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DUALITY OF SINGULARITIES FOR FLAT SURFACES IN EUCLIDEAN SPACE

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Dedicated to Professor Goo Ishikawa on the occasion of his 60th birthday.

Abstract. In this paper, we shall discuss the duality of singularities for a class of flat surfaces in Euclidean space. After introducing the definition of the conjugate of a tangent developable, we show that, if a tangent developable admits a swallowtail, its conjugate has a cuspidal cross cap. Similarly, we prove that the conjugate of a tangent developable having cuspidal S_1^+ singularities has cuspidal butterflies, and that cuspidal beaks have self-duality. We also show that cuspidal edges do not possess such a property, by exhibiting an example of a tangent developable with cuspidal edges whose conjugate has 5/2-cuspidal edges. Finally, we prove that conjugates of complete flat fronts with embedded ends cannot be complete flat fronts.

1. Introduction

We denote Euclidean 3-space by $\boldsymbol{R}^3$. It is well-known that, for a minimal surface

$$f = (x_1, x_2, x_3) : M \to \boldsymbol{R}^3,$$

its coordinate functions x_j $(j = 1, 2, 3)$ are harmonic functions on M. Then, the harmonic conjugates $x_j^\sharp$ $(j = 1, 2, 3)$ define another minimal surface $f^\sharp = (x_1^\sharp, x_2^\sharp, x_3^\sharp)$, which is called the *conjugate* minimal surface. Similarly, for maximal surfaces in the Lorentz-Minkowski 3-space $\boldsymbol{L}^3$, we can define the conjugate. Since the only complete maximal surfaces are spacelike planes [2], we need to consider maximal surfaces with singular points. Umehara–Yamada [23] introduced a class of maximal surfaces with admissible singularities called *maxfaces*, which satisfy the following property so-called the *duality of singularities*:

Fact 1.1 ([23, 4])**.** *Let $f : M \to \boldsymbol{L}^3$ be a maxface, $f^\sharp : M \to \boldsymbol{L}^3$ its conjugate, and $p \in M$ a singular point. Then, f at p is $\mathcal{A}$-equivalent to the cuspidal edge* (*resp. swallowtail, cuspidal cross cap*) *if and only if $f^\sharp$ at p is $\mathcal{A}$-equivalent to the cuspidal edge* (*resp. cuspidal cross cap, swallowtail*)*.*

The property as in Fact 1.1 is called the *duality of singularities*. Let S_1^3 (resp. S^3) be the de Sitter 3-space (resp. the 3-sphere) of constant sectional curvature 1. Also, let H_1^3 (resp. H^3) be the anti-de Sitter 3-space (resp. the hyperbolic 3-space) of constant sectional curvature -1, and Q^3 be the 3-lightcone. It is known that such a duality of singularities holds for various classes of surfaces as follows:

- timelike minimal surfaces (so-called *minfaces*) in $\boldsymbol{L}^3$ [21] (cf. [1]),
- spacelike surfaces of non-zero constant mean curvature in $\boldsymbol{L}^3$ [7],
- spacelike surfaces of constant mean curvature 1 in S_1^3 [4],
- timelike surfaces of constant mean curvature 1 in H_1^3 [24],

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- spacelike surfaces of zero extrinsic curvature in S^3_1, H^3 and Q^3 [13],
- surfaces of zero extrinsic curvature in S^3 [12].

We remark that such a duality is known for more degenerate singularities, such as cuspidal beaks, cuspidal butterflies and cuspidal S_1^- singularities ([12, 18]).

In this article, we shall study the duality of singularities in the case of flat surfaces with singularities in $\boldsymbol{R}^3$. Murata–Umehara [17] investigated the global properties of flat surfaces with singularities called *flat fronts* (cf. Fact 2.1). In particular, they proved that complete flat fronts with non-empty singular sets must be *tangent developables*. Ishikawa [10] investigated the singularities of tangent developables from the view point of the (real) *projective geometry*. In particular, Ishikawa [10] used the Scherbak's dual curves [20] in the dual projective space to define the dual tangent developables, and proved the duality of singularities. For more details, see [10, 11] (cf. [3]). However, to the best of the author's knowledge, there was no notion like the conjugate of flat surfaces $\boldsymbol{R}^3$ in the setting of *Euclidean geometry*. Thus, we shall find a suitable definition of the conjugate of flat fronts which satisfy the duality of singularities.

This paper is organized as follows. In Section 2, we review some basic facts on flat fronts, singularities of frontals in $\boldsymbol{R}^3$, and frontals in the 2-sphere S^2. Then, in Section 3, after reviewing a-orientable admissible developable frontals introduced by Murata–Umehara [17], we apply the criteria for cuspidal cross caps to such developable frontals. Comparing the condition for swallowtails and that for cuspidal cross caps, we give a definition of the conjugates for tangent developables (Definition 3.6, cf. Corollary 3.9). In Section 4, applying the criteria for other singularities (cuspidal beaks, cuspidal butterfly, cuspidal $S_1^{\pm}$, 5/2-cuspidal edge) to such tangent developables (cf. Propositions 4.2, 4.4, 4.6 and 4.9), we obtain the duality of singularities (Theorem 4.10). In the case of the cuspidal edge, we exhibit an example which does not satisfy the desired duality (see Example 4.11). Finally, in Section 5, we glance a global property of such conjugate operation, by proving that the conjugate of a complete flat front with embedded ends cannot be a complete flat front (Proposition 5.1).

2. Preliminaries

We denote by $\boldsymbol{R}^3$ the Euclidean 3-space. Let M be a connected smooth 2-manifold and

$$f : M \longrightarrow \boldsymbol{R}^3$$

a smooth map. A point $p \in M$ is called a *singular point* if f is not an immersion at p. Otherwise, we say p a *regular point*. Denote by $S(f)$ $(\subset M)$ the singular set. If $S(f)$ is empty, we call f a (regular) surface. In this case, at least locally, we can take a smooth unit normal vector field ν along f, that is, for every point $p \in M$, there exist an open neighborhood U of p and a smooth map $\nu : U \to S^2$ such that

$$df_q(\boldsymbol{v}) \cdot \nu(q) = 0 \quad \text{holds for each } q \in U \text{ and } \boldsymbol{v} \in T_qM, \tag{2.1}$$

where the dot '$\cdot$' is the canonical inner product on $\boldsymbol{R}^3$ and S^2 is the unit sphere

$$S^2 := \{\boldsymbol{x} \in \boldsymbol{R}^3 \,;\, \boldsymbol{x} \cdot \boldsymbol{x} = 1\}.$$

2.1. **Flat fronts.** A smooth map $f : M \to \boldsymbol{R}^3$ is called a *frontal* if, for each point $p \in M$, there exist a neighborhood U of p and a smooth map $\nu : U \to S^2$ which satisfies (2.1). Such a ν is called the unit normal vector field or the Gauss map of f. If ν can be defined throughout M, f is called *co-orientable*. If $(L :=)$ $(f, \nu) : U \to \boldsymbol{R}^3 \times S^2$ gives an immersion, f is called a *wave front* (or a *front*, for short).

A front f with a unit normal ν is called *flat* if $\operatorname{rank}(d\nu) \leq 1$ on M. Denote by $ds^2 := df \cdot df$ the first fundamental form of f. In the case that f is regular, f is flat as a front if and only if

f is flat as a regular surface (namely, the Gaussian curvature K of ds^2 is identically zero $K = 0$ on M).

A smooth map $f : M \to \boldsymbol{R}^3$ is called *complete* if there exists a symmetric covariant tensor T on M with compact support such that $ds^2 + T$ gives a complete Riemannian metric on M. If f is complete and the singular set $S(f)$ is non-empty, then $S(f)$ must be compact.

Murata-Umehara [17] proved the following.

Fact 2.1 ([17])**.** *Let $\boldsymbol{\xi} : S^1 \to S^2$ be a regular curve without inflection points, and $\alpha = a(t)dt$ a 1-form on $S^1 = \boldsymbol{R}/2\pi\boldsymbol{Z}$ such that $\int_{S^1} \boldsymbol{\xi}\, \alpha = 0$ holds. Then, $f : S^1 \times \boldsymbol{R} \to \boldsymbol{R}^3$ defined by*

$$f(t, v) := \sigma(t) + v\,\boldsymbol{\xi}(t) \qquad \left(\sigma(t) := \int_0^t a(\tau)\,\boldsymbol{\xi}(\tau)d\tau\right) \tag{2.2}$$

is a complete flat front with non-empty singular set. Conversely, let $f : M \to \boldsymbol{R}^3$ be a complete flat front defined on a connected smooth 2-manifold M. If the singular set $S(f)$ of f is not empty, then f is umbilic-free, co-orientable, M is diffeomorphic to $S^1 \times \boldsymbol{R}$, and f is given by (2.2)*. Moreover, if the ends of f are embedded, f has at least four singular points other than cuspidal edges.*

For the definition of umbilic points, see [17] (cf. [5, 6, 8]). The final statement of Fact 2.1 may be regarded as a variant of four vertex theorem for plane curves.

2.2. **Singularities of frontals.** Fix a smooth 2-manifold M and take two points $p_i \in M$ $(i = 1, 2)$. Let $f_i : (M, p_i) \to (\boldsymbol{R}^3, f(p_i))$ $(i = 1, 2)$ be two map germs. We say f_1 is $\mathcal{A}$*-equivalent* to f_2 if there exist diffeomorphism germs

$$\varphi : (M, p_1) \to (M, p_2) \quad \text{and} \quad \Phi : (\boldsymbol{R}^3, f(p_1)) \to (\boldsymbol{R}^3, f(p_2))$$

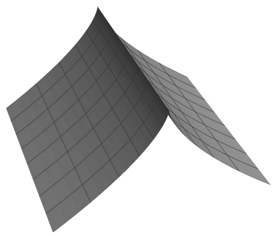
such that $\Phi \circ f_1 \circ \varphi^{-1} = f_2$. We set f_{CE}, f_{SW}, f_{CCR}, f_{CBK}, f_{CBF}, $f_{CS_k^{\pm}}$, f_{rCE} to be the germs from $(\boldsymbol{R}^2, 0)$ to $(\boldsymbol{R}^3, 0)$ given by:

$$\begin{aligned} f_{CE}(u, v) &:= (u, v^2, v^3), \\ f_{SW}(u, v) &:= (4u^3 + 2uv, 3u^4 + u^2v, -v), \\ f_{CCR}(u, v) &:= (u, v^2, uv^3), \\ f_{CBK}(u, v) &:= (v, -2u^3 + uv^2, -3u^4 + u^2v^2), \\ f_{CBF}(u, v) &:= (u, 5v^4 + 2uv, 4v^5 + uv^2 - u^2), \\ f_{CS_k^{\pm}}(u, v) &:= (u, v^2, v^3(u^{k+1} \pm v^2)), \\ f_{rCE}(u, v) &:= (u, v^2, v^5), \end{aligned} \tag{2.3}$$

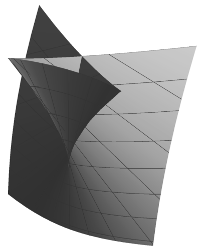
respectively, where k is a positive integer. We call the map germ f_{CE} (resp. f_{SW}, f_{CCR}, f_{CBK}, f_{CBF}, $f_{CS_k^{\pm}}$, f_{rCE}) the *cuspidal edge* (resp. *swallowtail, cuspidal cross cap, cuspidal beaks, cuspidal butterfly, cuspidal $S_k^{\pm}$ singularity, $5/2$-cuspidal edge*).

Kokubu–Rossman–Saji–Umehara–Yamada [15] gave a useful criteria for cuspidal edge and swallowtail. Similar useful criteria for other singularities are given in the following: [4] for cuspidal cross cap (cf. Fact 3.4); [14] for cuspidal beaks (cf. Fact 4.1); [13] for cuspidal butterfly (cf. Fact 4.3); [19] for cuspidal $S_k^{\pm}$ singularity (cf. Fact 4.5); and [9] for 5/2-cuspidal edge (cf. Fact 4.8). To state such criteria, we shall review some basic notions for frontals.

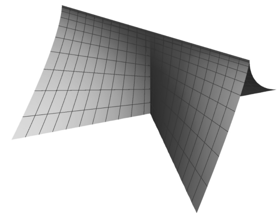
Let $f : M \to \boldsymbol{R}^3$ be a frontal with the (locally defined) unit normal ν. Take a point $p \in M$. Let $(U; u, v)$ be a coordinate neighborhood of p. We call $\lambda := \det(f_u, f_v, \nu)$ the *signed area density function*. Remark that p is a singular point of f if and only if $\lambda(p) = 0$. If $d\lambda(p) \neq 0$, a singular



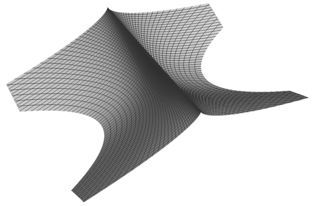
Cuspidal edge f_{CE}



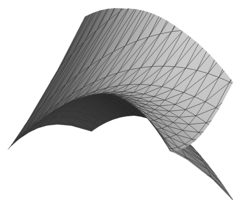
Swallowtail f_{SW}



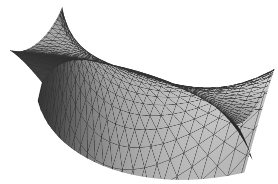
Cuspidal cross cap f_{CCR}



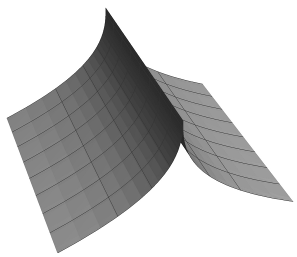
Cuspidal S_1^+ singularity $f_{CS_1^+}$



Cuspidal butterfly f_{CBF}



Cuspidal beaks f_{CBK}



5/2-cuspidal edge f_{rCE}

FIGURE 1. The images of standard models of the singularities (f_{CE}, f_{SW}, f_{CCR}, $f_{CS_1^+}$, f_{CBF}, f_{CBK}, f_{rCE}) given in (2.3).

point p is called *non-degenerate*. We remark that if p is non-degenerate, then $\operatorname{rank}(df)_p = 1$ holds. By the implicit function theorem, there exists a regular curve $\gamma(t)$ ($|t| < \varepsilon$) on the uv-plane such that $\gamma(0) = p$ and the image of γ coincides with the singular point set $S(f)$ near p, where $\varepsilon > 0$. We call $\gamma(t)$ the *singular curve* and $\gamma' = d\gamma/dt$ the singular direction. Then, there exists a non-zero smooth vector field $\zeta(t)$ along $\gamma(t)$ such that $\zeta(t)$ is a null vector (i.e., $df(\zeta(t)) = 0$) for each t. Such a vector field $\zeta(t)$ is called a *null vector field*. On the other hand, a non-vanishing smooth vector field $\zeta = \zeta(u, v)$ on U so that $\zeta|_{S(f)}$ gives a kernel direction of f is also called a *null vector field*. We set the functions $\delta(t)$ and $\psi_{ccr}(t)$ as

$$\delta(t) := \det\left(\gamma'(t),\, \zeta(t)\right), \quad \psi_{ccr}(t) := \det\left((f \circ \gamma)'(t),\, (\nu \circ \gamma)(t),\, d\nu(\zeta(t))\right), \tag{2.4}$$

respectively. Later we use these functions in the criteria for various singularity types (cf. Facts 3.4, 4.1, 4.3, 4.5 and 4.8).

2.3. **Frontals in 2-sphere.** Let J be an open interval of $\boldsymbol{R}$. A smooth map $\boldsymbol{\xi} : J \to S^2$ is called a *frontal* if there exists a smooth unit vector field $\boldsymbol{n}$ along $\boldsymbol{\xi}$ such that $\boldsymbol{\xi}' \cdot \boldsymbol{n} = 0$ holds. We call $\boldsymbol{n}$ the *unit normal vector field* or the *spherical dual*. The pair $(\boldsymbol{\xi}, \boldsymbol{n})$ gives a Legendre curve in the unit tangent bundle

$$T_1S^2 = \{(p, v) \in S^2 \times S^2\,;\, p \cdot v = 0\}$$

with respect to the canonical contact structure. Since $\boldsymbol{\xi} \cdot \boldsymbol{n}' = 0$, there exist smooth 1-forms ρ, ω such that

$$d\boldsymbol{\xi} = \rho\,\boldsymbol{\eta}, \qquad d\boldsymbol{n} = -\omega\,\boldsymbol{\eta}, \tag{2.5}$$

where we set $\boldsymbol{\eta} := \boldsymbol{n} \times \boldsymbol{\xi}$. Then, the frame $\mathcal{F}(t) := \{\boldsymbol{\xi}(t), \boldsymbol{\eta}(t), \boldsymbol{n}(t)\}$ satisfies

$$\mathcal{F}^{-1} d\mathcal{F} = \begin{pmatrix} 0 & -\rho & 0 \\ \rho & 0 & -\omega \\ 0 & \omega & 0 \end{pmatrix}, \tag{2.6}$$

where we used the identity $d\boldsymbol{\eta} = -\rho\,\boldsymbol{\xi} + \omega\,\boldsymbol{n}$. Conversely, the following holds.

Fact 2.2 ([22, Theorem 2.5])**.** *Let ρ, ω be smooth 1-forms on an interval J. Then, there exists a frontal $\boldsymbol{\xi} : J \to S^2$ with the spherical dual $\boldsymbol{n}$ such that (2.5) holds.*

Therefore, we may conclude that there exists a one-to-one correspondence between frontals with spherical duals and pairs of smooth 1-forms. We call the pair of 1-forms (ρ, ω) the *data* of the frontal $\boldsymbol{\xi} : J \to S^2$.

3. Conjugates of tangent developables

In this section, comparing the criteria for swallowtail and cuspidal cross cap, we give a definition of the conjugates of developable frontals.

3.1. **Developable frontals.** Let J be an open interval including $0 \in J$. Take 1-forms α, β on J and a frontal $\boldsymbol{\xi} : J \to S^2$ with the spherical dual $\boldsymbol{n}$. Then a smooth map $f : J \times \boldsymbol{R} \to \boldsymbol{R}^3$ defined by

$$f(t, v) := \sigma(t) + v\,\boldsymbol{\xi}(t) \qquad \left(\sigma(t) := \int_0^t (\alpha\,\boldsymbol{\xi} + \beta\,\boldsymbol{\eta}),\ \boldsymbol{\eta} := \boldsymbol{n} \times \boldsymbol{\xi}\right) \tag{3.1}$$

is a co-orientable frontal in $\boldsymbol{R}^3$ so that $\nu(t, v) := \boldsymbol{n}(t)$ is a unit normal. We shall call $f(t, v)$ an *a-orientable admissible developable frontal* and v is called the *asymptotic parameter*. The quadruple of the 1-forms $(\alpha, \beta, \rho, \omega)$ is independent of the choice of the parameter t on J as a 1-dimensional manifold, which we call the *data* of $f(t, v)$. Here, (ρ, ω) is the data corresponding to a frontal $\boldsymbol{\xi}$ in S^2 with the spherical dual $\boldsymbol{n}$ (cf. (2.5)).

Remark 3.1. We remark that Murata–Umehara defined a-orientable admissible developable frontals in [17, Definition 2.3], where 'a-orientable' means 'asymptotically orientable', see [17, page 289]. They gave a representation formula in [17, Theorem 2.8]. Our definition is based on [17, Theorem 2.8].

If f is a cylinder, then $\boldsymbol{\xi} : J \to S^2$ is a constant map, that is, $r(t) = 0$ holds for all $t \in J$, where $\rho = r(t)dt$. We call a point $p_0 = (t_0, v_0)$ a *cylindrical point* of $f(t, v)$ if $\boldsymbol{\xi}'(t_0) = 0$ (i.e., $r(t_0) = 0$) holds[1]. We denote by $S_c(f)$ (resp. $S_{nc}(f)$) the set of cylindrical singular points (resp. non-cylindrical singular points).

Lemma 3.2 (cf. [17, Proposition 2.16])**.** *Let $f(t, v)$ be an a-orientable admissible developable frontal whose data is given by $(\alpha, \beta, \rho, \omega) = (a(t)\,dt, b(t)\,dt, r(t)\,dt, w(t)\,dt)$. Then, a point*

$$p_0 = (t_0, v_0) \in J \times \boldsymbol{R}$$

is a singular point of f if and only if $b(t_0) + v_0\,r(t_0) = 0$. Moreover,

- *f is a front at a singular point $p_0 = (t_0, v_0)$ if and only if $w(t_0) \neq 0$.*

[1] Cylindrical singular points are *linear singular points* in the sense of [17, Definition 2.15].

- *$p_0 = (t_0, v_0)$ is a cylindrical singular point of f if and only if $b(t_0) = r(t_0) = 0$. Such a $p_0 \in S_c(f)$ is non-degenerate if and only if $b'(t_0) + v_0\, r'(t_0) \neq 0$. Setting*

$$\gamma_c(v) := (t_0, v), \qquad \zeta_c(v) := \partial_t - a(t_0)\,\partial_v, \tag{3.2}$$

we have that $\gamma_c(v)$ is a singular curve passing through $\gamma_c(v_0) = p_0$, and $\zeta_c(v)$ is a null vector field along $\gamma_c(v)$. Moreover, we have (cf. (2.4))

$$\delta_c(v) := \det\left(\gamma_c'(v), \zeta_c(v)\right) = -1. \tag{3.3}$$

- *$p_0 = (t_0, v_0)$ is a non-cylindrical singular point of f if and only if $r(t_0) \neq 0$ and $v_0 = -b(t_0)/r(t_0)$. Such a $p_0 \in S_{nc}(f)$ is non-degenerate, and setting*

$$\gamma_{nc}(t) := \left(t, -\frac{b(t)}{r(t)}\right), \qquad \zeta_{nc}(t) := \partial_t - a(t)\,\partial_v, \tag{3.4}$$

we have that $\gamma_{nc}(t)$ is a singular curve passing through $\gamma_{nc}(t_0) = p_0$, and $\zeta_{nc}(t)$ is a null vector field along $\gamma_{nc}(t)$. Moreover, we have (cf. (2.4))

$$\delta_{nc}(t) := \det\left(\gamma_{nc}'(t), \zeta_{nc}(t)\right) = -a(t) + \left(\frac{b(t)}{r(t)}\right)'. \tag{3.5}$$

Proof. By (2.5), we have

$$f_t = a(t)\,\boldsymbol{\xi}(t) + (b(t) + v\,r(t))\,\boldsymbol{\eta}(t), \quad f_v = \boldsymbol{\xi}(t). \tag{3.6}$$

So, the signed area density function λ is given by

$$\lambda = \det(f_t, f_v, \nu) = (b(t) + v\,r(t))\det\left(\boldsymbol{\eta}(t), \boldsymbol{\xi}(t), \boldsymbol{n}(t)\right) = -b(t) - v\,r(t). \tag{3.7}$$

Thus, we have $S(f) = \{(t, v) \in J \times \boldsymbol{R}\,;\, b(t) + v\,r(t) = 0\}$ and

$$-\lambda_t = b'(t) + v\,r'(t), \qquad -\lambda_v = r(t). \tag{3.8}$$

On the singular set $S(f)$, $f_t - a(t)f_v = 0$ holds. Thus, setting $\zeta(t, v) := \partial_t - a(t)\partial_v$, we have $df(\zeta) = 0$ at a singular point p_0. Since f is front at $p_0 \in S(f)$ if and only if

$$(dL)_{p_0} = ((df)_{p_0}, (d\nu)_{p_0})$$

is injective, this condition is equivalent to $(d\nu)_{p_0}(\zeta) \neq 0$. Since $-d\nu(\zeta) = -\boldsymbol{n}' = w\boldsymbol{\eta}$, f is front at $p_0 \in S(f)$ if and only if $w(t_0) \neq 0$.

If p_0 is cylindrical, $r(t_0) = 0$ holds. Thus, p_0 is a cylindrical singular point if and only if $r(t_0) = 0$ and $b(t_0)\,(= b(t_0) + v_0\,r(t_0)) = 0$. By (3.8), p_0 is non-degenerate if and only if $b'(t_0) + v_0\,r'(t_0) \neq 0$. In this case, $\gamma_c(v)$ given in (3.2) is a singular curve passing through $\gamma_c(v_0) = p_0$. By (3.6), $f_t - a(t_0)\,f_v = 0$ holds along $\gamma_c(v)$, and hence we have $\zeta_c(v)$ given in (3.2) is a null vector field along $\gamma_c(v)$.

If p_0 is a non-cylindrical singular point, $r(t_0) \neq 0$ holds. By (3.8), p_0 must be non-degenerate. Then $\gamma_{nc}(t)$ given in (3.4) is a singular curve passing through $\gamma_{nc}(t_0) = p_0$. By (3.6),

$$f_t - a(t)\,f_v = 0$$

holds along $\gamma_{nc}(t)$, and hence we have $\zeta_{nc}(t)$ given in (3.4) is a null vector field along $\gamma_{nc}(t)$. $\square$

As we seen in Lemma 3.2, the cylindrical and non-cylindrical singular sets, $S_c(f)$ and $S_{nc}(f)$, are written as

$$S_c(f) = \{(t, v) \in J \times \boldsymbol{R}\,;\, b(t) = r(t) = 0\}, \tag{3.9}$$

$$S_{nc}(f) = \left\{(t, v) \in J \times \boldsymbol{R}\,;\, r(t) \neq 0,\ v = -\frac{b(t)}{r(t)}\right\}, \tag{3.10}$$

respectively.

Murata–Umehara [17] applied the criteria for cuspidal edge and swallowtail given in [15] to developable frontals as follows:

Fact 3.3 ([17, Proposition 2.16]). *Let $f(t,v)$ be an a-orientable admissible developable frontal whose data is given by $(\alpha,\beta,\rho,\omega)=(a(t)\,dt, b(t)\,dt, r(t)\,dt, w(t)\,dt)$. Then, a point*

$$p_0=(t_0,v_0)\in J\times \boldsymbol{R}$$

is a singular point of f if and only if $b(t_0)+v_0\,r(t_0)=0$. Moreover, for a singular point $p_0=(t_0,v_0)$ of f, we have that

- *f at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if*

$$r(t_0)\neq 0,\quad a(t_0)\neq \left.\left(\frac{b(t)}{r(t)}\right)'\right|_{t=t_0},\quad w(t_0)\neq 0,$$

or

$$r(t_0)=0,\quad b'(t_0)+v_0\,r'(t_0)\neq 0,\quad w(t_0)\neq 0.$$

- *f at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if*

$$r(t_0)\neq 0,\quad a(t_0)=\left.\left(\frac{b(t)}{r(t)}\right)'\right|_{t=t_0},\quad a'(t_0)\neq \left.\left(\frac{b(t)}{r(t)}\right)''\right|_{t=t_0},\quad w(t_0)\neq 0. \tag{3.11}$$

We can observe that swallowtails never appear on the cylindrical singular set $S_c(f)$.

3.2. **Cuspidal cross cap.** Here we review the criterion for the cuspidal cross cap given by Fujimori–Saji–Umehara–Yamada [4].

Fact 3.4 (Criterion for cuspidal cross cap [4]). *Let $f:U\to\boldsymbol{R}^3$ be a frontal defined on a domain U of $\boldsymbol{R}^2$, with the unit normal ν, and $p\in U$ a non-degenerate singular point of f. And let $\gamma(t)$ be a singular curve such that $\gamma(0)=p$, $\zeta(t)$ a null vector field, $\delta(t)$ and $\psi_{ccr}(t)$ be the functions defied by (2.4). Then, the map germ f at p is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if $\delta(0)\neq 0$, $\psi_{ccr}(0)=0$ and $\psi'_{ccr}(0)\neq 0$.*

Now, we shall apply Fact 3.4 to a-orientable admissible developable frontals.

Proposition 3.5. *Let $f(t,v)$ be an a-orientable admissible developable frontal whose data is given by $(\alpha,\beta,\rho,\omega)=(a(t)\,dt, b(t)\,dt, r(t)\,dt, w(t)\,dt)$. For a singular point $p_0=(t_0,v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if*

$$r(t_0)\neq 0,\quad a(t_0)\neq \left.\left(\frac{b(t)}{r(t)}\right)'\right|_{t=t_0},\quad w(t_0)=0,\quad w'(t_0)\neq 0. \tag{3.12}$$

Proof. First, assume that $p_0\in S_{nc}(f)$. By Lemma 3.2, $\gamma_{nc}(t)$ is a singular curve passing through $\gamma_{nc}(t_0)=p_0$, and $\zeta_{nc}(t)$ is a null vector field along $\gamma_{nc}(t)$, where $\gamma_{nc}(t)$ and $\zeta_{nc}(t)$ are given by (3.4). Let $\delta_{nc}(t)$ be the function given in (3.5). By Lemma 3.2, the function δ given in (2.4) coincides with $\delta_{nc}(t)$. On the other hand, setting $\hat{\gamma}_{nc}(t):=f(\gamma_{nc}(t))$, we have

$$\hat{\gamma}'_{nc}(t)=-\delta_{nc}(t)\boldsymbol{\xi}(t).$$

Hence, the function ψ_{ccr} given in (2.4) is

$$\psi_{ccr}(t)=-\delta_{nc}(t)\det\left(\boldsymbol{\xi}(t),\,\boldsymbol{n}(t),\,\boldsymbol{n}'(t)\right)=-\delta_{nc}(t)w(t). \tag{3.13}$$

Therefore, f at p_0 is $\mathcal{A}$-equivalent to cuspidal cross cap if and only if (3.12) holds.

Next, we shall prove that, if $p_0\in S_c(f)$, f at p_0 cannot be $\mathcal{A}$-equivalent to the cuspidal cross cap. By Lemma 3.2, $\gamma_c(v)$ is a singular curve passing through $\gamma_c(v_0)=p_0$, and $\zeta_c(v)$ is a null vector field along $\gamma_c(v)$, where $\gamma_c(v)$ and $\zeta_c(v)$ are given by (3.2). By Lemma 3.2, the function δ given in (2.4) is identically -1. On the other hand, setting $\hat{\gamma}_c(v) := f(\gamma_c(v))$, we have $\hat{\gamma}'_c(v) = \boldsymbol{\xi}(t_0)$. Hence, the function ψ_{ccr} given in (2.4) is

$$\psi_{ccr}(v) = \det\left(\boldsymbol{\xi}(t),\, \boldsymbol{n}(t_0),\, w(t_0)\boldsymbol{\eta}(t_0)\right) = w(t_0). \tag{3.14}$$

Therefore, $\psi_{ccr}(v_0) = 0$ and $\psi'_{ccr}(v_0) \neq 0$ do not occur at the same time. Thus, f at p_0 cannot be $\mathcal{A}$-equivalent to the cuspidal cross cap. □

3.3. **Observation and definition.** For an a-orientable admissible developable frontal $f = f(t, v)$, we would like to find its conjugate $f^\sharp$ which satisfies the so-called *duality of singularities* as in Fact 1.1 in the introduction.

We shall compare the condition (3.11) for swallowtail and that (3.12) for cuspidal cross cap. If $\beta = b(t)\,dt$ is identically zero, (3.11) is equivalent to

$$r(t_0) \neq 0, \quad a(t_0) = 0, \quad a'(t_0) \neq 0, \quad w(t_0) \neq 0, \tag{3.15}$$

and (3.12) is equivalent to

$$r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) = 0, \quad w'(t_0) \neq 0. \tag{3.16}$$

Thus, for an a-orientable admissible developable frontal $f = f(t, v)$ with the data $(\alpha, 0, \rho, \omega)$, if we set $f^\sharp$ to be the a-orientable admissible developable frontal whose data is given by

$$(\alpha^\sharp, 0, \rho^\sharp, \omega^\sharp) := (\omega, 0, \rho, \alpha),$$

we have that *f at p is $\mathcal{A}$-equivalent to the swallowtail if and only if $f^\sharp$ at p is $\mathcal{A}$-equivalent to the cuspidal cross cap*. Namely, f and $f^\sharp$ satisfy the duality of singularities.

A-orientable admissible developable frontals with $\beta = 0$ are *tangent developables*. In fact, when $\beta = 0$, f given in (3.1) is written as

$$f(t, v) := \sigma(t) + v\,\boldsymbol{\xi}(t) \qquad \left(\sigma(t) := \int_0^t \alpha\,\boldsymbol{\xi},\ \boldsymbol{\eta} := \boldsymbol{n} \times \boldsymbol{\xi}\right). \tag{3.17}$$

Since $\sigma'(t)$ and $\boldsymbol{\xi}(t)$ are linearly dependent, we may conclude that f is a tangent developable.

Definition 3.6 (A-tangent developable). We call an a-orientable admissible developable frontal with $\beta = 0$ an *a-tangent developable*. For an a-tangent developable f, the triplet of the 1-forms (α, ρ, ω) is also called the *data*. Then, the a-tangent developable $f^\sharp$ whose data is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha)$ is called the *conjugate* of f.

We remark that, by Lemma 3.2 and $\rho = \rho^\sharp$, the singular set of an a-tangent developable f coincides with that of the conjugate $f^\sharp$ of f, namely $S(f) = S(f^\sharp) = \{(t, v) \in J \times \boldsymbol{R}\,;\, v\,r(t) = 0\}$ holds. In the case that the a-tangent developable $f = f(t, v)$ is defined on $M := S^1 \times \boldsymbol{R}$, the domain of the conjugate $f^\sharp$ is the universal covering $\tilde{M} = \boldsymbol{R}^2$ of M.

Remark 3.7. An a-orientable admissible developable frontal without cylindrical points is an a-tangent developable. If (t, v) is non-cylindrical, by changing the parameter $v \mapsto v - b(t)/r(t)$, f can be written as

$$f = \sigma(t) + \left(v - \frac{b(t)}{r(t)}\right)\boldsymbol{\xi}(t) = \tilde{\sigma}(t) + v\,\boldsymbol{\xi}(t).$$

Here we set $\tilde{\sigma}(t) := \sigma(t) - (b(t)/r(t))\boldsymbol{\xi}(t)$, which satisfies that $\tilde{\sigma}'(t)$ and $\boldsymbol{\xi}(t)$ are linearly dependent.

Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ be the canonical orthonormal basis of $\boldsymbol{R}^3$, namely, $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3) = Id$, where Id is the identity matrix $Id := \mathrm{diag}(1, 1, 1)$. The procedure of constructing the a-tangent developable from a given data (α, ρ, ω) is as follows:

- Take $\mathcal{F}_0 \in \mathrm{SO}(3)$ arbitrarily.

- Let $\mathcal{F} = \mathcal{F}(t)$ be a solution of (2.6) with the initial value $\mathcal{F}(t_0) = \mathcal{F}_0$.
- Setting $\boldsymbol{\xi}(t) := \mathcal{F}(t)\boldsymbol{e}_1$, then,

$$f(t,v) = \sigma(t) + v\,\boldsymbol{\xi}(t) \qquad \left(\sigma(t) := \int_{t_0}^{t} \alpha\,\boldsymbol{\xi}\right)$$

is an a-tangent developable whose data is given by (α, ρ, ω) such that $\boldsymbol{n}(t) := \mathcal{F}(t)\boldsymbol{e}_3$ is a unit normal.

Taking account of the data of the conjugate $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) = (\omega, \rho, \alpha)$, we have the following.

Lemma 3.8. *Let $f = f(t,v)$ be the a-tangent developable defined on $J \times \boldsymbol{R}$ whose data is given by (α, ρ, ω). Fix $t_0 \in J$. Take a solution $\mathcal{F}^\sharp = \mathcal{F}^\sharp(t)$ of the following initial value problem*

$$(\mathcal{F}^\sharp)^{-1} d\mathcal{F}^\sharp = \begin{pmatrix} 0 & -\rho & 0 \\ \rho & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix}, \qquad \mathcal{F}^\sharp(t_0) = \mathit{Id}. \tag{3.18}$$

Then setting $\boldsymbol{\xi}^\sharp(t) := \mathcal{F}^\sharp(t)\boldsymbol{e}_1$, the conjugate $f^\sharp$ is given by

$$f^\sharp(t,v) = \sigma^\sharp(t) + v\,\boldsymbol{\xi}^\sharp(t) \qquad \left(\sigma^\sharp(t) := \int_{t_0}^{t} \omega\,\boldsymbol{\xi}^\sharp\right) \tag{3.19}$$

such that the data of $f^\sharp$ is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha)$, and $\boldsymbol{n}^\sharp(t) := \mathcal{F}^\sharp(t)\boldsymbol{e}_3$ gives a unit normal of $f^\sharp$.

By [17, Proposition 2.16] (cf. Fact 3.3) and Proposition 3.5, we have the following:

Corollary 3.9. *Let $f(t,v)$ be an a-tangent developable whose data is given by*

$$(\alpha, \rho, \omega) = (a(t)\,dt, r(t)\,dt, w(t)\,dt).$$

Take a singular point $p_0 = (t_0, v_0) \in S(f)$. Then,

- *f at p_0 is $\mathcal{A}$-equivalent to the cuspidal edge if and only if*

$$v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) \neq 0,$$

or

$$v_0 \neq 0, \quad r(t_0) = 0, \quad r'(t_0) \neq 0, \quad w(t_0) \neq 0.$$

- *f at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if*

$$v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) = 0, \quad a'(t_0) \neq 0, \quad w(t_0) \neq 0. \tag{3.20}$$

- *f at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap if and only if*

$$v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) = 0, \quad w'(t_0) \neq 0. \tag{3.21}$$

In particular, f at p_0 is $\mathcal{A}$-equivalent to the swallowtail if and only if $f^\sharp$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap, where $f^\sharp$ is the conjugate of f.

As an example, we calculate the conjugate of the standard swallowtail.

Example 3.10 (Conjugate of the standard swallowtail). Let f_{SW} be the standard swallowtail given in (2.3). By a parameter change $(u,v) \mapsto (t, y - 6t^2)$, we have

$$f_{SW}(t,y) = (-8t^3, -3t^4, 6t^2) + y\,(2t, t^2, -1).$$

Thus, setting $v := y\sqrt{1+4t^2+t^4}$, f_{SW} is an a-tangent developable $f_{SW}(t,v) = \sigma(t) + v\,\boldsymbol{\xi}(t)$, where

$$\sigma(t) := (-8t^3, -3t^4, 6t^2), \qquad \boldsymbol{\xi}(t) := \frac{1}{\sqrt{1+4t^2+t^4}}(2t, t^2, -1).$$

Since $\sigma'(t) = -12t\sqrt{1+4t^2+t^4}\,\boldsymbol{\xi}(t)$, we have

$$a(t) = -12t\sqrt{1+4t^2+t^4}. \tag{3.22}$$

Then the spherical dual $\boldsymbol{n}(t)$ of $\boldsymbol{\xi}(t)$ and $\boldsymbol{\eta}(t) = \boldsymbol{n}(t) \times \boldsymbol{\xi}(t)$ are given by

$$\begin{aligned}\boldsymbol{n}(t) &= \frac{1}{\sqrt{1+t^2+t^4}}\left(t, -1, t^2\right),\\ \boldsymbol{\eta}(t) &= \frac{1}{\sqrt{1+4t^2+t^4}\sqrt{1+t^2+t^4}}\left(1-t^4, t+2t^3, 2t+t^3\right),\end{aligned}$$

respectively. Hence we have

$$r(t) = \frac{2\sqrt{1+t^2+t^4}}{1+4t^2+t^4}, \qquad w(t) = -\frac{\sqrt{1+4t^2+t^4}}{1+t^2+t^4}, \tag{3.23}$$

where $r(t) = \boldsymbol{\xi}'(t)\cdot\boldsymbol{\eta}(t)$, $w(t) = -\boldsymbol{n}'(t)\cdot\boldsymbol{\eta}(t)$.

Then, applying Lemma 3.8 with $(\alpha, \rho, \omega) = (a(t)\,dt, r(t)\,dt, w(t)\,dt)$ and $t_0 = 0$, we obtain the conjugate $f^{\sharp}_{SW}(t, v)$, where $a(t)$, $r(t)$, $w(t)$ are given by (3.22) and (3.23), respectively (cf. Figure 2).

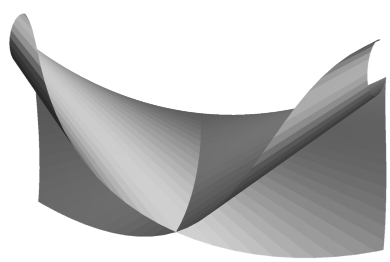


FIGURE 2. The a-tangent developable $f^{\sharp}_{SW}$ which is the conjugate of the standard swallowtail f_{SW} given by (2.3) (cf. Figure 1). By Corollary 3.9, we have that $f^{\sharp}_{SW}$ at $(t, v) = (0, 0)$ is $\mathcal{A}$-equivalent to the cuspidal cross cap. This figure is plotted by integrating (3.18) and (3.19) numerically.

4. Other singularities

Here, we shall write down the criteria for other singularities (cuspidal beaks, cuspidal butterfly, cuspidal $S_1^{\pm}$ singularity, 5/2-cuspidal edge) on a-tangent developables in terms of their data.

4.1. **Cuspidal beaks.** First, we review the criterion for the cuspidal beaks given by Izumiya–Saji–Takahashi [14].

Fact 4.1 (Criterion for cuspidal beaks [14])**.** *Let $f : U \to \boldsymbol{R}^3$ be a front defined on a domain U of $\boldsymbol{R}^2$ with the unit normal ν. Also let $p \in U$ be a singular point of f and ζ a null vector field. Then, the map germ f at p is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if* $\operatorname{rank}(df)_p = 1$, $d\lambda(p) = 0$, $\det \operatorname{Hess}\lambda(p) < 0$ *and* $\zeta\zeta\lambda(p) \neq 0$ *hold.*

Applying Fact 4.1 to a-tangent developables, we have the following.

Proposition 4.2. *Let $f(t,v)$ be an a-tangent developable whose data is given by*

$$(\alpha,\rho,\omega)=(a(t)\,dt, r(t)\,dt, w(t)\,dt).$$

Then, for a singular point $p_0=(t_0,v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if

$$v_0=0,\quad r(t_0)=0,\quad r'(t_0)\neq 0,\quad a(t_0)\neq 0,\quad w(t_0)\neq 0. \tag{4.1}$$

Proof. We remark that for any singular point p_0 of f, $\mathrm{rank}(df)_{p_0}=1$ holds (cf. (3.6)). Hence, by Fact 4.1, f at p_0 is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if $d\lambda(p_0)=0$, $\det\mathrm{Hess}\lambda(p_0)<0$, $\zeta\zeta\lambda(p_0)\neq 0$, and f is a front at p_0. By (3.8), $d\lambda(p_0)=0$ if and only if $r(t_0)=0$ (i.e., p_0 is cylindrical) and $v_0\,r'(t_0)=0$. Since the signed area density function λ is given by

$$\lambda(t,v)=-v\,r(t)$$

(cf. (3.7)), we have

$$\det\mathrm{Hess}\lambda=\det\begin{pmatrix}\lambda_{tt} & \lambda_{tv}\\ \lambda_{tv} & \lambda_{vv}\end{pmatrix}=-\lambda_{tv}^2=-(r')^2.$$

Hence, $\det\mathrm{Hess}\lambda(p_0)<0$ if and only if $r'(t_0)\neq 0$. As we see in the proof of Lemma 3.2, $\zeta(t,v):=\partial_t-a(t)\partial_v$ gives a null vector field. Since $\zeta\lambda=v\,r'(t)-a(t)r(t)$, we have

$$\zeta^2\lambda=v\,r''(t)-a'(t)r(t)-2a(t)r'(t). \tag{4.2}$$

Therefore, f at p_0 is $\mathcal{A}$-equivalent to the cuspidal beaks if and only if (4.1) holds. □

4.2. **Cuspidal butterfly.** Next, we review the criterion for the cuspidal butterfly given by Izumiya–Saji [13].

Fact 4.3 (Criterion for cuspidal butterfly [13])**.** *Let $f:U\to\boldsymbol{R}^3$ be a front defined on a domain U of $\boldsymbol{R}^2$ with the unit normal ν. Take a non-degenerate singular point $p\in U$ of f. Let $\gamma(t)$ be a singular curve such that $\gamma(0)=p$ and $\zeta(t)$ a null vector field. Then, the map germ f at p is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if $\delta(0)=\delta'(0)=0$ and $\delta''(0)\neq 0$ hold.*

Applying Fact 4.3 to a-tangent developables, we have the following.

Proposition 4.4. *Let $f(t,v)$ be an a-tangent developable whose data is given by*

$$(\alpha,\rho,\omega)=(a(t)\,dt, r(t)\,dt, w(t)\,dt).$$

Then, for a singular point $p_0=(t_0,v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if

$$v_0=0,\quad r(t_0)\neq 0,\quad a(t_0)=a'(t_0)=0,\quad a''(t_0)\neq 0,\quad w(t_0)\neq 0. \tag{4.3}$$

Proof. By (4.2), we have

$$\zeta^3\lambda=v\,r'''(t)-a''(t)r(t)-3a'(t)r'(t)-3a(t)r''(t). \tag{4.4}$$

Hence, by Fact 4.3, f at p_0 is $\mathcal{A}$-equivalent to the cuspidal butterfly if and only if

(i) f is a front at $p_0=(t_0,v_0)$ (i.e., $w(t_0)\neq 0$),
(ii) $p_0=(t_0,v_0)$ is non-degenerate (i.e., $r(t_0)\neq 0$ or $r(t_0)=0$, $v_0\,r'(t_0)\neq 0$),
(iii) $v_0\,r(t_0)=0$,
(iv) $v_0\,r'(t_0)-a(t_0)r(t_0)=0$,
(v) $v_0\,r''(t_0)-a'(t_0)r(t_0)-2a(t_0)r'(t_0)=0$,
(vi) $v_0\,r'''(t_0)-a''(t_0)r(t_0)-3a'(t_0)r'(t_0)-3a(t_0)r''(t_0)\neq 0$.

If we assume that $p_0=(t_0,v_0)$ is cylindrical (i.e., $r(t_0)=0$), the condition (i) implies $v_0\,r'(t_0)\neq 0$. This contradicts the condition (iv), $v_0\,r'(t_0)=0$. Thus, we have $r(t_0)\neq 0$. Then, we can check that the conditions (i)–(vi) are equivalent to (4.3). □

4.3. **Cuspidal $S_1^{\pm}$ singularity.** Now, we review the criterion for the cuspidal $S_1^{\pm}$ singularity given by Saji [19].

Fact 4.5 (Criterion for cuspidal $S_1^{\pm}$ singularity [19])**.** *Let $f : U \to \boldsymbol{R}^3$ be a frontal defined on a domain U of $\boldsymbol{R}^2$ with the unit normal ν. Take a non-degenerate singular point $p \in U$ of f. Let $\gamma(t)$ be a singular curve such that $\gamma(0) = p$ and ζ a null vector field. Then, the map germ f at p is $\mathcal{A}$-equivalent to the cuspidal S_1^+ singularity* (*resp. the cuspidal S_1^- singularity*) *if and only if the following* (i)-(iv) *hold:*

(i) $\delta(0) \neq 0$,

(ii) $\psi_{ccr}(0) = \psi'_{ccr}(0) = 0$ *and*

$$(d_1 :=)\, \psi''_{ccr}(0) \neq 0, \tag{4.5}$$

(iii) *there exist a regular curve $c : (-\varepsilon, \varepsilon) \to U$ and $\ell \in \boldsymbol{R}$ such that $c(0) = p$, $c'(0)$ is parallel to $\zeta(0)$, $\hat{c}''(0) \neq 0$, $\hat{c}'''(0) = \ell\, \hat{c}''(0)$ and*

$$(d_2 :=) \det\Big(df_p(\xi_p),\, \hat{c}''(0),\, 3\hat{c}^{(5)}(0) - 10\ell\, \hat{c}^{(4)}(0)\Big) \neq 0 \tag{4.6}$$

hold, where $\hat{c} := f \circ c$ and $\xi_p := \gamma'(0)$,

(iv) *the product $d_1 d_2$ is positive* (*resp. negative*)*, where d_1, d_2 are given by* (4.5)*,* (4.6)*, respectively. Here, we choose ζ and c so that $c'(0)$ points the same direction as the null vector $\zeta(0)$ and that $\{\gamma'(0), \zeta(0)\}$ is positively oriented.*

Applying Fact 4.5 to a-tangent developables, we have the following.

Proposition 4.6. *Let $f(t, v)$ be an a-tangent developable whose data is given by*

$$(\alpha, \rho, \omega) = (a(t)\, dt, r(t)\, dt, w(t)\, dt).$$

Then, for a singular point $p_0 = (t_0, v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the cuspidal S_1^+ singularity if and only if

$$v_0 = 0, \quad r(t_0) \neq 0, \quad a(t_0) \neq 0, \quad w(t_0) = w'(t_0) = 0, \quad w''(t_0) \neq 0. \tag{4.7}$$

Remark 4.7. It is known that, by Ishikawa's theorem [10], developable surfaces do not admit any cuspidal $S_k^{\pm}$ singularities for $k > 1$. We also remark that, by Mond [16] and Saji [19, Theorem 4.1], tangent developable surfaces of a regular space curve do not admit cuspidal S_1^- singularity, as in the following proof.

Proof of Proposition 4.6. We first show that p_0 is non-cylindrical. If we assume $p_0 = (t_0, v_0)$ is cylindrical, we have $\gamma_c(v) = (t_0, v)$ is a singular curve passing through $\gamma_c(v_0) = p_0$. Then the function ψ_{ccr} defined as (2.4) is given by $\psi_{ccr}(v) = w(t_0)$ (cf. (3.14)). Thus,

$$\psi_{ccr}(v_0) = \psi'_{ccr}(v_0) = 0$$

and $\psi''_{ccr}(v_0) \neq 0$ do not occur at the same time. Therefore, p_0 must be non-cylindrical.

Since $r(t_0) \neq 0$ and $0 = \lambda(t_0, v_0) = -v_0\, r(t_0)$, we have $v_0 = 0$. Then, $\gamma_{nc}(t) = (t, 0)$ is a singular curve passing through $\gamma_{nc}(t_0) = p_0$, and $\zeta_{nc}(t) = \partial_t - a(t)\partial_v$ is a null vector field along $\gamma_{nc}(t)$. Then we have $\delta(t) = \det(\gamma'_{nc}(t), \zeta_{nc}(t)) = -a(t)$ (cf. (3.5)). Thus, the condition (i) of the criterion in Fact 4.5 implies $a(t_0) \neq 0$.

Now, assume that $a(t_0) < 0$, namely, $\{\gamma'_{nc}(t_0), \zeta_{nc}(t_0)\}$ is positively oriented. The function ψ_{ccr} defined as (2.4) is given by $\psi_{ccr}(t) = a(t)w(t)$ (cf. (3.13)). Thus, under the condition (i), the condition (ii) in Fact 4.5 implies $w(t_0) = w'(t_0) = 0$ and $w''(t_0) \neq 0$ hold. The constant d_1 in

(4.5) is given by $d_1 = a(t_0)w''(t_0)$. With respect to the condition (iii) in Fact 4.5, by a parallel translation of $\boldsymbol{R}^3$, we may assume that $\sigma(t_0) = 0$ without loss of generality. Then, setting

$$c(\tau) := (\tau, -\varphi(\tau)) \qquad \left(\varphi(\tau) := \frac{\sigma(\tau)\cdot\boldsymbol{\xi}(t_0)}{\boldsymbol{\xi}(\tau)\cdot\boldsymbol{\xi}(t_0)}\right), \tag{4.8}$$

we have $c(t_0) = p_0$. Differentiating $\varphi(\tau)$, we have that $c'(t_0) = \zeta_{nc}(t_0)$. Since

$$\hat{c}''(t_0) = -a(t_0)\rho(t_0)\boldsymbol{\eta}(t_0)$$

and $\hat{c}'''(t_0) = -(2a(t_0)\rho'(t_0) + a'(t_0)\rho(t_0))\boldsymbol{\eta}(t_0)$ under the conditions (i) and (ii) in Fact 4.5, we have

$$\hat{c}'''(t_0) = \ell\,\hat{c}''(t_0) \qquad \left(\ell := \frac{2a(t_0)\rho'(t_0) + a'(t_0)\rho(t_0)}{a(t_0)\rho(t_0)}\right).$$

Moreover, by a direct calculation, we can check that $\hat{c}^{(4)}(t_0)$ is a constant multiple of $\boldsymbol{\eta}(t_0)$ and

$$\hat{c}^{(5)}(t_0) = k_1\boldsymbol{\eta}(t_0) - 4a(0)\rho(0)\omega''(0)\boldsymbol{n}(t_0)$$

holds, where $k_1 \in \boldsymbol{R}$ is a constant. Thus, the constant d_2 in (4.6) is given by

$$\begin{aligned} d_2 &= \det\left(a(t_0)\boldsymbol{\xi}(t_0),\ -a(t_0)r(t_0)\boldsymbol{\eta}(t_0),\ -4a(0)r(t_0)w''(t_0)\boldsymbol{n}(t_0)\right) \\ &= 12a(t_0)^3 r(t_0)^2 w''(t_0). \end{aligned}$$

Hence, under the conditions (i) and (ii) in Fact 4.5, the condition (iii) is always satisfied.

In the case of $a(t_0) > 0$, we take the null vector field as $\zeta_{nc}(t) := -\partial_t + a(t)\partial_v$ and the curve $c(\tau)$ as $c(\tau) := (-\tau, \varphi(\tau))$, where $\varphi(\tau)$ is given by (4.8). Then, by a similar calculation as above, the constant d_1 in (4.5) is given by $d_1 = -a(t_0)w''(t_0)$, and the constant d_2 in (4.6) is $d_2 = -12a(t_0)^3 r(t_0)^2 w''(t_0)$. Therefore, regardless of the sign of $a(t_0)$, we have

$$d_1 d_2 = 12a(t_0)^4 r(t_0)^2 w''(t_0)^2 > 0.$$

Thus, Fact 4.5 implies that any a-tangent developable does not admit cuspidal S_1^- singularities, and that f at $p_0 = (t_0, v_0)$ is $\mathcal{A}$-equivalent to the cuspidal S_1^+ singularity if and only if (4.7) holds. □

4.4. 5/2**-cuspidal edge.** Finally, we review the criterion for the 5/2-cuspidal edge given in [9].

Fact 4.8 (Criterion for 5/2-cuspidal edge [9])**.** *Let $f : U \to \boldsymbol{R}^3$ be a frontal defined on a domain U of $\boldsymbol{R}^2$ with the unit normal ν. Take a non-degenerate singular point $p \in U$ of f. Let $\gamma(t)$ ($|t| < \varepsilon$) be a singular curve such that $\gamma(0) = p$ and ζ a null vector field. Then, the map germ f at p is $\mathcal{A}$-equivalent to the 5/2-cuspidal edge if and only if the following* (i)-(iii) *hold:*

(i) $\delta(0) \neq 0$,

(ii) $\det\left(\hat{\gamma}',\, \zeta^2 f,\, \zeta^3 f\right)\big|_{(u,v)=\gamma(t)} = 0$ *holds for each* $t \in (-\varepsilon, \varepsilon)$,

(iii) $\det\left(\hat{\gamma}'(0),\, \bar{\zeta}^2 f(p),\, 3\bar{\zeta}^5 f(p) - 10C\,\bar{\zeta}^4 f(p)\right) \neq 0$.

Here $\bar{\zeta}$ is a special null vector field such that

$$\hat{\gamma}'(0)\cdot\bar{\zeta}^2 f(p) = \hat{\gamma}'(0)\cdot\bar{\zeta}^3 f(p) = 0, \qquad \bar{\zeta}^3 f(p) = C\,\bar{\zeta}^2 f(p), \tag{4.9}$$

where $C \in \boldsymbol{R}$ is a constant.

We remark that if $\hat{\gamma}'(0)\cdot\bar{\zeta}^2 f(p) = \hat{\gamma}'(0)\cdot\bar{\zeta}^3 f(p) = 0$ holds, then there exists a constant $C \in \boldsymbol{R}$ which satisfies $\bar{\zeta}^3 f(p) = C\,\bar{\zeta}^2 f(p)$. Applying Fact 4.8 to a-tangent developables, we have the following.

Proposition 4.9. *Let $f(t,v)$ be an a-tangent developable whose data is given by*

$$(\alpha, \rho, \omega) = (a(t)\,dt, r(t)\,dt, w(t)\,dt).$$

Then, for a singular point $p_0 = (t_0, v_0)$ of f, we have that f at p_0 is $\mathcal{A}$-equivalent to the 5/2-cuspidal edge if and only if

$$v_0 \neq 0, \quad r'(t_0) \neq 0, \quad r(t_0) = w(t_0) = 0, \quad \left(\frac{w'}{r'}\right)'\bigg|_{t=t_0} \neq -\frac{2a(t_0)w'(t_0)}{v_0 r'(t_0)}. \tag{4.10}$$

Proof. We first show that p_0 is cylindrical. If we assume that $p_0 = (t_0, v_0)$ is non-cylindrical, we have that $r(t_0) \neq 0$ and $v_0 = 0$. As we have seen in Lemma 3.2, $\gamma_{nc}(t) = (t, 0)$ is a singular curve passing through $\gamma_{nc}(t_0) = p_0$ and $\zeta_{nc}(t) = \partial_t - a(t)\partial_v$ is a null vector field. Since the function δ defined as (2.4) is given by $\delta_{nc}(t) = -a(t)$ (cf. (3.5)), the condition (i) in Fact 4.8 is equivalent to $a(t_0) \neq 0$. On the other hand, since $\zeta_{nc}^2 f(\gamma_{nc}(t)) = -a(t)r(t)\boldsymbol{\eta}(t)$ and

$$\zeta_{nc}^3 f(\gamma_{nc}(t)) \quad = \quad 2a(t)r(t)^2\boldsymbol{\xi}(t) \quad - \quad (r(t)a'(t) + 2a(t)m'(t))\,\boldsymbol{\eta}(t) \quad + \quad 2a(t)r(t)w(t)\boldsymbol{n}(t),$$

we have that the condition (ii) in Fact 4.8 is equivalent to $w(t) = 0$ for all t. Then, setting

$$\bar{\zeta} := \left(1 - \frac{r(t_0)}{a(t_0)^2}v^2\right)\partial_t - a(t)\partial_v,$$

we have $\bar{\zeta}^2 f(p_0) = -a(t_0)r(t_0)\boldsymbol{\eta}(t_0)$ and $\bar{\zeta}^3 f(p_0) = -(a'(t_0)m(t_0) + 2a(t_0)m'(t_0))\boldsymbol{\eta}(t_0)$. Hence, $\bar{\zeta}$ is a null vector field satisfying (4.9) with the constant

$$C := (a'(t_0)m(t_0) + 2a(t_0)m'(t_0))/(a(t_0)m(t_0)).$$

Then, by a direct calculation, we have $\bar{\zeta}^4 f(p_0),\, \bar{\zeta}^5 f(p_0) \in \mathrm{Span}(\boldsymbol{\xi}(t_0), \boldsymbol{\eta}(t_0))$, which implies

$$\det\left(df(\gamma_{nc}'(0)),\, \bar{\zeta}^2 f(p_0),\, 3\bar{\zeta}^5 f(p_0) - 10C\,\bar{\zeta}^4 f(p_0)\right) = 0.$$

Hence, p_0 must be cylindrical.

As we have seen in Lemma 3.2, a non-degenerate cylindrical singular point $p_0 = (t_0, v_0)$ satisfies $r(t_0) = 0$, $r'(t_0) \neq 0$, $v_0 \neq 0$. Then, $\gamma_c(v) = (t_0, v)$ is a singular curve passing through $\gamma_c(v_0) = p_0$ and $\zeta_c = \partial_t - a(t)\partial_v$ is a null vector field. Since the function δ defined as (2.4) is given by $\delta_c(v) = -1$ (cf. (3.3)), the condition (i) in Fact 4.8 is always satisfied. On the other hand, since $\zeta_c^2 f(\gamma_c(v)) = vm'(t_0)\boldsymbol{\eta}(t_0)$ and

$$\zeta_c^3 f(\gamma_c(v)) = (-2a(t_0)m'(t_0) + vm''(t_0))\,\boldsymbol{\eta}(t_0) - 2vw(t_0)m'(t_0)\boldsymbol{n}(t_0),$$

we have that the condition (ii) in Fact 4.8 is equivalent to $w(t_0) = 0$. Then, $\zeta_c = \partial_t - a(t)\partial_v$ is a null vector field satisfying (4.9) with the constant $C := (v_0 m''(t_0) - 2a(t_0)m'(t_0))/(vm'(t_0))$. By a direct calculation, we have

$$\begin{aligned}&\det\left(df(\gamma_c'(0)),\, \zeta_c^2 f(p_0),\, 3\zeta_c^5 f(p_0) - 10C\,\zeta_c^4 f(p_0)\right)\\ &\qquad = -12v_0 m'(t_0)\left(2a(t_0)m'(t_0)w'(t_0) - v_0 m''(t_0)w'(t_0) + v_0 m'(t_0)w''(t_0)\right).\end{aligned}$$

Hence, by Fact 4.8, we have that f at $p_0 = (t_0, v_0)$ is $\mathcal{A}$-equivalent to the 5/2-cuspidal edge if and only if (4.10) holds. □

	Criteria
Cuspidal edge	$v_0 = 0,\ r(t_0) \neq 0,\ a(t_0) \neq 0,\ w(t_0) \neq 0$ or $v_0 \neq 0,\ r(t_0) = 0,\ r'(t_0) \neq 0,\ w(t_0) \neq 0$
Swallowtail	$v_0 = 0,\ r(t_0) \neq 0,\ a(t_0) = 0,\ a'(t_0) \neq 0,\ w(t_0) \neq 0$
Cuspidal cross cap	$v_0 = 0,\ r(t_0) \neq 0,\ a(t_0) \neq 0,\ w(t_0) = 0,\ w'(t_0) \neq 0$
Cuspidal beaks	$v_0 = 0,\ r(t_0) = 0,\ r'(t_0) \neq 0,\ a(t_0) \neq 0,\ w(t_0) \neq 0$
Cuspidal butterfly	$v_0 = 0,\ r(t_0) \neq 0,\ a(t_0) = a'(t_0) = 0,$ $a''(t_0) \neq 0,\ w(t_0) \neq 0$
Cuspidal S_1^+ singularity	$v_0 = 0,\ r(t_0) \neq 0,\ a(t_0) \neq 0,$ $w(t_0) = w'(t_0) = 0,\ w''(t_0) \neq 0$
5/2-cuspidal edge	$v_0 \neq 0,\ r'(t_0) \neq 0,\ r(t_0) = w(t_0) = 0,$ $\left(\dfrac{w'}{r'}\right)'\Bigg\vert_{t=t_0} \neq -\dfrac{2a(t_0)w'(t_0)}{v_0 r'(t_0)}$

TABLE 1. The criterion for singularities of a-tangent developables. See Corollary 3.9, Propositions 4.2, 4.4, 4.6 and 4.9.

4.5. **Duality of singularities.** Here, we give a summary of the criterion for singularities of a-tangent developables. Let $f(t, v)$ be an a-tangent developable defined on $J \times \boldsymbol{R}$ whose data is given by $(\alpha, \rho, \omega) = (a(t)\,dt, r(t)\,dt, w(t)\,dt)$. In Corollary 3.9, Propositions 4.2, 4.4, 4.6 and 4.9, we proved that the singularity type of the germ f at $p_0 = (t_0, v_0) \in J \times \boldsymbol{R}$ is determined by the data as in Table 1.

Since the conjugate $f^\sharp$ of an a-tangent developable f is given by the data

$$(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha),$$

exchanging the roles α and ω we have the following.

Theorem 4.10 (Duality of singularities for a-tangent developables)**.** *Let $f : M \to \boldsymbol{R}^3$ be an a-tangent developable, $f^\sharp$ the conjugate of f, and $p_0 \in M$ a singular point, where $M := J \times \boldsymbol{R}$. Then, f at p_0 is $\mathcal{A}$-equivalent to the swallowtail* (*resp. cuspidal cross cap, cuspidal beaks, cuspidal butterfly, cuspidal S_1^+ singularity*) *if and only if $f^\sharp$ at p_0 is $\mathcal{A}$-equivalent to the cuspidal cross cap* (*resp. swallowtail, cuspidal beaks, cuspidal S_1^+ singularity, cuspidal butterfly*)*.*

In the case of the cuspidal edge, there exist examples which do not satisfy the desired duality of singularities.

Example 4.11. Let $f = f(t, v)$ be an a-tangent developable whose data is given by

$$(\alpha, \rho, \omega) = (2t\,dt, t\,dt, dt).$$

By Corollary 3.9, f at $(0, v)$ is cuspidal edge for $v \neq 0$ (see Figure 3). The conjugate $f^\sharp = f^\sharp(t, v)$ of f is given by the data $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) = (dt, t\,dt, 2t\,dt)$. By Proposition 4.9, $f^\sharp$ at $(0, v)$ is 5/2-cuspidal edge for $v \neq 0$ (see Figure 4).

5. CONJUGATE OF COMPLETE FLAT FRONTS

Finally, we observe a global behavior of the conjugate operations among a-tangent developables.

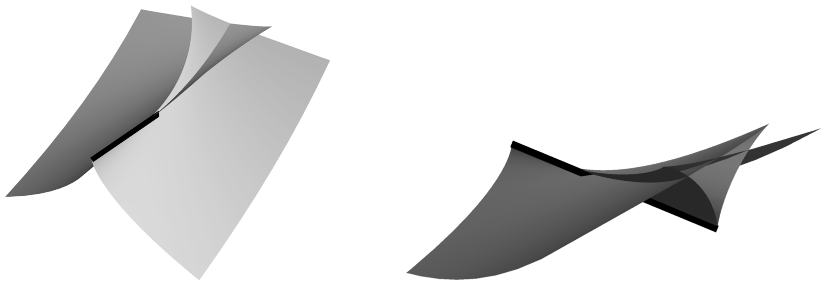


FIGURE 3. The image of the a-tangent developable $f = f(t, v)$ whose data is given by $(\alpha, \rho, \omega) = (2t\,dt, t\,dt, dt)$. By Corollary 3.9, we have that f at $(0, v)$ is $\mathcal{A}$-equivalent to the cuspidal edge for $v \neq 0$. This figure is plotted by integrating (2.6) and (3.17) numerically. The black line is the image of the cylindrical singular set $S_c(f) = \{(0, v)\,;\, v \neq 0\}$.

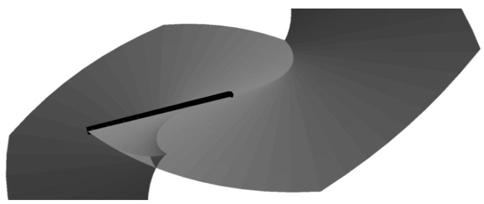


FIGURE 4. The image of the conjugate $f^\sharp = f^\sharp(t, v)$ of the a-tangent developable with the data $(\alpha, \rho, \omega) = (2t\,dt, t\,dt, dt)$. Since the data of $f^\sharp$ is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) = (dt, t\,dt, 2t\,dt)$, Proposition 4.9 yields that f at $(0, v)$ is $\mathcal{A}$-equivalent to the 5/2-cuspidal edge for $v \neq 0$. This figure is plotted by integrating (3.18) and (3.19) numerically. The black line is the image of the cylindrical singular set $S_c(f^\sharp) = \{(0, v)\,;\, v \neq 0\}$.

Proposition 5.1. *Let $f : M \to \boldsymbol{R}^3$ be an a-tangent developable such that f is a complete flat front with embedded ends, where $M := S^1 \times \boldsymbol{R}$. Then, the conjugate $f^\sharp$ of f is not a front. In particular, the conjugate of a complete flat front with embedded ends cannot be a complete flat front.*

Proof. Let (α, ρ, ω) be the data of f. By Fact 2.1, f has at least four singular points other than cuspidal edges. In fact, if we denote by $\alpha = a(t)\,dt$, it is proved in [17, pp. 311–312] that $a(t)$ changes signs at least four times on S^1. Since the data of $f^\sharp$ is given by $(\alpha^\sharp, \rho^\sharp, \omega^\sharp) := (\omega, \rho, \alpha)$, and $f^\sharp$ is front if and only if $\omega^\sharp$ never vanishes, we have that $f^\sharp$ cannot be a front. □

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