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FINITE TYPE ξ-ASYMPTOTIC LINES OF PLANE FIELDS IN $\mathbb{R}^3$

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Abstract. We prove that a finite type curve is a ξ-asymptotic line (without parabolic points) of a suitable plane field. It is also given an explicit example of a hyperbolic closed finite type ξ-asymptotic line. These results obtained here are generalizations, for plane fields, of the results of V. Arnold.

1. Introduction

A regular plane field in $\mathbb{R}^3$ is usually defined by the kernel of a differential one form or a unit vector field $\xi\colon \mathbb{R}^3 \to \mathbb{R}^3$. In this last case $\xi(p)$ is the normal vector to the plane at point p. The classical and germinal work about plane fields in $\mathbb{R}^3$ is [14].

The normal curvature of a plane field is defined by (see [2] and [5])

$$k_n(p, dr) = -\frac{\langle d\xi(p), dr\rangle}{\langle dr, dr\rangle}.$$

For integrable plane fields the normal curvature is the usual concept of curves on surfaces.

The regular curves $\gamma\colon I \to \mathbb{R}^3$ such that $k_n(\gamma(t), \gamma'(t)) = 0$ are called *ξ-asymptotic lines* and the directions dr such $k_n(p, dr) = 0$ are called *ξ-asymptotic directions*.

Recall that asymptotic lines on surfaces are regular curves γ such that $k_n(\gamma(t), \gamma'(t)) = 0$. Also, asymptotic lines are the curves γ such that the osculating plane of γ coincides with the tangent plane of the surface along it, so asymptotic lines are of extrinsic nature.

The local study, and singular aspects of asymptotic lines on surfaces in $\mathbb{R}^3$, near parabolic points, is a very classical subject, see [3, 6, 7, 8], [9] and references therein.

The study of closed asymptotic lines of surfaces in $\mathbb{R}^3$ under the viewpoint of qualitative theory of differential equations is more recent, see [6, 7, 8]. It is worth to mention that existence of closed asymptotic lines on the tubes of "T-surfaces" is still an open problem. See [1, page 107] and [11].

Also, it is not known if there is a surface in $\mathbb{R}^3$ having a cylindrical region foliated by closed asymptotic lines (see [13, page 110]). In $\mathbb{S}^3$, all asymptotic lines of the Clifford torus are globally defined, and they are the Villarceau circles.

V. Arnold in [4] studied the topology of asymptotic lines being curves of type (t, t^m, t^n) near $t = 0$, which are called of finite type. Also, it was shown in [4] that the projection of a closed asymptotic line of a hyperbolic surface of graph type $(x, y, h(x, y))$ in the horizontal plane (x, y) cannot be a starlike curve.

The main results of this work are the following.

The Theorem 3.1 states that any finite type curve is a ξ-asymptotic line (without parabolic points) of a suitable plane field in $\mathbb{R}^3$.

The Theorem 4.3 gives an example of a hyperbolic closed finite type ξ-asymptotic line of a plane field in $\mathbb{R}^3$.

2. Preliminaries and Previous Results

In this paper, the space $\mathbb{R}^3$ is endowed with the Euclidean norm $|\cdot| = \langle \cdot, \cdot \rangle^{\frac{1}{2}}$.

Definition 2.1 ([10, Definition 5.15])**.** A subset $\Omega \subset \mathbb{R}^2$ is called a *starlike convex set* if there is a point $p \in \Omega$, called the *star point*, such that, for every $q \in \Omega$, the segment $\overline{pq}$ lies in Ω. The boundary of a starlike convex set is called a *starlike curve*.

Theorem 2.2 (D. Panov, see [4])**.** *The projection of a closed asymptotic line of a surface $z = \varphi(x, y)$ to the plane $\{z = 0\}$ cannot be a starlike curve (in particular, this projection cannot be a convex curve).*

Definition 2.3 ([4])**.** A smoothly immersed curve $\gamma : I \to \mathbb{R}^3$ is said to be of *finite type* at a point x, if $\{\gamma'(x), \gamma''(x), \ldots, \gamma^{(k)}(x)\}$ generate all the tangent space $T_{\gamma(x)}\mathbb{R}^3$ for some $k \in \mathbb{N}$. Here $\gamma^{(k)}(x)$ denotes the derivative of order k of γ. In a neighborhood of this point, the curve is parametrized locally by $\gamma(x) = (x, a_m x^m + \mathcal{O}^{m+1}(x), b_n x^n + \mathcal{O}^{n+1}(x))$, where $m, n \in \mathbb{N}$, $a_m b_n \neq 0$ and $1 < m < n$.

The set $\{1, m, n\}$, $(1 < m < n)$, of the degrees of γ is called the *symbol* of the point. If $n = m + 1$, then γ is said to be of *rotating type* at the point.

If a curve is of finite type (resp. rotating type) at every point, then it is called of *finite type curve* (resp. *rotating type curve*).

A finite type curve γ can have inflection points, i.e., points where the curvature of γ vanishes.

Arnold's Theorem (See [4])**.** *An asymptotic curve of finite type on a hyperbolic surface is a rotating curve.*

Every rotating space curve of finite type is an asymptotic line on a suitable hyperbolic surface.

A new proof of Arnold's Theorem will be given in the appendix.

2.1. Plane fields in $\mathbb{R}^3$.

Let $\xi : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field of class C^k, where $k \geq 3$.

Definition 2.4. A *plane field* ξ in $\mathbb{R}^3$, orthogonal to the vector field ξ, is defined by the 1-form $\langle \xi, dr \rangle = 0$, where dr is a direction in $\mathbb{R}^3$. See Fig. 1.

Theorem 2.5 ([2, Jacobi Theorem, p.2])**.** *There exists a family of surfaces orthogonal to ξ if, and only if, $\langle \xi, curl(\xi) \rangle \equiv 0$.*

A plane field ξ is said to be *completely integrable* if $\langle \xi, curl(\xi) \rangle \equiv 0$. A surface of the family of surfaces orthogonal to ξ is called an integral surface.

2.2. Normal curvature of a plane field.

Definition 2.6 ([2, p. 8])**.** The *normal curvature* k_n of a plane field in the direction dr orthogonal to ξ is defined by

$$k_n = \frac{\langle \xi, d^2 r \rangle}{\langle dr, dr \rangle} = -\frac{\langle d\xi, dr \rangle}{\langle dr, dr \rangle}.$$

This definition agrees with the classical one given by L. Euler, see [5].

The geometric interpretation of k_n is given by means of the curvature of a plane curve, which we shall now describe.

In the plane $\pi(p_0, dr)$ generated by $\xi(p_0)$ and dr (direction orthogonal to $\xi(p_0)$) we have a line field $\ell(p)$ orthogonal to vector $\bar{\xi}(p) \in \pi(p_0, dr)$ obtained projecting $\xi(p)$ in the plane $\pi(p_0, dr)$, with $p \in \pi(p_0, dr)$. The integral curves $\varphi_p(t)$ of the line field ℓ are regular curves and $k_n(p_0, dr)$ is the plane curvature of $\varphi_{p_0}(t)$ at $t = 0$. See Fig. 2.

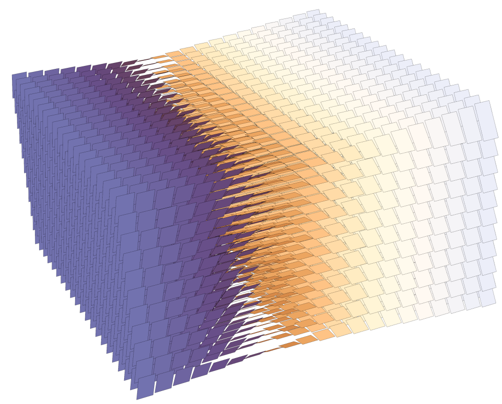


FIGURE 1. Plane field ξ in $\mathbb{R}^3$ defined by the 1-form $\langle \xi, dr \rangle = dz - ydx = 0$, where $\xi(x, y, z) = (-y, 0, 1)$ and $dr = (dx, dy, dz)$.

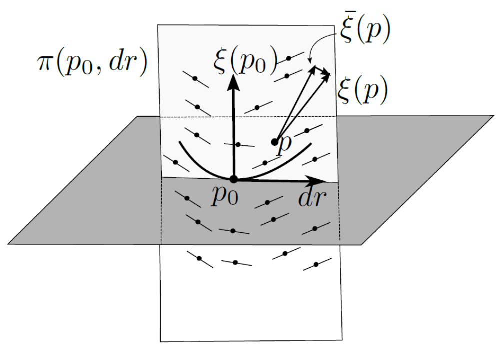


FIGURE 2. Line field and normal curvature $k_n(p_0, dr)$.

2.3. **ξ-asymptotic lines and parabolic points of a plane field.** The ξ-asymptotic directions of a plane field ξ are defined by the following implicit differential equation

$$\langle \xi, dr \rangle = 0, \quad \langle d\xi, dr \rangle = 0. \tag{2.1}$$

and will referred as the implicit differential equation of the ξ-asymptotic lines.

A solution dr of equation (2.1) is called a ξ-*asymptotic direction*. A curve γ in $\mathbb{R}^3$ is a ξ-*asymptotic line* if γ is an integral curve of equation (2.1). Analogously to the case of asymptotic lines on surfaces, for plane fields the osculating plane of a $\xi-$asymptotic line coincides with the plane of the distribution of planes passing through the point of the curve. See also [2, page 29].

Definition 2.7. If at a point r there exists two real distinct ξ-asymptotic directions (resp. two complex ξ-asymptotic directions), then r is called a *hyperbolic point* (resp. *elliptic point*).

Definition 2.8. If at r the two ξ-asymptotic directions coincide or all the directions are ξ-asymptotic directions then r is called a *parabolic point*.

Example 2.9. The circle in $\mathbb{R}^3$ given by $x^2 + y^2 = 1$, $z = 0$, is a ξ-asymptotic line without parabolic points of the plane field ξ defined by the orthogonal vector field $\xi = (\rho, \varrho, \sigma)$, where $\rho = x^2yz + y^3z - x^2y - y^3 + xz - 2yz + y$, $\varrho = x^3 - x^3z - xy^2z + xy^2 + 2xz + yz - x$ and $\sigma = -x^2 - y^2$. See Fig. 3. The plane field ξ is not completely integrable. By the Theorem 2.2, this circle cannot be an asymptotic line of a regular surface $z = \varphi(x, y)$.

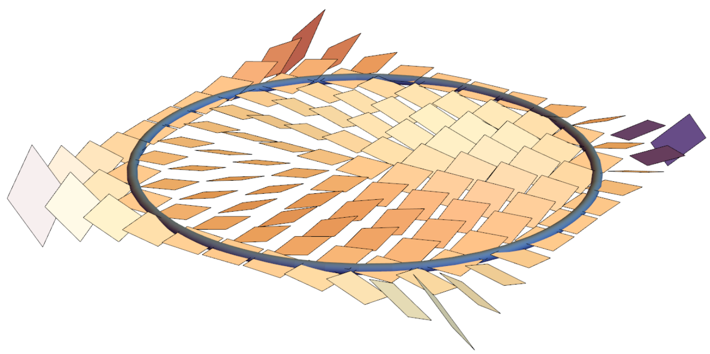


FIGURE 3. The circle is a ξ-asymptotic line without parabolic points of the plane field defined by the orthogonal vector field $\xi = (\rho, \varrho, \sigma)$, where $\rho = x^2yz + y^3z - x^2y - y^3 + xz - 2yz + y$, $\varrho = x^3 - x^3z - xy^2z + xy^2 + 2xz + yz - x$ and $\sigma = -x^2 - y^2$.

Proposition 2.10. *Given a plane field ξ, let $\varphi : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable nonvanishing function. Then a curve γ is a ξ-asymptotic line if, and only if, γ is a ξ-asymptotic line of the plane field $\widetilde{\xi}$ orthogonal to the vector field $\widetilde{\xi} = \varphi\xi$.*

Proof. The implicit differential equation of ξ-asymptotic lines of $\widetilde{\xi}$ is given by

$$\langle \widetilde{\xi}, dr \rangle = \varphi \langle \xi, dr \rangle = 0, \quad \langle d\widetilde{\xi}(dr), dr \rangle = d\varphi(dr)\langle \xi, dr \rangle + \varphi \langle d\xi(dr), dr \rangle = 0.$$

Then γ is a ξ-asymptotic line of ξ if, and only if, γ is a ξ-asymptotic line of the plane field $\widetilde{\xi}$. □

2.4. **Tubular neighborhood of an integral curve of a plane field.** Let ξ be a plane field orthogonal to a vector field $\xi(x, y, z)$. Then $d\xi = \xi_x dx + \xi_y dy + \xi_z dz$. Let

$$\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$$

be a curve such that $(\gamma_1'(x), \gamma_2'(x)) \neq (0, 0)$ for all x. Set $X(x) = \gamma'(x)$, $Y(x) = (\gamma_2'(x), -\gamma'(x), 0)$, $Z(x) = (X \wedge Y)(x)$ and $\alpha : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\alpha(x, y, z) = \gamma(x) + yY(x) + zZ(x). \tag{2.2}$$

The map (2.2) is a parametrization of a tubular neighborhood of γ. At this neighborhood, the position point is given by $r = \alpha(x, y, z)$ and then $dr = d\alpha = \alpha_x dx + \alpha_y dy + \alpha_z dz$. It follows that the implicit differential equation (2.1) of the ξ-asymptotic lines is given by

$$\begin{aligned} &\langle \xi, d\alpha \rangle = adx + bdy + cdz = 0, \\ &\langle d\xi, d\alpha \rangle = L_1 dx^2 + L_2 dxdy + L_3 dy^2 + L_4 dxdz + L_5 dydz + L_6 dz^2 = 0, \end{aligned} \tag{2.3}$$

where,

$$a = \langle \xi, \alpha_x \rangle, \quad b = \langle \xi, \alpha_y \rangle, \quad c = \langle \xi, \alpha_z \rangle,$$

and

$$L_1 = \langle \xi_x, \alpha_x \rangle, \quad L_2 = \langle \xi_x, \alpha_y \rangle + \langle \xi_y, \alpha_x \rangle, \quad L_3 = \langle \xi_y, \alpha_y \rangle,$$
$$L_4 = \langle \xi_x, \alpha_z \rangle + \langle \xi_z, \alpha_x \rangle, \quad L_5 = \langle \xi_y, \alpha_z \rangle + \langle \xi_z, \alpha_y \rangle, \quad L_6 = \langle \xi_z, \alpha_z \rangle.$$

Proposition 2.11. *Let $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$ be a curve such that, for all x,*

$$(\gamma_1'(x), \gamma_2'(x)) \neq (0, 0).$$

Consider a tubular neighborhood of γ parametrized by equation (2.2). If ξ is a plane field such that $\frac{a}{c}$ and $\frac{b}{c}$ are well defined in a neighborhood of γ, where a, b, c are given by (2.3), then the implicit differential equation of the ξ-asymptotic lines, in this neighborhood, is given by

$$dz = -\left(\frac{a}{c}\right) dx - \left(\frac{b}{c}\right) dy, \quad e dx^2 + 2f dx dy + g dy^2 = 0, \tag{2.4}$$

where,

$$e = L_1 - \frac{aL_4}{c} + \frac{a^2 L_6}{c^2}, \quad g = L_3 - \frac{bL_5}{c} + \frac{b^2 L_6}{c^2}, \quad f = \frac{L_2}{2} - \frac{(aL_5 + bL_4)}{2c} + \frac{abL_6}{2c^2}.$$

Furthermore, in this neighborhood, the parabolic set of ξ is given by $eg - f^2 = 0$.

Proof. In a neighborhood of γ, solve the first equation of (2.3) in the variable dz to get the first equation of (2.4). Replace this dz in the second equation of (2.3) to get the second equation of (2.4).

If $eg - f^2 < 0$ at a point (resp. $eg - f^2 > 0$), then the equations (2.4) define two distinct ξ-asymptotic directions at this point (resp. two complex ξ-asymptotic directions).

If $eg - f^2 = 0$ at a point, then at it the ξ-asymptotic directions coincide or, if $e = g = f = 0$, all directions are ξ-asymptotic directions. □

Definition 2.12 ([2, p. 11])**.** Let ξ be a plane field satisfying the assumptions of Lemma 2.11. The function defined by $\mathcal{K} = eg - f^2$ is called the *Gaussian curvature* of ξ.

Lemma 2.13. *Let $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$ be a ξ-asymptotic line of a plane field ξ, such that $(\gamma_1'(x), \gamma_2'(x)) \neq (0, 0)$ for all x. Consider a tubular neighborhood of γ parametrized by equation (2.2). Then, in a neighborhood of γ, the vector field ξ is given by*

$$\begin{aligned}\xi(x, y, z) &= l_0(x) Y(x) + k_0(x) Z(x) \\ &+ \left(y k_1(x) + z l_1(x) + \left(\frac{y^2}{2}\right) \widetilde{k}_1(x) + yz \widetilde{j}_1(x) + \left(\frac{z^2}{2}\right) \widetilde{l}_1(x) + \widetilde{A}(x, y, z)\right) X(x) \\ &+ \left(y k_2(x) + z l_2(x) + \left(\frac{y^2}{2}\right) \widetilde{k}_2(x) + yz \widetilde{j}_2(x) + \left(\frac{z^2}{2}\right) \widetilde{l}_2(x) + \widetilde{B}(x, y, z)\right) Y(x) \\ &+ \left(y k_3(x) + z l_3(x) + \left(\frac{y^2}{2}\right) \widetilde{k}_3(x) + yz \widetilde{j}_3(x) + \left(\frac{z^2}{3}\right) \widetilde{l}_3(x) + \widetilde{C}(x, y, z)\right) Z(x),\end{aligned} \tag{2.5}$$

where

$$X(x) = \gamma'(x), \; Y(x) = (\gamma_2'(x), -\gamma_1'(x), 0), \; Z(x) = (X \wedge Y)(x),$$
$$\widetilde{A}(x, 0, 0) = \widetilde{B}(x, 0, 0) = \widetilde{C}(x, 0, 0) = 0$$

and

$$[(\gamma_3'\gamma_1'' - \gamma_1'\gamma_3'')\gamma_1' + (\gamma_3'\gamma_2'' - \gamma_2'\gamma_3'')\gamma_2'] k_0 - (\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'') l_0 = 0. \tag{2.6}$$

Furthermore, if

$$k_0 = \gamma_1'\gamma_2'' - \gamma_2'\gamma_1'', \quad l_0 = (\gamma_3'\gamma_1'' - \gamma_1'\gamma_3'')\gamma_1' + (\gamma_3'\gamma_2'' - \gamma_2'\gamma_3'')\gamma_2' \tag{2.7}$$

and $\gamma_1'(x)\gamma_2''(x) - \gamma_2'(x)\gamma_1''(x) \neq 0$ for all x, then the implicit differential equation of the ξ-asymptotic lines is given by (2.4).

Proof. The expression (2.5) holds, since γ is an integral curve of the plane field defined by ξ. Also, as γ is a ξ-asymptotic line, $\langle \xi(x), \gamma''(x)\rangle = 0$ for all x, which gives the equation (2.6).

If $\gamma_1'(x)\gamma_2''(x) - \gamma_2'(x)\gamma_1''(x) \neq 0$, then $c(x,0,0) \neq 0$. The conclusion then follows from Proposition 2.11. □

3. Finite type ξ-asymptotic lines of plane fields

In this section the following result is established.

Theorem 3.1. *Any finite type curve is a ξ-asymptotic line (without parabolic points) of a suitable plane field.*

Proof. Let $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x)) = (x, a_m x^m + \mathcal{O}^{m+1}(x), a_n x^n + \mathcal{O}^{n+1}(x))$ be a finite type curve. Consider a tubular neighborhood of γ parametrized by equation (2.2) and the vector field ξ given by (2.5). Set $k_0(x) \equiv 1$ and solve (2.6) for $l_0(x)$. Then γ is a ξ-asymptotic line of the plane field orthogonal to ξ.

We have that

$$a(x,0,0) = 0, \quad b(0,0,0) = 0, \quad \text{and} \quad c(0,0,0) = a_m m(m-1) \neq 0.$$

By Proposition 2.11, in a neighborhood of $(0,0,0)$, the equation of ξ-asymptotic lines are given by (2.4).

Set $l_1(x) \equiv 0$ and define $k_1(x)$ by

$$\begin{aligned} k_1 = & \frac{((\gamma_1')^2 + (\gamma_2')^2)^2[(\gamma_2''\gamma_3''' - \gamma_3''\gamma_2''')\gamma_1' + (\gamma_3''\gamma_1''' - \gamma_1''\gamma_3''')\gamma_2' + (\gamma_1''\gamma_2''' - \gamma_2''\gamma_1''')\gamma_3']}{((\gamma_1')^2 + (\gamma_2')^2 + (\gamma_3')^2)(\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')} \\ & + \frac{2(\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')}{((\gamma_1')^2 + (\gamma_2')^2 + (\gamma_3')^2)}. \end{aligned}$$

Then $\mathcal{K}(x,0,0) = -1$. □

4. Hyperbolic closed finite type ξ-asymptotic line

Examples of hyperbolic asymptotic lines on surfaces are given in [6, 7, 8].

In this section it will be given an example of a hyperbolic closed ξ-asymptotic line of finite type for a suitable plane field.

Proposition 4.1. *Let γ, $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$, be a curve such that*

$$(\gamma_1'(x), \gamma_2'(x)) \neq (0,0), \quad \gamma_1'(x)\gamma_2''(x) - \gamma_2'(x)\gamma_1''(x) \neq 0$$

for all x. Consider the tubular neighborhood α given by (2.2) and the vector field ξ given by (2.5), with $k_0(x)$, $l_0(x)$ given by (2.7). Let $H(x)$ be a nonvanishing function and define $k_1(x)$ by

$$\begin{aligned} k_1 = & \frac{((\gamma_1')^2 + (\gamma_2')^2)^2[(\gamma_2''\gamma_3''' - \gamma_3''\gamma_2''')\gamma_1' + (\gamma_3''\gamma_1''' - \gamma_1''\gamma_3''')\gamma_2' + (\gamma_1''\gamma_2''' - \gamma_2''\gamma_1''')\gamma_3']}{((\gamma_1')^2 + (\gamma_2')^2 + (\gamma_3')^2)(\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')} \\ & + \frac{[(\gamma_1'\gamma_1'' + \gamma_2'\gamma_2'')\gamma_3' - ((\gamma_1)^2 + (\gamma_2)^2)\gamma_3'']l_1 + 2(\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')H}{((\gamma_1')^2 + (\gamma_2')^2 + (\gamma_3')^2)(\gamma_1'\gamma_2'' - \gamma_2'\gamma_1'')}. \end{aligned} \tag{4.1}$$

Then, γ is a ξ-asymptotic line, without parabolic points, of the plane field orthogonal to the vector field ξ.

Furthermore, $\mathcal{K}(x,0,0) = -(H(x))^2$.

Proof. By direct calculations, we can see that γ is a ξ-asymptotic line. The implicit differential equation of the ξ-asymptotic lines are given by (2.4) and $e(x,0,0) = 0$, $f(x,0,0) = H(x)$. Since $e(x,0,0) = 0$, then $\mathcal{K}(x,0,0) = -(H(x))^2$ for all x.

□

4.1. **Poincaré map associated to a closed ξ-asymptotic line.** Let $\gamma : [0,l] \to \mathbb{R}^3$, $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$, be a closed ξ-asymptotic line, without parabolic points, of a plane field ξ, such that $\gamma(0) = \gamma(l)$, $(\gamma_1'(x), \gamma_2'(x)) \neq (0,0)$, $\gamma_1'(x)\gamma_2''(x) - \gamma_2'(x)\gamma_1''(x) \neq 0$ for all x, and consider the tubular neighborhood α given by (2.2).

This means that γ is a regular curve having a projection in a plane which is a strictly locally convex curve.

By the Proposition 2.13, ξ is given by (2.5) and the implicit differential equations of the ξ-asymptotic lines is given by (2.4).

Let $\Sigma_{x_0} = \{(x_0, y, z)\}$ be a transversal section. Then $\alpha(\Sigma_{x_0})$ is the plane spanned by $Y(x_0)$ and $Z(x_0)$. By Lemma 2.13, in a neighborhood of γ, the ξ-asymptotic line passing through $\alpha(x_0, y_0, z_0)$ intersects $\alpha(\Sigma_{x_0})$ again at the point

$$\alpha(x_0 + l, y(x_0 + l, y_0, z_0), z(x_0 + l, y_0, z_0)),$$

where $(y(x, y_0, z_0), z(x, y_0, z_0))$ is solution of the following Cauchy problem

$$\begin{aligned} &\frac{dz}{dx} = -\frac{a}{c} - \left(\frac{b}{c}\right)\frac{dy}{dx} = A + B\frac{dy}{dx}, \\ &e + 2f\frac{dy}{dx} + g\left(\frac{dy}{dx}\right)^2 = 0, \\ &(y(x_0, y_0, z_0), z(x_0, y_0, z_0)) = (y_0, z_0). \end{aligned} \tag{4.2}$$

The *Poincaré map* $\mathcal{P}$, also called *first return map*, associated to γ is defined by $\mathcal{P} : \mathcal{U} \subset \Sigma \to \Sigma$, $\mathcal{P}(y_0, z_0) = (y(l, y_0, z_0), z(l, y_0, z_0))$. See Fig. 4.

A closed ξ-asymptotic line γ is said to be *hyperbolic* if the eigenvalues of $d\mathcal{P}_{(0,0)}$ does not belong to $\mathbb{S}^1$. See [12] for the generic properties of the Poincaré map associated to closed orbits of vector fields.

We will denote by $d\mathcal{P}_{(0,0)}$ the matrix of the first derivative of the Poincaré map evaluated at $(y_0, z_0) = (0, 0)$.

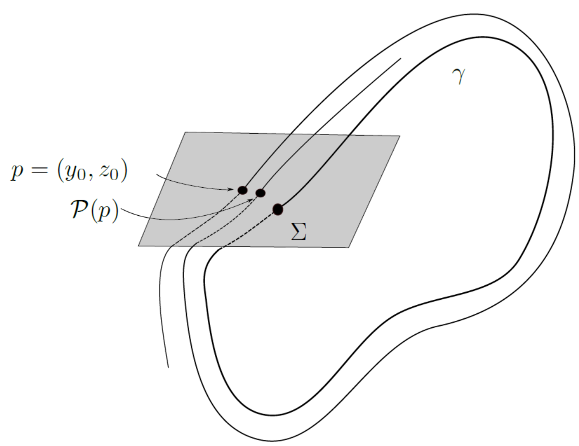


FIGURE 4. Poincaré return map.

Proposition 4.2. *Let $\gamma : [0,l] \to \mathbb{R}^3$, $\gamma(x) = (\gamma_1(x), \gamma_2(x), \gamma_3(x))$, be a closed ξ-asymptotic line, having a projection in a plane which is a locally strictly convex curve.*

Let $\mathcal{P}$ be the Poincaré map associated to γ. Then $d\mathcal{P}_{(0,0)} = \mathcal{Q}(l)$, where $\mathcal{Q}(x)$ is solution of the following Cauchy problem:

$$\frac{d}{dx}\left(\mathcal{Q}(x)\right) = \mathcal{M}(x)\mathcal{Q}(x), \quad \mathcal{Q}(0) = \mathcal{I}, \tag{4.3}$$

where $\mathcal{I}$ is the identity matrix, and $\mathcal{M}(x)$, $\mathcal{Q}(x)$ are the matrices given by

$$\mathcal{M}(x) = \begin{pmatrix} -\frac{e_y(x,0,0)}{2f(x,0,0)} & -\frac{e_z(x,0,0)}{2f(x,0,0)} \\ (A)_y(x,0,0) & (A)_z(x,0,0) \end{pmatrix}, \mathcal{Q}(x) = \begin{pmatrix} \frac{dy}{dy_0}(x,0,0) & \frac{dy}{dz_0}(x,0,0) \\ \frac{dz}{dy_0}(x,0,0) & \frac{dz}{dz_0}(x,0,0) \end{pmatrix},$$

where $A = -\frac{a}{c}$.

Proof. To fix the notation suppose that

$$\gamma(0) = \gamma(l), \quad (\gamma_1'(x), \gamma_2'(x)) \neq (0,0), \quad \text{and} \quad \gamma_1'(x)\gamma_2''(x) - \gamma_2'(x)\gamma_1''(x) \neq 0 \quad \text{for all} \quad x$$

.

Let $(y(x, y_0, z_0), z(x, y_0, z_0))$ be solution of the Cauchy problem given by equation (4.2). Then, at $(y, z) = (0, 0)$, $\frac{dy}{dx}(x, 0, 0) = \frac{dz}{dx}(x, 0, 0) = 0$.

Differentiating the first equation of (4.2) with respect to y_0 (resp. z_0), it results that:

$$\frac{d}{dx}\left(\frac{dz}{dy_0}\right) = A_y\frac{dy}{dy_0} + A_z\frac{dz}{dy_0} + B\frac{d}{dx}\left(\frac{dy}{dy_0}\right) + \left(B_y\frac{dy}{dy_0} + B_z\frac{dz}{dy_0}\right)\frac{dy}{dx}, \tag{4.4}$$

respectively,

$$\frac{d}{dx}\left(\frac{dz}{dz_0}\right) = A_y\frac{dy}{dz_0} + A_z\frac{dz}{dz_0} + B\frac{d}{dx}\left(\frac{dy}{dz_0}\right) + \left(B_y\frac{dy}{dz_0} + B_z\frac{dz}{dz_0}\right)\frac{dy}{dx}. \tag{4.5}$$

Differentiating the second equation of (4.2) with respect to y_0 (resp. z_0), it results that:

$$\begin{aligned} &e_y\frac{dy}{dy_0} + e_z\frac{dz}{dy_0} + 2f\frac{d}{dx}\left(\frac{dy}{dy_0}\right) + 2\left(f_y\frac{dy}{dy_0} + f_z\frac{dz}{dy_0} + g\frac{d}{dx}\left(\frac{dy}{dy_0}\right)\right)\frac{dy}{dx} \\ &+ \left(g_y\frac{dy}{dy_0} + g_z\frac{dz}{dy_0}\right)\left(\frac{dy}{dx}\right)^2 = 0, \end{aligned} \tag{4.6}$$

respectively,

$$\begin{aligned} &e_y\frac{dy}{dz_0} + e_z\frac{dz}{dz_0} + 2f\frac{d}{dx}\left(\frac{dy}{dz_0}\right) + 2\left(f_y\frac{dy}{dz_0} + f_z\frac{dz}{dz_0} + g\frac{d}{dx}\left(\frac{dy}{dz_0}\right)\right)\frac{dy}{dx} \\ &+ \left(g_y\frac{dy}{dz_0} + g_z\frac{dz}{dz_0}\right)\left(\frac{dy}{dx}\right)^2 = 0. \end{aligned} \tag{4.7}$$

Evaluating (4.4), (4.5), (4.6), (4.7) at $(y, z) = (0, 0)$, it follows that:

$$\begin{aligned} A_y\frac{dy}{dy_0} + A_z\frac{dz}{dy_0} = \frac{d}{dx}\left(\frac{dz}{dy_0}\right), &\quad e_y\frac{dy}{dy_0} + e_z\frac{dz}{dy_0} + 2f\frac{d}{dx}\left(\frac{dy}{dy_0}\right) = 0, \\ A_y\frac{dy}{dz_0} + A_z\frac{dz}{dz_0} = \frac{d}{dx}\left(\frac{dz}{dz_0}\right), &\quad e_y\frac{dy}{dz_0} + e_z\frac{dz}{dz_0} + 2f\frac{d}{dx}\left(\frac{dy}{dz_0}\right) = 0. \end{aligned}$$

Then $\frac{d}{dx}\left(\mathcal{Q}(x)\right) = \mathcal{M}(x)\mathcal{Q}(x)$. Since $(y(0, y_0, z_0), z(0, y_0, z_0)) = (y_0, z_0)$, it follows that $\mathcal{Q}(0) = \mathcal{I}$.

Since $\mathcal{P}(y_0, z_0) = (y(l, y_0, z_0), z(l, y_0, z_0))$, the first derivative $d\mathcal{P}_{(0,0)}$ is given by $\mathcal{Q}(l)$. □

4.2. **Example of a hyperbolic closed finite type ξ-asymptotic line.** An explicit example of a hyperbolic closed ξ-asymptotic line is given in the next result.

Theorem 4.3. *Let $\gamma : [0, 2\pi] \to \mathbb{R}^3$, $\gamma(x) = (sin(x), cos(x), sin^3(x))$, see Fig. 5. Then it is a hyperbolic finite type ξ-asymptotic line of a suitable plane field.*

Proof. Let ξ be a plane field orthogonal to the vector field ξ given by (2.5), where $k_0(x)$ and $l_0(x)$ are given by (2.7). Let $k_1(x)$ given by (4.1), with $H(x) \equiv 1$. Then

$$k_1(x) = \frac{3(3cos^2(x) - 1)sin(x)l_1(x) + 24cos^3(x) - 18cos(x) - 2}{9cos^6(x) - 18cos^4(x) + 9cos^2(x) + 1}.$$

By Proposition 4.1, γ is a ξ-asymptotic line without parabolic points and $\mathcal{K}(x, 0, 0) = -1$. Performing the calculations, $e_z(x, 0, 0) = \mathcal{E}(x) + l_2(x)$. Solve $e_z(x, 0, 0) = 0$ for $l_2(x)$. This vanishes the entry $\left(-\frac{e_z(x,0,0)}{2f(x,0,0)}\right)$ of $\mathcal{M}(x)$ given by Theorem 4.2. From (4.3), it follows that the eigenvalues of $d\mathcal{P}_{(0,0)}$ are given by

$$\exp\left(\int_0^{2\pi} -\frac{e_y(x, 0, 0)}{2f(x, 0, 0)}dx\right) \quad \text{and} \quad \exp\left(\int_0^{2\pi} A_z(x, 0, 0)dx\right).$$

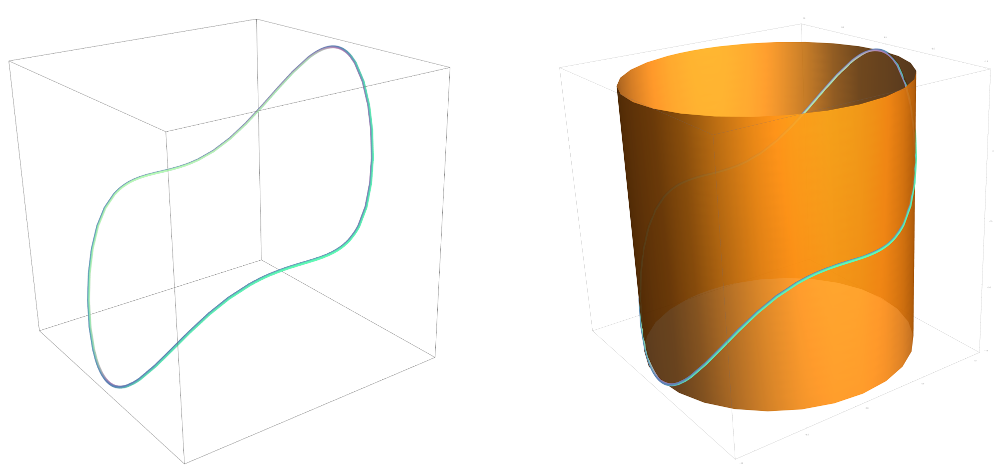
Set $l_1(x) = cos(x)$. Then

$$\begin{aligned} A_z(x, 0, 0) = {} & 9sin(x)cos^8(x) + 54sin(x)cos^6(x) - 9cos^6(x) - 117sin(x)cos^4(x) \\ & + 18cos^4(x) + 55cos^2(x)sin(x) - 9cos^2(x) - 1. \end{aligned}$$

It follows that $\int_0^{2\pi} A_z(x, 0, 0)dx = -\frac{25\pi}{8}$. Let $k_3(x) = 0$ and $k_2(x)$ a solution of the equation $e_y(x, 0, 0) + 2f(x, 0, 0) = 0$. It follows that

$$\int_0^{2\pi} \left(-\frac{e_y(x, 0, 0)}{2f(x, 0, 0)}\right) dx = 2\pi.$$

□



(A) Curve $\gamma(x) = (sin(x), cos(x), sin^3(x))$.

(B) Curve $\gamma(x)$ on the cylinder $\beta(x, y) = (sin(x), cos(x), y)$.

FIGURE 5. Finite type curve $\gamma(x) = (sin(x), cos(x), sin^3(x))$.

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Appendix

A new proof of Arnold's Theorem

The proof of Arnold's Theorem [4] is given on graph surfaces $z = z(x, y)$. Using affine coordinates, the surface takes the form $z = xy + \dots$, where the dots denote the terms of higher order. Arnold showed that an asymptotic line $x = x(t)$, $y = y(t)$, $z = z(t)$ of finite type is a rotating curve.

After that, he proves that given a rotating curve $x = x(t)$, $y = y(t)$, $z = z(t)$ then there exists an appropriated function $H(x, y)$ such that the rotating curve is an asymptotic line of the surface $z = H(x, y)$.

Below, will be given a geometric proof of Arnold's Theorem, with an explicit parametrization of the surface.

Proof. Let γ be a curve of finite type (u, u^m, u^n), $n \geq m$. Set $N(u) = (\gamma_2'(u), -\gamma_1'(u), 0)$. Consider the local surface parametrized by

$$\alpha(u,v) = \gamma(u) + vN(u) + (k_1(u)v + k_2(u)v^2 + k_3(u)v^3 + \mathcal{O}^4(v))(\gamma' \wedge N)(u).$$

Let N_α be the unit normal vector

$$N_\alpha = \frac{\alpha_u \wedge \alpha_v}{|\alpha_u \wedge \alpha_v|}.$$

The implicit differential equation of the asymptotic lines of α is given by

$$edu^2 + 2fdudv + gdv^2 = 0,$$

where $e = \langle \alpha_{uu}, N_\alpha \rangle$, $f = \langle \alpha_{uv}, N_\alpha \rangle$ and $g = \langle \alpha_{vv}, N_\alpha \rangle$.

Supposing that γ is an asymptotic line of α, and parametrized by $v = 0$, we have that $e(u, 0) = 0$. Then by equation (4.1) it follows that

$$k_1(u) = \frac{[(n-m)m^2u^{2(m-1)} + n - 1]nu^{n-m}}{[1 + m^2u^{2(m-1)} + n^2u^{2(n-1)}](m-1)m}. \tag{A.8}$$

Direct calculations show that

$$f(u, 0) = \frac{(n-m)(n-1)n(1 + m^2u^{2(m-1)})^2u^{n-m-1}}{(m-1)m}.$$

It follows that $f(0, 0) \neq 0$ if, and only if, $n = m + 1$, i.e., γ is a rotating curve.

If γ is a rotating space curve of finite type (u, u^m, u^{m+1}), $m \geq 2$, set $N(u) = (\gamma_2'(u), -\gamma_1'(u), 0)$ and let

$$\beta(u, v) = \gamma(u) + vN(u) + k_1(u)v(\gamma' \wedge N)(u),$$

where $k_1(u)$ is given by (A.8) with $n = m + 1$. Therefore, $e(u, 0) = [\beta_u, \beta_v, \beta_{uu}](0, 0) = 0$ and $f(0, 0) = [\beta_u, \beta_v, \beta_{uv}](0, 0) = \frac{m+1}{m-1} \neq 0$. Then γ is an asymptotic line, without parabolic points, of the surface parametrized by β in a neighborhood of $(u, v) = (0, 0)$. $\square$

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