, the Milnor tube fibration and the sphere fibration, in the real case were extended later for non-isolated singular map germs under the assumption that the discriminant set is 0-dimensional . In order to state properly these results we need to provide new definitions and notations.

Let us consider $U \subset \mathbb{R}^m$ an open subset such that $0 \in U$ and let $\rho : U \to \mathbb{R}_{\geq 0}$ be a non-negative proper function which defines the origin.

Definition 2.5. Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ. We denote by

$$M_\rho(G) := \{x \in U \mid \rho \not\pitchfork_x G\}$$

the set of *ρ-nonregular points* of G, sometimes also called the *Milnor set* of G.

The transversality of the fibers of a map G to the levels of ρ is called *ρ-regularity* and we will see below that it is a condition for the existence of a locally trivial smooth fibration. It was used in the local (stratified) setting by Thom, Milnor, Mather, Looijenga, Bekka, e.g. [Be, Lo1, Mi, Th1, Th2] and more recently in [ACT1, AT1, AT2], and [CSS1, CSS3] under a different name *d-regularity*, as well as at infinity in the references [ACT2, DRT, NZ, Ti1, Ti2].

It follows from Definition 2.5 that the Milnor set $M_\rho(G)$ is the set of points $x \in U$ such that the vectors $\{\nabla\rho(x), \nabla G_1(x), \dots, \nabla G_p(x)\}$ are linearly dependent over $\mathbb{R}$, i.e., $M_\rho(G)$ is the singular locus Sing (G, ρ) of the pair of map $(G, \rho) : U \to \mathbb{R}^p \times \mathbb{R}$. Hence, the singular set Sing G is included in $M_\rho(G)$.

For the sake of simplicity, in what follows ρ is the square of the Euclidean distance function $\rho(x) = \|x\|^2$, and we write $M(G) := M_\rho(G)$ for short. However, all results carry out easily over any other function ρ as considered above.

Consider the following condition:

$$\overline{M(G) \setminus V_G} \cap V_G \subseteq \{0\} \tag{8}$$

where the closure of the set $\overline{M(G) \setminus V_G}$ is thought as a germ of a set at the origin. See Figure 3 for an example.

Condition (8) was used in [ACT1, AT1, AT2], where it was shown that it insures the existence of the Milnor tube fibration. More recently, this condition was adapted by the authors in [ART1] and used in a stratified sense to ensure the existence of a singular Milnor tube fibration (see

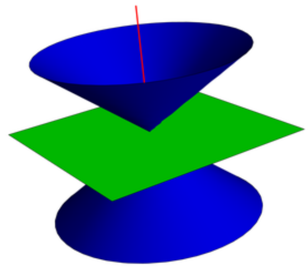


FIGURE 3. From Example 2.8, $M(G)$ is the cone and the plane, while V_G is the plane and the line. Hence G satisfies Condition (8).

Section 5.1 below). Note that this condition is equivalent to saying that for all small enough $\varepsilon > 0$ and $0 < \eta \ll \varepsilon$, the map:

$$G_| : S_\varepsilon^{m-1} \cap G^{-1}(\overline{B}_\eta^p \setminus \{0\}) \to \overline{B}_\eta^p \setminus \{0\}$$

is a locally trivial smooth fibration.

In [Ma] D. Massey considered Condition (8) but with different notation and called it the *Milnor condition (b)*. Massey used the condition to prove the existence of the Milnor tube fibration in the local setting, as in Theorem 2.6 below. Here we shall use the same notation of [ACT1] and [ART1].

Theorem 2.6. [Ma, Existence of the (full) Milnor's tube fibration] *Let $G : U \to \mathbb{R}^p$ be as above and assume that it has isolated critical value at origin, i.e.* $\operatorname{Disc} G = \{0\}$, *and satisfies Condition* (8). *Then there exists $\varepsilon_0 > 0$ such that, for each ε, $0 < \varepsilon \le \varepsilon_0$, there exists η, $0 < \eta \ll \varepsilon$, such that*

$$G_| : \overline{B}_\varepsilon^m \cap G^{-1}(\overline{B}_\eta^p \setminus \{0\}) \to \overline{B}_\eta^p \setminus \{0\} \tag{9}$$

is the projection of a locally trivial smooth fibration.

Corollary 2.7. [Ma, Existence of the tube fibration] *Given G with the conditions of Theorem 2.6, for any small enough $\varepsilon > 0$, there exists η, $0 < \eta \ll \varepsilon$, such that*

$$G_| : \overline{B}_\varepsilon^m \cap G^{-1}(S_\eta^{p-1}) \to S_\eta^{p-1}$$

is the projection of a locally trivial smooth fibration.

In this case we also denote $M_G = \overline{B}_\varepsilon^m \cap G^{-1}(S_\eta^{p-1})$ and also call it the Milnor tube.

EXAMPLE 2.8. Let $G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ given by $G(x, y, z) = (xy, xz)$. Consider $\mathbf{v} := (x, y, z)$. One has that

$$\mathrm{J}G(\mathbf{v}) = \begin{bmatrix} y & x & 0 \\ z & 0 & x \end{bmatrix}$$

and

$$\mathrm{J}G(\mathbf{v})[\mathrm{J}G(\mathbf{v})]^t = \begin{bmatrix} x^2 + y^2 & yz \\ yz & x^2 + z^2 \end{bmatrix}$$

where $\mathrm{J}G(\mathbf{v})$ and $[\mathrm{J}G(\mathbf{v})]^t$ denote the Jacobian matrix of G in $\mathbf{v}$ and its transpose, respectively. We know that $\operatorname{Sing} G = \{\det(\mathrm{J}G(\mathbf{v})[\mathrm{J}G(\mathbf{v})]^t) = 0\}$ thus $\operatorname{Sing} G = \{x = 0\}$. Since

$$V_G = \{x = 0\} \cup \{y = z = 0\}$$

one gets that $\operatorname{Disc} G = \{0\}$. Now to compute the Milnor set $M(G)$ let us consider the matrix

$$B(\mathbf{v}) := \begin{bmatrix} y & x & 0 \\ z & 0 & x \\ x & y & z \end{bmatrix}.$$

The Milnor set $M(G) = \{\mathbf{v} \in \mathbb{R}^3 \mid \det(B(\mathbf{v})) = 0\}$. Consequently,

$$M(G) = \{x = 0\} \cup \{x^2 - y^2 - z^2 = 0\},$$

and G satisfies Condition (8). Therefore, by Theorem 2.6, G has a Milnor tube fibration.

In Figure 4 below one can see that the Milnor tube M_G consists of two connected components. Compare with Figure 1.

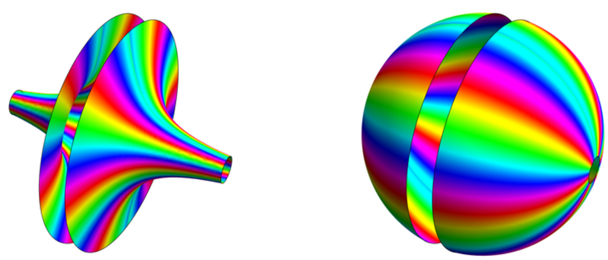


FIGURE 4. Milnor tube and Milnor sphere fibrations for $G(x, y, z) = (xy, xz)$.

2.4. **Existence of the Sphere fibration.** Several authors have worked on the problem of fibration over spheres in the real setting, for isolated and non-isolated singularities, e.g. [A1, ACT1, AT1, CSS1, CSS3, RA, RSV]. In [ACT1, AT1, AT2] the authors generalized all previous results as we describe below. In order to explain their main results, define the map $\Psi : \mathbb{R}^m \setminus V_G \to S^{p-1}$ through the diagram:

$$\begin{array}{ccc} \mathbb{R}^m \setminus V_G & \xrightarrow{G} & \mathbb{R}^p \setminus \{0\} \\ & \underset{\Psi}{\searrow} & \downarrow \pi_1 \\ & & S^{p-1} \end{array}$$

where π_1 is radial projection: $\pi_1(x) = x/\|x\|$. Given a neighborhood $U \in \mathbb{R}^m$ of 0, define the set of ρ-nonregular points of Ψ as the set

$$M(\Psi) = \{x \in U \setminus V_G \mid \rho \not\pitchfork_x \Psi\}.$$

Definition 2.9. The map germ Ψ *is ρ-regular* when $M(\Psi) = \emptyset$, as a germ of a set at the origin.

The set $M(\Psi)$ was characterized as follows.

Lemma 2.10. [AT1, AT2, ACT1, S] *Let $G := (G_1, \dots, G_p) : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ. Then on the open set $\{G_1(x) \neq 0\}$*[6] *one has that*

$$M(\Psi) = \left\{ x \in U \setminus V_G \,|\, rank \begin{bmatrix} \Omega_2(x) \\ \vdots \\ \Omega_p(x) \\ \nabla \rho(x) \end{bmatrix} < p \right\},$$

where $\Omega_k = G_1 \nabla G_k - G_k \nabla G_1$, for $k = 2, \dots, p$.

Remark 2.11. We notice that for any $x \notin V_G$, if $\rho \pitchfork_x G$ then $\rho \pitchfork_x \Psi$. Hence, $M(\Psi) \subset M(G) \setminus V_G$.

Since the $\rho-$regularity is a measurement of transversality between the normal spaces of the fibers of ρ and Ψ, the set $M(\Psi)$ does not depend on the particular choice of the open set $\{G_1(x) \neq 0\}$. In general, for $G_i(x) \neq 0$, $1 \leq i \leq p$, one can find appropriate generators for the normal space of the fibers $X_y = \Psi^{-1}(y)$, $y = \Psi(x)$, considering the collection of vectors $\Omega_{i,k}(x) = G_i \nabla G_k(x) - G_k \nabla G_i(x)$, $k = 1, 2, 3, \dots, \hat{i}, \dots, p$, where $\hat{i}$ means that the index i is omitted. See [DACA, Lemma 3.3 and Remark 3.4] for more details.

It also follows from [AT1] that the condition $M(\Psi) = \emptyset$ is equivalent to saying that for small enough $\varepsilon > 0$, the projection $\Psi : S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^{p-1}$ is a smooth submersion. However, since the map is not proper (unless the link is empty), it might not be a fibration.

In [ACT1] the authors used Condition (8) to ensure that the map Ψ is a projection of a locally trivial smooth fibration. In this setting where $\operatorname{Disc} G = \{0\}$ their result can be read as:

Theorem 2.12. [ACT1, Theorem 1.3] *Let $G : U \to \mathbb{R}^p$, $m > p \geq 2$ be an analytic map germ such that $codim\, V_G = p$. Suppose G satisfies Condition (8), i.e.,*

$$\overline{M(G) \setminus V_G} \cap V_G \subseteq \{0\}.$$

If Ψ is ρ-regular, then for any ε, $0 < \varepsilon \leq \varepsilon_0$, the map projection

$$\Psi : S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^{p-1} \tag{10}$$

is a locally trivial smooth fibration, independent (up to isotopies) of small enough $\varepsilon > 0$.

Example 2.13 ([Han], p. 35). Let $G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, $G(x, y, z) = (x^2 + y^2, (x^2 + y^2)z)$. By hand calculations, one can see that $\operatorname{Sing} G = V_G = \{x = y = 0\}$, hence $\operatorname{Disc} G = \{0\}$. Moreover, by Lemma 2.10, $M(\Psi) = \emptyset$ and therefore Ψ is ρ-regular. Also, $M(G) = \mathbb{R}^3$,

$$\overline{M(G) \setminus V_G} \cap V_G = V_G \neq \{0\}$$

and Condition (8) fails. Therefore we cannot prove that Ψ is a locally trivial fibration. Indeed, the topological type of the fibers of Ψ changes along S^1; sometimes the fiber is a circle, sometimes the fiber is empty (see Figure 5). This shows that the hypothesis in Theorem 2.12 (or, Theorem 1.3 of [ACT1]) can not be weakened and therefore it is sharp!

Example 2.14 (Revising the sphere fibration for holomorphic functions). Let

$$f : \left(\mathbb{C}^{n+1}, 0\right) \to (\mathbb{C}, 0)$$

be a germ of a holomorphic function. We see that the hypothesis of Theorem 2.12 are naturally satisfied if we consider f as a real map germ from $\mathbb{R}^{2n+2}$ to $\mathbb{R}^2$. Indeed, it is well known that any holomorphic function satisfies the Łojasiewicz inequality

$$\|f(z)\|^\theta \leq c \|\nabla f(z)\|,$$

[6] Here, this set means $\{x \in U \setminus V_G \,|\, G_1(x) \neq 0\}$.

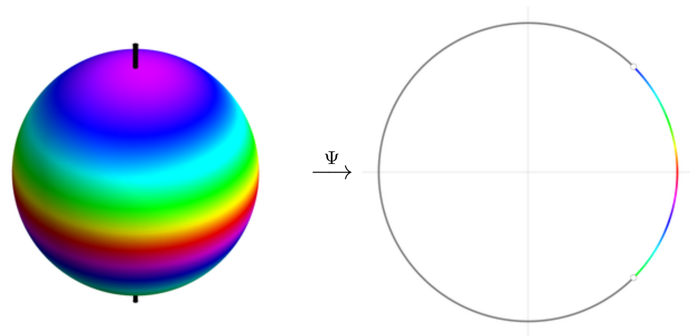


FIGURE 5. Ψ for $G(x,y,z) = ((x^2+y^2),(x^2+y^2)z)$. Colored points on S^1 have circles for fibers, while gray points have empty fibers.

where $0 < \theta < 1$, $c > 0$, and for any z in a small open neighborhood of the origin. So the isolated critical value condition is already satisfied. Moreover, Hamm and Lê in [HL, Theorem 1.2.1 p. 322] have proved that the Łojasiewicz inequality implies that f is Thom regular at V_f and hence f satisfies Condition (8). Finally, by [Mi, Lemma 4.3], one gets that for all $\varepsilon > 0$ small enough, $M(f/\|f\|) = \emptyset$, as a germ of a set. Therefore, from Theorem 2.12 the Milnor fibration on the sphere follows.

Let us point out some important facts.

In the paper [S1] published in 1997, the author used the method known as *Pencil* to construct examples of real analytic map germs with isolated singular point at the origin, which induces the so-called "Open book decomposition on the sphere" (see Definition 3.3), and hence the Milnor fibration on the sphere. Such construction was also used by the authors in [RSV]. In the paper [RA] published in 2005, the authors used this technique and tools from Stratification theory to ensure the existence of the Milnor fibration for real map germs $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^2, 0)$ with $m > 2$. Inspired by [RA], in the paper [AT1] on arXiv (2008) and in the paper [AT2] published in 2010, the authors used the technique of blow-up to provide a generalization of the method for map germs $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with $m > p \geq 2$, and with that, they were able to prove two results which were generalized later in [ACT1].

In order to produce a new class of purely real examples, the authors in [ACT1] used the theory of mixed functions (see [Oka1, Oka2, Oka3] and Chapter 3 of [Ri] for definitions and properties), and proved Theorem 2.16 below. Before stating the theorem, let us consider the following definition.

Definition 2.15. [CT, CT1, CSS3, Oka2, Oka3, PT] A mixed polynomial function $f : \mathbb{C}^n \to \mathbb{C}$ is called *polar weighted-homogeneous* if there are non-zero integers $p_1, ..., p_n$ and d, such that $gcd(p_1, ..., p_n) = 1$ and

$$\sum_{j=1}^{n} p_j \left(\nu_j - \mu_j\right) = d$$

for any monomial of the expansion $f(\mathbf{z}, \bar{\mathbf{z}}) = \sum_{\nu,\mu} c_{\nu,\mu} \mathbf{z}^{\nu} \bar{\mathbf{z}}^{\mu}$. We call $(p_1, ..., p_n)$ the polar weight of f and d the polar degree of f. More precisely, f is polar weighted homogeneous of type $(p_1, ..., p_n; d)$ if and only if it satisfies the following equation for all $\lambda \in S^1$:

$$f(\lambda \cdot (\mathbf{z}, \bar{\mathbf{z}})) = \lambda^d f(\mathbf{z}, \bar{\mathbf{z}}),$$

where the corresponding S^1-action on $\mathbb{C}^n$ is:

$$\lambda \cdot (\mathbf{z}, \bar{\mathbf{z}}) = \left(\lambda^{p_1} z_1, ..., \lambda^{p_n} z_n, \lambda^{-p_1} \bar{z}_1, ..., \lambda^{-p_n} \bar{z}_n\right), \ \lambda \in S^1.$$

Theorem 2.16. [ACT1, Theorem 1.4] *Let $f : \mathbb{C}^n \to \mathbb{C}$ be a non-constant mixed polynomial which is polar weighted-homogeneous, $n \geq 2$, such that $codim_{\mathbb{R}} V_f = 2$. Then for any $\varepsilon > 0$ small enough, the projection*

$$f/\|f\| : S_\varepsilon^{2n-1} \setminus K_\varepsilon \to S^1$$

is a locally trivial smooth fibration, independent (up to isotopies) of small enough $\varepsilon > 0$.

Moreover, they proved the result below where now no control on the projection of the fibration is required outside a neighborhood of the link in the sphere.

Theorem 2.17. [ACT1, Theorem 2.1] *Let $G : U \to \mathbb{R}^p$, $m > p \geq 2$ be an analytic map such that $codim\, V_G = p$ and $\operatorname{Disc} G = \{0\}$ which satisfies Condition (8). Then there exists a locally trivial smooth fibration*

$$S_\varepsilon^{m-1} \setminus K_\varepsilon \to S^{p-1}$$

which is independent of small enough $\varepsilon > 0$, up to isotopies.

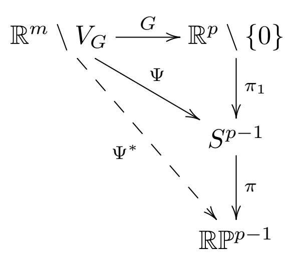
The control of the projection of the fibration is directly related to the ρ-regularity of the map Ψ, as has been seen in Theorem 2.12 and in the discussion that precedes it. This point is the main difference between Theorem 2.12 and Theorem 2.17 (for further details see [ACT1, Section 2]).

2.5. **Fibration on sphere under Thom regularity condition.** In the sequence of papers [CSS1, CSS3], the authors considered maps germs $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m > p \geq 2$, with isolated critical value and satisfying a condition called *d-regularity* which, together with the Thom regularity, ensured the existence of the sphere fibrations. To do that, the authors associated to G *a pencil*, as we explain below. We follow the notations and the construction as described in the paper [CSS1], published in 2010.

For each $l \in \mathbb{RP}^{p-1}$ consider the line $\mathcal{L}_l \subset \mathbb{R}^p$ through the origin and set

$$X_l = \{x \in U | G(x) \in \mathcal{L}_l\}.$$

In particular, if we consider the commutative diagram



where π_1 is radial projection and π is the canonical double covering, then $X_l = (\Psi^*)^{-1}(l) \cup V_G$.

Each X_l is a real analytic variety that contains V_G, and since G has an isolated critical value, then each $X_l \setminus V_G$ is either empty or it is an $(m-p+1)$-dimensional smooth submanifold of U. The family $\left\{X_l : l \in \mathbb{RP}^{p-1}\right\}$ is called *the canonical pencil* of G.

Definition 2.18. [CSS1, Definition of d-regularity] The map G is said to be *d-regular* at 0 if there exist a metric d induced by some positive-definite quadratic form and an $\varepsilon > 0$ such that every sphere (for the metric d) of radius $\leq \varepsilon$ centered at 0 meets each $X_l \setminus V_G$ transversely, whenever the intersection is not empty. We shall also say that G is d-regular with respect to the metric d.

In order to study the existence of Milnor fibrations associated to a map G, the authors introduced an auxiliary function $\mathfrak{G} : B^m_\varepsilon \setminus V_G \to B^p_\varepsilon$ called the *Spherification map* of G. This function was defined by

$$\mathfrak{G}(x) = \|x\| \frac{G(x)}{\|G(x)\|}$$

and it was used to characterize the d-regularity as follows.

Proposition 2.19. [CSS1, Proposition 3.2] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be an analytic map germ with an isolated critical value at the origin. The following statements are equivalent:*

(i) The map G is d-regular at 0.

(ii) For each sphere S^{m-1}_ε of small enough radius $\varepsilon > 0$, the restriction map

$$\mathfrak{G} : S^{m-1}_\varepsilon \setminus V_G \to S^{p-1}_\varepsilon$$

is a submersion.

(iii) The spherification map $\mathfrak{G}$ is a submersion at each $x \in B^m_\varepsilon \setminus V_G$.

(iv) The map $\Psi_| : S^{m-1}_\varepsilon \setminus K_\varepsilon \to S^{p-1}$ is a submersion for any small enough sphere S^{m-1}_ε.

This proposition shows that when d is the square of the Euclidean metric, then d-regularity of G is equivalent to ρ-regularity of Ψ. The main result of [CSS1] is the following.

Theorem 2.20. [CSS1, Theorem 5.3] *Assume either V_G is a point or $\dim V_G > 0$ and G has the Thom regularity. The following statements are equivalent:*

(i) The map G is d-regular at 0.

(ii) One has a commutative diagram of smooth fiber bundles on $S^{m-1}_\varepsilon \setminus K_\varepsilon$ for any small enough sphere S^{m-1}_ε:

$$\begin{array}{ccc} S^{m-1}_\varepsilon \setminus K_\varepsilon & \xrightarrow{\phi} & S^{p-1} \\ & {\scriptstyle\psi}\searrow \quad \swarrow{\scriptstyle\pi} & \\ & \mathbb{RP}^{p-1} & \end{array}$$

where $\psi := (G_1(x) : \cdots : G_p(x))$ and $\phi := G/\|G\| : S^{m-1}_\varepsilon \setminus K_\varepsilon \to S^{p-1}$ is the Milnor fibration on G.

(iii) For any small enough sphere S^{m-1}_ε, the restriction $\mathfrak{G} : S^{m-1}_\varepsilon \setminus V_G \to S^{p-1}_\varepsilon$ is a smooth fiber bundle and this is the Milnor fibration ϕ up to multiplication by a constant.

2.6. **Comparing the fibration structure on spheres under Thom regularity at V_G and Condition (8).** One can show that if a map germ G is Thom regular at V_G then G satisfies Condition (8). Example 2.21 below shows that the converse in not true in general. Therefore, Theorem 2.12 is more general than Theorem 2.20.

Example 2.21. [Han, Example 1.4.9] Consider $G(x, y, z) = (x, y(x^2 + y^2) + xz^2)$ in three real variables. One has that $\operatorname{Sing} G = V_G = \{x = y = 0\}$ and $M(G) = \{x = y = 0\} \cup \{z = 0\}$. Hence, $\overline{M(G) \setminus V_G} \cap V_G = \{0\}$ and Condition (8) holds. We claim that $M(\Psi) = \emptyset$. Indeed, let $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$ and consider the matrix

$$B(\mathbf{v}) := \begin{bmatrix} \Omega_2(\mathbf{v}) \\ \mathbf{v} \end{bmatrix},$$

where

$$\Omega_2(\mathbf{v}) = (x(2xy + z^2) - y(x^2 + y^2) - xz^2, x(x^2 + 3y^2), 2x^2z).$$

By Lemma 2.10,

$$M(\Psi) = \{\mathbf{v} \in B^3_\varepsilon \setminus V_G \mid \det\big(B(\mathbf{v})[B(\mathbf{v})]^t\big) = 0\}.$$

Since

$$\det\big(B(\mathbf{v})[B(\mathbf{v})]^t\big) = (x^2 + y^2)(x^6 + 3x^4y^2 + 5x^4z^2 - 8x^3yz^2 + 3x^2y^4 + 6x^2y^2z^2 + y^6 + y^4z^2)$$

and $M(\Psi) \subset M(G) \setminus V_G$, then $M(\Psi) = \emptyset$. By Theorem 2.12, we get the sphere fibration $\Psi : S^{m-1}_\varepsilon \setminus K_\varepsilon \to S^{p-1}$.

On the other hand, for any value $z \neq 0$, consider the point $p = (0, 0, z)$, $\mathrm{T}_pV_G = \mathrm{span}\,\{(0, 0, 1)\}$, and the sequence $p_n = (\frac{1}{n}, 0, z)$ which converges to p. One has that $\mathrm{T}_{p_n}G^{-1}(G(p_n)) = \mathrm{span}\,\{v_n\}$, where

$$v_n = \left(0, \frac{-2z}{\sqrt{4z^2 + \frac{1}{n^2}}}, \frac{1}{\sqrt{4z^2n^2 + 1}}\right);$$

hence $v_n \to (0, \pm1, 0)$, where plus and minus depends on the sign of z. Therefore,

$$\lim_n (T_{p_n}G^{-1}(G(p_n))) = \mathrm{span}\,\{(0, 1, 0)\}$$

and G is not Thom regular at V_G.

Remark 2.22. Another source of examples of maps with Milnor tube and sphere fibration without the Thom regularity can be found in the recent paper [Ri2].

3. Open Book Structures on semialgebraic sets

The *classical open book structures* with smooth binding appear in the literature relative to 3-manifolds and in different branches of mathematics under many names like *Lefschetz pencils* (Algebraic and Symplectic Geometry), *fibered links*, *Neuwirth-Stallings pairs*, or *spinnable structures* (Topology).

As explained by the authors in [AT1], this consists of a pair (K, θ) where $K \subset M$ is a 2-codimensional submanifold of a real manifold M and $\theta : M \setminus K \to S^1$ with $S^1 := \partial B^2$, is a locally trivial smooth fibration such that K admits a neighborhood N diffeomorphic to $B^2 \times K$ for which K is identified with $\{0\} \times K$ and the restriction $\theta_{|N\setminus K}$ is the following composition with the natural projections:

$$N \setminus K \overset{\text{diffeo}}{\simeq} (B^2 \setminus \{0\}) \times K \overset{\text{proj}}{\to} B^2 \setminus \{0\} \overset{s/\|s\|}{\to} S^1. \tag{11}$$

In that case, K is the *binding* and the closure of the fibers of θ are the pages of the open book.

As described in the introduction, an important example of classical open book structure on a small sphere S^{2n-1}_ε can be obtained if we consider a germ of a holomorphic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, under the condition that $\mathrm{Sing}\, f = \{0\}$.

Milnor noted that if $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p \geq 2$, has an isolated critical point at $0 \in \mathbb{R}^m$, then for any small enough $\varepsilon > 0$, the complement $S^{m-1}_\varepsilon \setminus K_\varepsilon$ of the link K_ε is the total space of a smooth fiber bundle over the unit sphere S^{p-1}. In such a case, one can conclude from Milnor's comment that the sphere S^{m-1}_ε is endowed with an open book structure with binding K_ε, where now the binding is of *higher codimension* $p \geq 2$ instead of 2.

These structures were extended later, as follows:

Definition 3.1. [AT2, Definition 2.1] A *higher open book structure* of a real manifold M is a pair (K, θ), where K is a p-codimensional non-empty submanifold of M and $\theta : M \setminus K \to S^{p-1}$ is a locally trivial smooth fibration over the sphere $S^{p-1} = \partial B^p$, such that K admits a neighborhood N diffeomorphic to $B^p \times K$ for which K is identified to $\{0\} \times K$ and the restriction $\theta_{|N\setminus K}$ is the composition

$$N \setminus K \overset{\text{diffeo}}{\simeq} (B^p \setminus \{0\}) \times K \overset{\text{proj}}{\to} B^p \setminus \{0\} \overset{s/\|s\|}{\to} S^{p-1}.$$

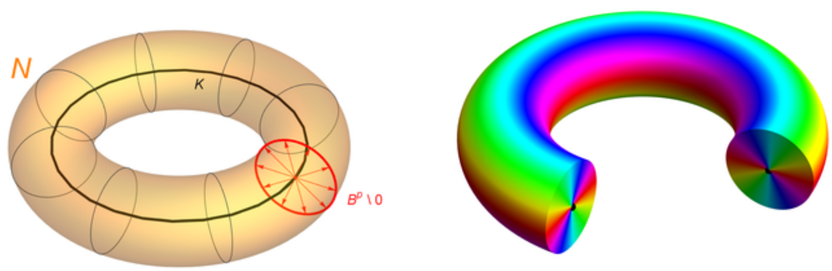


FIGURE 6. Left: an example of N and K from Definition 3.1. Right: a cross section of the corresponding open book structure.

REMARK 3.2. In this case E. Looijenga in [Lo1] called this structure a Neuwirth-Stallings pair, or NS-pair, and denoted them by $(S_\varepsilon^{m-1}, K_\varepsilon)$.

In [AT1], the authors presented a general criterion for the existence of these structures associated to a real map germ G with isolated critical point at $0 \in \mathbb{R}^m$ and with $\theta = G/\|G\|$ (see [AT1, Theorem 1.1]). In [AT2], they focused on the existence of higher open book structures defined by map germs which satisfies the condition $\operatorname{Sing} G \cap V_G \subset \{0\}$, which is the most general one under which open book structures with non-singular binding K may exist. Finally, in [ACT1], the authors introduced the notion of *singular open book structure* as follows.

Definition 3.3. [ACT1, Definition 1.1]. The pair (K, θ) is a *higher open book structure with singular binding* on an analytic manifold M of dimension $m - 1 \geq p \geq 2$, if $K \subset M$ is a singular real subvariety of codimension p and $\theta : M \setminus K \to S^{p-1}$ is a locally trivial smooth fibration such that K admits a neighborhood N for which the restriction $\theta_{|N\setminus K}$ is the composition $N \setminus K \overset{h}{\to} B^p \setminus \{0\} \overset{s/\|s\|}{\to} S^{p-1}$, where h is a locally trivial fibration.

They investigated the case when V_G contains non-isolated singularities and thus the link K_ε is not a manifold. Under the hypothesis of Theorem 2.12, they ensured the pair (K_ε, Ψ) is an open book structure with singular binding on S_ε^{m-1} having extended all previous results related to the existence of open book structures of [AT1] and [AT2]. In addition, they found important classes of genuine real analytic mappings which yield such structures (see for instance Theorem 2.16).

REMARK 3.4. Based on the results obtained in [ACT1], the authors in [ACT2] considered polynomial maps $G : \mathbb{R}^m \to \mathbb{R}^p$, $m \geq p \geq 1$. Under certain adapted conditions defined in terms of the Milnor sets $M(G)$ and $M(\Psi)$, they ensured the existence of an *open book decomposition at infinity with singular binding* (i.e., on spheres of large enough radius R).

Motivated by recent techniques developed in [ACT1, AT1, AT2] and [ACT2], the authors in [DACA] guaranteed the existence of a fibration structure associated to a more general class of maps and sets. Actually, they have considered C^2-semi-algebraic maps $G : \mathbb{R}^m \to \mathbb{R}^p$ and embedded compact semi-algebraic manifolds without boundary $W \subset \mathbb{R}^m$ of dimension $n-1 \geq p$. In this new setting, they introduced sufficient conditions in order to ensure the existence of an open book structure on W and, as a consequence, extended both previous open book structures on local and global cases. For that, the first step was to consider an appropriate extension of the Milnor set as below.

Definition 3.5. [DACA]
Let $G : \mathbb{R}^m \to \mathbb{R}^p$ be a C^2-semi-algebraic map, $W \subset \mathbb{R}^m$ a compact semi-algebraic $(n-1)$-dimensional submanifold embedded in $\mathbb{R}^m$ and

$$\bar{G} := \frac{G}{\|G\|} : \mathbb{R}^m \setminus V_G \to S^{p-1}.$$

Consider $\bar{G}_{|W} : W \setminus V_W(G) \to S^{p-1}$ where $V_W(G) = V_G \cap W$, and

(i) Σ_G the set of critical points of G;
(ii) $\Sigma_{\bar{G}}$ the set of critical points of $\bar{G}$;
(iii) Σ_G^W the set of critical points of $G_{|W}$;
(iv) $\Sigma_{\bar{G}}^W$ the set of critical points of $\bar{G}_{|W}$.

The map G satisfies the *generalized Milnor condition* (b) whenever $\overline{\Sigma_G^W \setminus V_W(G)} \cap V_W(G) = \emptyset$. Moreover, G satisfies the *generalized Milnor condition* (a) when $\Sigma_{\bar{G}}^W = \emptyset$.

With the notations above, the authors in [DACA] stated and proved the following result.

Theorem 3.6 (Structural Theorem)**.** *Let $G : \mathbb{R}^m \to \mathbb{R}^p$ be a C^2-semi-algebraic map such that G satisfies the generalized Milnor condition (a). Then the following statements are equivalent:*

(i) $\bar{G}_{|W}$ is a locally trivial smooth fibration induced by G on W;
(ii) The map G satisfies the generalized Milnor condition (b).

Let us point out that the proof of Theorem 3.6 follows similar arguments used in [ACT1, ACT2, AT2], and consequently also guarantee the existence of an open book structure on W. The Structural Theorem generalizes the analogues for local and global cases.

In addition, considering the canonical projection $\pi_j : \mathbb{R}^p \to \mathbb{R}^{p-1}$ for $p \geq 2$, and

$$\pi_j(x_1, \dots, x_p) = (x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_p),$$

where $j = 1, \dots, p$, the authors also have shown that the composition $\hat{G}_j := \pi_j \circ G : \mathbb{R}^m \to \mathbb{R}^{p-1}$ provides a new open book structures for W, (see [DACA, Lemma 3.5]). Moreover, the fibers of new and old structure are related as follows: if F_G and $F_{\hat{G}_j}$ are the fibers of locally trivial smooth fibrations induced by G and $\hat{G}_j$ on W, respectively, then $F_{\hat{G}_j}$ is homotopically equivalent to the product $F_G \times [0,1]$. This ensures that one can, without loss of generality, reduce the study of the topology of the fibers of a C^2-semi-algebraic map $G = (G_1, \dots, G_p) : \mathbb{R}^m \to \mathbb{R}^p$ satisfying generalized Milnor conditions to the study of the singularity type of $G_i, i = 1, \dots, p$, i.e., any coordinate function.

4. Positive dimensional discriminant set

Let

$$G : U \subset \mathbb{R}^m \to \mathbb{R}^p, \quad m > p \geq 2,$$

be a representative of a map germ $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with positive dimensional discriminant set $\operatorname{Disc} G$. Consider a Whitney stratification $\mathbb{W} = \{\mathcal{C}_j\}_{j=1}^r$ of $\operatorname{Disc} G$ with the origin a single stratum. Let us assume that the complement $\mathbb{R}^p \setminus \operatorname{Disc} G$ is equal to union $\cup_{i=1}^k \mathcal{D}_i$, where on each connected component $\mathcal{D}_i$ the topology of the fibers of G does not change.

Let us consider the following situation: for $i \neq j$ such that $\mathcal{C}_k \subset \overline{\mathcal{D}_i} \cap \overline{\mathcal{D}_j} \setminus \{0\}$, let $p_i \in \mathcal{D}_i$ and $p_j \in \mathcal{D}_j$ and let $l_{i,j}$ be a path connecting them, with $l_{i,j}$ intersecting $\mathcal{C}_k$ once and is in general position[7] (see Figure 7).

The problem is: How do we describe the topological changes of the topology of the fibers over p_i and over p_j as we travel along $l_{i,j}$?

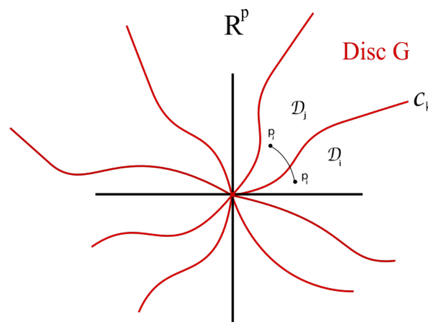


Figure 7. Positive dimensional discriminant set and the complementary set $\mathbb{R}^p \setminus \operatorname{Disc} G$.

Maybe this problem is too hard to approach as it is stated. However, it motivates one to think of a natural way to extend the Milnor fibrations for map germs with positive dimensional discriminant sets as done by H. Hamm in [Ham] (see Theorem 1.3).

As explained in detail in [ART1] and [ART2], in this new setting the following problems have to be taken into account so that the fibration problem can be well posed:

a) The local fibration must be independent of the small enough neighborhood data, like in Equations (1) and (5). This does not come automatically for map germs with positive dimensional discriminant set outside the ICIS case (see Examples 4.2 and 2.13).
b) The image of the map germ G may not be a neighborhood of $\{0\}$ in $\mathbb{R}^p$ (see Example 5.9). Moreover, it may not be independent of the radius ε of the ball $B_\varepsilon^m \subset \mathbb{R}^m$, and thus the image of G may not be well defined as a set germ in $(\mathbb{R}^p, 0)$ (see Examples 4.2 and 2.13).
c) The set $G(\operatorname{Sing} G)$ may not be well defined as a set germ. In case the image $G(\operatorname{Sing} G)$ of the singular locus is a set germ, and when the image $\operatorname{Im} G$ is a set germ too and has a

[7] It means that the tangent vector of $l_{i,j}$ at the point of intersection is not contained in the tangent space of the stratum $\mathcal{C}_k$

boundary[8] which contains the origin $\{0\}$, then in this new setting it seems appropriate that the "discriminant set" $\operatorname{Disc} G$ should contain this boundary (see Definition 4.7).

Recall that, given subsets $V, W \subset \mathbb{R}^p$ containing the origin and denoting $(V, 0)$ and $(W, 0)$ their respective germs at $\{0\}$, then one has $(V, 0) = (W, 0)$ as a germ of a set if and only if there exists some open ball $B_\varepsilon \subset \mathbb{R}^p$ centered at 0 and of radius $\varepsilon > 0$ such that $V \cap B_\varepsilon = W \cap B_\varepsilon$.

Definition 4.1. [ART1] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$, be a continuous map germ. We say that the image $G(K)$ of a set $K \subset \mathbb{R}^m$ containing 0 is a *well-defined set germ* at $0 \in \mathbb{R}^p$ if, for any open balls $B_\varepsilon, B_{\varepsilon'}$ centered at 0, with $\varepsilon, \varepsilon' > 0$, we have the equality of germs $[G(B_\varepsilon \cap K)]_0 = [G(B_{\varepsilon'} \cap K)]_0$.

Whenever the images $\operatorname{Im} G$ and $G(\operatorname{Sing} G)$ are well-defined as germs, we say that G is a *nice map germ*.

Example 4.2. [ART1, Example 2.1] Let $G : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$, $G(x, z) = (x, xz)$. For the 2-disks

$$D_t := \{|x| < t, |z| < t\}$$

as a basis of open neighborhoods of 0 for $t > 0$, we get that the image $A_t := G(D_t)$ is the full angle with vertex at 0, having the horizontal axis as bisector, and of slope $< t$. Since the relations defining A_t depend of t, it means that the image of G is not well-defined as a germ (see Figure 8). A similar behavior happens over $\mathbb{C}$ instead of $\mathbb{R}$.

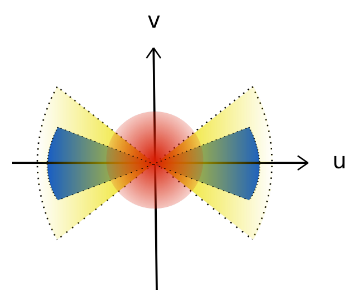


Figure 8. Images A_{t_1} and A_{t_2} with $t_1 \neq t_2$ in the yellow and blue color, respectively.

Remark 4.3. The authors in [ART1] point out that even if the image $\operatorname{Im} G$ of a map G is well-defined as a germ, the restriction of G to some subset might not be (see [ART1, Remark 2.3]). Therefore, in the definition of a nice map germ, it is necessary to ask that the set $G(\operatorname{Sing} G)$ is well-defined as a germ as well.

Example 4.4. Given $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p \geq 2$ with $\operatorname{Disc} G = \{0\}$. If Condition (8) holds true, then G is a nice map germ (see [Ma, Corollary 4.7]). In particular, any non-constant germ of a holomorphic function is nice.

Remark 4.5. One can do similar calculations as in Example 4.2 on the map germ

$$G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0), \quad G(x, y, z) = (x^2 + y^2, (x^2 + y^2)z)$$

[8][ART1]: Whenever $\operatorname{Im} G$ is well-defined as a set germ, its boundary $\partial\overline{\operatorname{Im} G} := \overline{\operatorname{Im} G} \setminus \operatorname{int}(\operatorname{Im} G)$ is a closed subanalytic proper subset of $\mathbb{R}^p$, where $\operatorname{int} A := \mathring{A}$ denotes the p-dimensional interior of a subanalytic set $A \subset \mathbb{R}^p$ (hence it is empty whenever $\dim A < p$), and $\overline{A}$ denotes the closure of it. One considers here $\partial\overline{\operatorname{Im} G}$ as a set germ at $0 \in \mathbb{R}^p$; this is of course empty if (and only if) the equality $(\operatorname{Im} G, 0) = (\mathbb{R}^p, 0)$ holds.

(Example 2.13), and find that $\operatorname{Im} G$ is not well-defined as a set germ, and thus G is not nice. Note that while $\operatorname{Disc} G = \{0\}$, Condition (8) is not satisfied, so we cannot conclude that G is nice (like we could in Example 4.4).

EXAMPLE 4.6. In [ART1] the authors found sufficient conditions for an analytic map germ with positive dimensional discriminant set to be a nice germ and have introduced a good class of maps with this property, namely the map germs of type

$$f\bar{g} : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0),$$

where $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ are holomorphic germs such that the meromorphic function f/g is irreducible.

The authors in [ART1] gave an appropriate definition of the discriminant set as the locus where the topology of the fibers may change.

Definition 4.7. For a nice map germ G, the *discriminant* is the following set

$$\operatorname{Disc}^* G := \overline{G(\operatorname{Sing} G)} \cup \partial \overline{\operatorname{Im} G} \tag{12}$$

which is a closed subanalytic set of dimension strictly less than p, well-defined as a germ since G is nice.

Usually the discriminant set $\operatorname{Disc} G$ is just $G(\operatorname{Sing} G)$. However, in this new setting where $\dim \operatorname{Disc} G > 0$, the complement of the discriminant set may consist of several connected components through the origin (see Figure 7), and hence the base space of the fibration may not be a connected space and the topological type of the fibers may not be unique. Consequently, the classical definition of discriminant is not sufficient to detect the change of the topological type of the fibers. We also note that when $\operatorname{Disc} G = \{0\}$ (like in the previous sections) and G satisfies Condition (8), then $\operatorname{Disc}^* G = \operatorname{Disc} G$.

5. Singular Milnor tube fibration

Definition 5.1. Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$, $m \geq p > 0$, be a non-constant analytic nice map germ. We say that G has a *Milnor-Hamm (tube) fibration* if, for any $\varepsilon > 0$ small enough, there exists $0 < \eta \ll \varepsilon$ such that the restriction:

$$G_| : B^m_\varepsilon \cap G^{-1}(B^p_\eta \setminus \operatorname{Disc}^* G) \to B^p_\eta \setminus \operatorname{Disc}^* G \tag{13}$$

is a locally trivial fibration over each connected component $\mathcal{C}_i$ included in $B^p_\eta \setminus \operatorname{Disc}^* G$, such that it is independent of the choices of ε and η up to diffeomorphisms.

In order to guarantee the existence of fibration (13), the authors in [ART1] considered the following condition

$$\overline{M(G) \setminus G^{-1}(\operatorname{Disc}^* G)} \cap V_G \subseteq \{0\} \tag{14}$$

where the closure of the analytic set $M(G) \setminus G^{-1}(\operatorname{Disc}^* G)$ is considered as a set germ at the origin. Condition (14) is a direct extension of Condition (8). Therefore, the next result is a natural extension of Theorem 2.6 for the case where $\dim \operatorname{Disc}^* G > 0$.

Theorem 5.2. [ART1, Lemma 3.3] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant nice analytic map germ, $m \geq p > 0$. If G satisfies Condition (14), then G has a Milnor-Hamm (tube) fibration (13).*

A similar type of fibration but with the stronger assumptions of Thom regularity have been studied in [CGS]. In the article, the authors considered a real analytic map germ $G : (U, 0) \to (\mathbb{R}^p, 0)$, where $U \subset \mathbb{R}^m$ is an open set, $m > p \geq 2$, G has a critical point at 0, and V_G has dimension ≥ 2. They considered a fixed closed ball $\bar{B}^m_\varepsilon$ as a stratified set with strata the interior B^m_ε and the boundary $S^{m-1}_\varepsilon = \partial\bar{B}^m_\varepsilon$, the restriction map $G_| : \bar{B}^m_\varepsilon \to \mathbb{R}^p$ and its discriminant set as $\Delta^\varepsilon_G := G(\mathcal{C}(B^m_\varepsilon) \cup \mathcal{C}(S^{m-1}_\varepsilon))$, where $\mathcal{C}(B^m_\varepsilon)$ and $\mathcal{C}(S^{m-1}_\varepsilon)$ stand for the set of critical points of G on the open ball and on the sphere, respectively. With these notations, they used the Thom Isotopy Theorem to get that the map

$$G_| : \bar{B}^m_\varepsilon \cap G^{-1}(\mathbb{R}^p \setminus \Delta^\varepsilon_G) \to \mathbb{R}^p \setminus \Delta^\varepsilon_G$$

is a locally trivial fibration (see [CGS, Proposition 2.1]). As a consequence for each fixed $\varepsilon > 0$ and $\eta > 0$ they obtained the following locally trivial fibration [CGS, Corollary 2.2]:

$$G_| : \bar{B}^m_\varepsilon \cap G^{-1}(B^p_\eta \setminus \Delta^\varepsilon_G) \to B^p_\eta \setminus \Delta^\varepsilon_G. \tag{15}$$

In order to ensure that the fibration (15) does not depend on $\varepsilon > 0$, they considered Whitney stratifications $\mathbb{W}$ and $\mathbb{S}$ of U and $G(U)$, respectively, such that V_G is a union of strata and both stratifications give the stratification of G. They further assume that G satisfies the Thom a_f-property with respect to such stratification of G i.e., $(\mathbb{W}, \mathbb{S}, G)$ is a Thom stratified mapping (see [CGS, Proposition 2.4]).

Since the Thom a_f-property implies Condition (14), the examples below show that [CGS, Proposition 2.4] under the nice condition is a particular case of Theorem 5.2.

Example 5.3. [ART1, Example 5.3] Let F be one of the mixed functions:

1) $F_1(x, y) = xy\bar{x}$ from [ACT1],
2) $F_2(x, y, z) = (x + z^k)\bar{x}y$ for a fixed $k \geq 2$ from [PT],
3) $F_3(w_1, \dots, w_n) = w_1 \left(\sum_{j=1}^k |w_j|^{2a_j} - \sum_{t=k+1}^n |w_t|^{2a_t}\right)$ from [Oka4].

They are all polar weighted-homogeneous and thus, by [ACT1, Theorem 1.4], one obtains that $\operatorname{Disc}^* F_j = \{0\}$ and that F_j is nice and has Milnor tube fibration. It was also proved in the respective papers that F_j is not Thom regular.

Let $G_j := (F_j, g)$, where $g(v) = v$ and note that $\operatorname{Disc}^* G_j = \{0\} \times \mathbb{C}$. By [ART1, Lemma 5.1] the map G_j satisfies Condition (14) and therefore, by Theorem 5.2, G_j has a Milnor-Hamm (tube) fibration. However, again by [ART1, Lemma 5.1] G_j is not a Thom stratified mapping.

Summing up, the authors in [ART1] have shown that the Thom regularity of the map G may fail whereas the Milnor-Hamm (tube) fibration still exists. Moreover, they present several classes of map germs with Milnor-Hamm fibration by introducing a weaker type of Thom regularity condition called *∂-Thom regularity condition.*

Remark 5.4. In article [MS], the authors defined a type of tube fibration in a more general setting and presented a necessary and sufficient condition on the fibers of coordinate functions to ensure its existence [MS, Proposition 2.5]. However, since their main objective was to study the topology of real analytic map germs with isolated critical value, i.e., $\operatorname{Disc} G = \{0\}$, they did not present examples in the more general case.

5.1. **Singular Milnor tube fibration.** In [ART1] the authors have defined a general notion of *stratified tube fibration* by considering all singular fibers over the stratified discriminant, and they have shown that such structure is a natural generalization of Milnor-Hamm fibration. In that case, the tube fibration is actually a collection of finitely many fibrations over path-connected subanalytic sets. In order to make this notion more precise, they made use of the classical stratification theory (see e.g. [GLPW]), and they considered the following definitions.

Definition 5.5. [ART1] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant analytic map germ, $m \geq p > 1$. Let $G_\varepsilon : B^m_\varepsilon \to \operatorname{Im} G_\varepsilon$ denote the restriction of G to a small ball. Consider a locally finite subanalytic Whitney stratifications $(\mathbb{W}, \mathbb{S})$ of the source of G_ε and of its target, respectively, such that $\overline{\operatorname{Im} G_\varepsilon}$ is a union of strata, that $\operatorname{Disc}^* G_\varepsilon$ is a union of strata, and that G_ε is a stratified submersion. In particular every stratum is a non-singular, open and connected subanalytic set at the respective origin, and moreover:

(i) The image by G_ε of a stratum of $\mathbb{W}$ is a single stratum of $\mathbb{S}$,

(ii) The restriction $G_| : W_\alpha \to S_\beta$ is a submersion, where $W_\alpha \in \mathbb{W}$, and $S_\beta \in \mathbb{S}$.

One calls $(\mathbb{W}, \mathbb{S})$ a *regular stratification of the map germ G.*

We say that G is *S-nice* whenever all the above subsets of the target are well-defined as subanalytic germs, independent of the radius ε.

Definition 5.6. [ART1] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant S-nice analytic map germ. We say that G has a *singular Milnor tube fibration* relative to some regular stratification $(\mathbb{W}, \mathbb{S})$, which is well-defined as a germ at the origin by our assumption, if for any small enough $\varepsilon > 0$ there exists $0 < \eta \ll \varepsilon$ such that the restriction:

$$G_| : B^m_\varepsilon \cap G^{-1}(B^p_\eta \setminus \{0\}) \to B^p_\eta \setminus \{0\} \tag{16}$$

is a stratified locally trivial fibration which is independent, up to stratified homeomorphisms, of the choices of ε and η.

The authors clarified the notion of stratified fibration by saying that *stratified locally trivial fibration* meant that for any stratum S_β, the restriction $G_{|G^{-1}(S_\beta)}$ is a locally trivial fibration.

In order to ensure the existence of stratified fibration (16), they defined the *stratwise Milnor set* $M(G)$ with respect to the stratifications $\mathbb{W}$ and $\mathbb{S}$, as the union of the Milnor sets of the restrictions of G to each stratum. Namely, $M(G) := \sqcup_\alpha M(G_{|W_\alpha})$, where

$$M(G_{|W_\alpha}) := \left\{ x \in W_\alpha \mid \rho_{|W_\alpha} \not\pitchfork_x G_{|W_\alpha} \right\},$$

with $W_\alpha \in \mathbb{W}$ the germ at the origin of some stratum, and $\rho_{|W_\alpha}$ the restriction of the distance function ρ to the subset W_α (see [ART1, Definition 6.4]). They then considered the following condition:

$$\overline{M(G) \setminus V_G} \cap V_G \subset \{0\}. \tag{17}$$

which restricted to $M(G) \setminus G^{-1}(\operatorname{Disc}^* G)$ is just Condition (14). Finally, with the notations and definitions above, the main result in this new setting is the following:

Theorem 5.7. [ART1] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant S-nice analytic map germ. If G satisfies Condition* (17)*, then G has a singular Milnor tube fibration* (16)*.*

The corollary below says that the singular Milnor tube fibration (16) generalizes the previous Milnor-Hamm fibration.

Corollary 5.8. [ART1] *Under the hypotheses of Theorem 5.7, the map G has a Milnor-Hamm fibration over $B^p_\eta \setminus \operatorname{Disc}^* G$, with nonsingular Milnor fiber over each connected component.*

Example 5.9. [ART1] Let $G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$, $G(x, y, z) = (xy, z^2)$. One has:

$V_G = \{x = z = 0\} \cup \{y = z = 0\}$	$\operatorname{Im} G = \mathbb{R} \times \mathbb{R}_{\geq 0} \subsetneq \mathbb{R}^2$
$\operatorname{Sing} G = \{x = y = 0\} \cup \{z = 0\}$	$G(\operatorname{Sing} G) = \{0\} \times \mathbb{R}_{\geq 0} \cup \mathbb{R} \times \{0\}$
$\operatorname{Disc}^* G = \{(0, \beta) \mid \beta \geq 0\} \cup \{(\lambda, 0) \mid \lambda \in \mathbb{R}\}$	$G^{-1}(\operatorname{Disc}^* G) = \{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$
$M(G) = \{x = \pm y\} \cup \{z = 0\}$	$\overline{M(G) \setminus G^{-1}(\operatorname{Disc}^* G)} = \{x = \pm y\}.$

It follows that G is nice and satisfies Condition (14). Indeed to check this, consider

$$p_0 = (x_0, y_0, z_0) \in \overline{M(G) \setminus G^{-1}(\mathrm{Disc}^* G)} \cap V_G.$$

Hence, there exists a sequence $p_n := (x_n, y_n, z_n) \in M(G) \setminus G^{-1}(\mathrm{Disc}^* G)$ such that $p_n \to p_0$ with $p_0 \in V_G$. Consequently, $z_0 = 0$ and $x_n = \pm y_n \neq 0$ because $p_n \notin G^{-1}(\mathrm{Disc}^* G)$. Since $x_0 = \lim x_n = \pm \lim y_n = y_0 = 0$, one concludes that $p_0 = (0, 0, 0)$. Thus G has a Milnor-Hamm fibration by Theorem 5.2. In particular, each fiber consists of four open segments, consisting of hyperbolas sitting in two planes parallel and equal distance to the xy-plane, (see Figure 9).

The complement $\mathbb{R}^2 \setminus \mathrm{Disc}^* G$ consists of 3 connected components. We have: the fiber over $\mathbb{R} \times \mathbb{R}_{<0}$ is empty; the fiber over $\mathbb{R}_{>0} \times \mathbb{R}_{>0}$ and the fiber over $\mathbb{R}_{<0} \times \mathbb{R}_{>0}$ are two non-intersecting hyperbolas, with 4 connected components.

Moreover, it follows that G is S-nice and satisfies Condition (17), thus it has a singular tube fibration by Theorem 5.7. The singular tube fibration fibers over three of the strata of the discriminant as follows: over the positive vertical axis, the fibers are two disconnected components each of which being two intersecting lines; over the positive and the negative horizontal axis, the fibers are both hyperbolas with two components (see Figure 9).

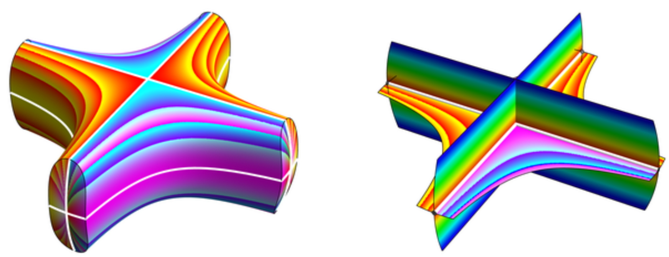


FIGURE 9. The Milnor-Hamm tube fibration (left) and the singular Milnor tube fibration over $\mathrm{Disc}^* G$ (right) for $G(x, y, z) = (xy, z^2)$. Each color scheme is a fibration over a connected component of the codomain.

In order to find good class singularities with the singular Milnor tube fibrations, the authors considered the following condition of regularity which does not require $\mathbb{W}$ to be a Thom regular stratification.

Definition 5.10. [ART1] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant analytic map germ. We say that G is *Thom regular at* V_G if there exists a Whitney stratification $(\mathbb{W}, \mathbb{S})$ like in Definition 5.5 such that 0 is a point stratum in $\mathbb{S}$, that V_G is a union of strata of $\mathbb{W}$, and that the Thom a_g-regularity condition is satisfied at any stratum of V_G.

Then they proved the following result

Theorem 5.11. [ART1] *Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a non-constant S-nice analytic map germ. If G is Thom regular at V_G, $\dim V_G > 0$, then G has a singular Milnor tube fibration (16). In particular, if $V_G \cap \mathrm{Sing}\, G = \{0\}$ and $\dim V_G > 0$, then G has a Milnor-Hamm fibration (13).* $\square$

EXAMPLE 5.12. Let $f, g : \mathbb{C}^2 \to \mathbb{C}$ given by

$$f(x, y) = x^2 + y^2 \quad \text{and} \quad g(x, y) = x^2 - y^2.$$

One has $V_{(f,g)} = \{(0, 0)\}$ and

$$\text{Sing}\,(f, g) = \{x = 0\} \cup \{y = 0\};$$

hence (f, g) is obviously Thom regular at $V_{(f,g)}$. It then follows from [ART1, Theorem 4.3] that $f\bar{g}$ is Thom regular at $V_{f\bar{g}}$ hence, by Theorem 5.11, it has a Milnor-Hamm fibration, and also a singular Milnor tube fibration.

6. Milnor-Hamm sphere fibration

Inspired by the techniques developed by Milnor [Mi] and detailed in [AT2], the authors in [ART2] considered the problem of existence of a fibration structure over small spheres under a general situation when the discriminant $\text{Disc}^* G$ has positive dimension. They introduced the *Milnor-Hamm sphere fibration*, gave natural sufficient conditions of singular maps that shows the fibration exists, and exhibited several such classes of singular maps. They then stated the problem of equivalence with the corresponding tube fibration and they showed how to solve it for some class of maps in the general setting under natural supplementary conditions.

First, the authors introduced a natural condition for a nice map germ G under which it was possible to define the sphere fibrations whenever $\text{Disc}^* G$ is positive dimensional.

Definition 6.1. [ART2] Let $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a real analytic map germ. We say that its discriminant $\text{Disc}^* G$ is *radial* if, as a set germ at the origin, it is a union of real half-lines or the origin only.

The next example is a natural way of building map germs with radial discriminants.

EXAMPLE 6.2. [ART2] Let $f : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ be a real analytic map germ and let $g : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a germ of a diffeomorphism, such that f and g are in separable variables, and consider the pair of map germs

$$G := (f, g) : (\mathbb{R}^m \times \mathbb{R}, 0) \to (\mathbb{R}^p \times \mathbb{R}, 0).$$

Since $\text{Sing}\, G = \text{Sing}\, f \times \mathbb{R}$, one has that if $\text{Disc}^* f$ is radial, then $\text{Disc}^* G$ is radial.

Let $G : U \to \mathbb{R}^p$ be a representative of the map germ G for some open set $U \ni 0$ and recall the definition of Ψ:

$$\Psi := \frac{G}{\|G\|} : U \setminus V_G \to S^{p-1}. \tag{18}$$

In order to define a new fibration structure associated to the nice map germ G under assumption of radial discriminant, the authors have shown [ART2] that the restriction

$$\Psi_| : S^{m-1}_\varepsilon \setminus G^{-1}(\text{Disc}^* G) \to S^{p-1} \setminus \text{Disc}^* G \tag{19}$$

is well defined for $\varepsilon > 0$ small enough.

Definition 6.3. [ART2] We say that the map germ $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$ with radial discriminant has a *Milnor-Hamm sphere fibration* whenever the restriction (19) is a locally trivial smooth fibration which is independent, up to diffeomorphisms, of the choice of ε provided it is small enough.

In this more general setting, in [ART2] the authors defined *ρ-regularity* of Ψ whenever the following inclusion of germs is satisfied: $M(\Psi) \subset G^{-1}(\mathrm{Disc}^* G)$.

Finally with the notations and definitions above, the most general result regarding the existence of fibration structures on a sphere associated to non-constant nice map germs has been enunciated and demonstrated in [ART2]. It is the direct extension of [ACT1, Theorem 1.3] and its proof follows from the case $\mathrm{Disc}^* G = \{0\}$.

Theorem 6.4. *Let* $G : (\mathbb{R}^m, 0) \to (\mathbb{R}^p, 0)$*,* $m > p \geq 2$*, be a non-constant nice analytic map germ with radial discriminant, satisfying Condition* (14)*. If* Ψ *is* ρ*-regular then* G *has a Milnor-Hamm sphere fibration.*

Example 6.5. [ART1, ART2] Let $G : (\mathbb{R}^3, 0) \to (\mathbb{R}^2, 0)$ given by $G(x, y, z) = (xy, z^2)$. It follows from Example 5.9 that $G^{-1}(\mathrm{Disc}^* G)$ is the union of the coordinates planes in $\mathbb{R}^3$, hence it intersects the sphere S^2_ε on three great circles. Since $M(\Psi_G) = \mathrm{Sing}\, G$, it follows that Ψ is ρ-regular. Therefore, by Theorem 6.4 G has a Milnor-Hamm sphere fibration (see Figure 10).

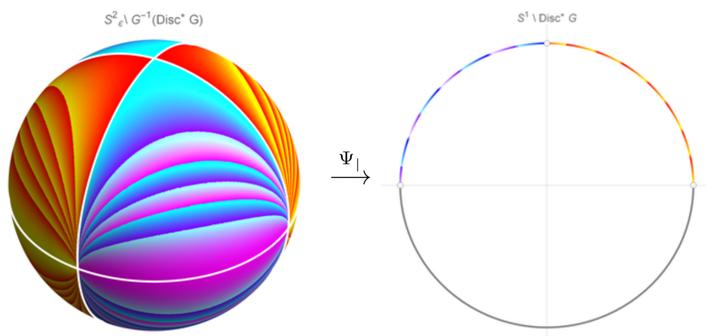


Figure 10. Milnor-Hamm sphere fibration for G. Each color scheme is a fibration over a connected component of the $S^1 \setminus \mathrm{Disc}^* G$.

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