Journal *of* Singularities
Volume 20 (2020), 342-370

received: 13 May 2020
in revised form: 3 October 2020
DOI: 10.5427/jsing.2020.20p

POSITIVE POPULATIONS

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ABSTRACT. A positive structure on the varieties of critical points of master functions for KZ equations is introduced. It comes as a combination of the ideas from classical works by G. Lusztig and a previous work by E. Mukhin and the second named author.

1. Introduction: Whitney-Lusztig patterns and Bethe populations

1.1. **Big cell and critical points.** The aim of the present note is to introduce a positive structure on varieties of critical points of master functions arising in the integral representation for solutions of KZ equations and the Bethe ansatz method.

Let $N = N_{r+1} \subset G = \mathrm{SL}_{r+1}(\mathbb{C})$ denote the group of upper triangular matrices with 1's on the diagonal. It may also be considered as a big cell in the flag variety $\mathrm{SL}_{r+1}(\mathbb{C})/B_-$, where B_- is the subgroup of lower triangular matrices. Let S_{r+1} denote the Weyl group of G, the symmetric group.

In this note two objects, related to N, will be discussed: on the one hand, what we call here the *Whitney-Loewner-Lusztig data on* N, on the other hand, a construction, introduced in [MV], which we call here *the Wronskian evolution* along the varieties of critical points.

An identification of these two objects may allow us to use cluster theory to study critical sets of master functions and may also bring some critical point interpretation of the relations in cluster theory.

In this note we consider only the case of the group $\mathrm{SL}_{r+1}(\mathbb{C})$, although the other reductive groups can be considered similarly.

1.2. **What is done in Introduction.** In Section 1.3 we recall the classical objects: *Whitney-Loewner charts*, these are collections of birational coordinate systems on N indexed by reduced decompositions of the longest element $w_0 \in S_{r+1}$, and *Lusztig transition maps* between them.

In Sections 1.4 - 1.8 the main ideas from [MV] are introduced. Namely, it is a *reproduction* recipe, called here a *Wronskian evolution*, which produces varieties of critical points for *master functions* appearing in integral representations for solutions of KZ equations, [SV].

In Section 1.10 the content of Sections 2 - 8 is described.

1.3. **Whitney-Loewner charts and Lusztig transition maps.** In the seminal papers [L, BFZ] Lusztig and Berenstein-Fomin-Zelevinsky have performed a deep study of certain remarkable coordinate systems on N, i.e. morphisms of algebraic varieties

$$\mathcal{L}_{\boldsymbol{h}} : \mathbb{C}^q \longrightarrow N,$$

2010 *Mathematics Subject Classification.* 13F60 (14M15, 82B23).
Key words and phrases. Totally positive matrices, Whitney-Lusztig charts, Wronskian differential equation, Wronski map, master functions, Bethe cells, positive populations.

$q = r(r+1)/2$, with dense image, which are birational isomorphisms. The main feature of these morphisms is that the restriction of them to $\mathbb{R}^q_{>0}$ induces an isomorphism

$$\mathcal{L}_{\boldsymbol{h}}:\ \mathbb{R}^q_{>0} \xrightarrow{\sim} N_{>0},$$

where $N_{>0}$ is the subspace of *totally positive* upper triangular matrices.

Recall that a matrix $g \in N$ is called *totally positive* if all its minors are strictly positive, except for those who are identically zero on the whole group N, see [BFZ][1].

The set of such coordinate systems, which we will be calling the *Whitney-Loewner charts*, is in bijection with the set $\mathrm{Red}(w_0)$ of reduced decompositions

$$\boldsymbol{h}:\ w_0 = s_{i_q} \dots s_{i_1} \tag{1.1}$$

of the longest element $w_0 \in S_{r+1}$.

For example, for $r = 2$ there are two such coordinate systems, $\mathcal{L}_{121}$ and $\mathcal{L}_{212}$ corresponding to the reduced words $s_1 s_2 s_1$ and $s_2 s_1 s_2$ respectively.

The construction of maps $\mathcal{L}_{\boldsymbol{h}}$ will be recalled below, see Section 4.1.

To every $\boldsymbol{h} \in \mathrm{Red}(w_0)$ and $\boldsymbol{a} = (a_q, \dots, a_1) \in \mathbb{C}^q$ there corresponds a matrix

$$N_{\boldsymbol{h}}(\boldsymbol{a}) = \mathcal{L}_{\boldsymbol{h}}(\boldsymbol{a}) \in N.$$

A theorem of A. Whitney[2], as reformulated by Ch. Loewner, see [W, Lo], says;

Theorem 1.1. *For any reduced decomposition* $\boldsymbol{h}$ *of the longest element* $w_0 \in S_{r+1}$ *the map* $\mathcal{L}_{\boldsymbol{h}} : \mathbb{C}^{r(r+1)/q} \to N_{r+1}$ *restricted to the positive cone* $\mathbb{R}^{r(r+1)/2}_{>0}$ *defines an isomorphism of the positive cone* $\mathbb{R}^{r(r+1)/2}_{>0}$ *and the space* $N_{>0}$ *of totally positive matrices .*

For any two words $\boldsymbol{h}, \boldsymbol{h}'$ Lusztig has defined a *birational* self-map of $\mathbb{A}^q$, i.e. an automorphism of the field of rational functions

$$\mathbb{F} := \mathbb{C}(\boldsymbol{a}) = \mathbb{C}(a_1, \dots, a_q) \cong \mathbb{C}(N)$$

(here we consider a_i as independent transcendental generators),

$$R_{\boldsymbol{h},\boldsymbol{h}'}:\ \mathbb{F} \xrightarrow{\sim} \mathbb{F}, \tag{1.2}$$

such that

$$\boldsymbol{a}' = R_{\boldsymbol{h},\boldsymbol{h}'}(\boldsymbol{a}), \quad \text{if} \quad N_{\boldsymbol{h}}(\boldsymbol{a}) = N_{\boldsymbol{h}'}(\boldsymbol{a}')\,.$$

For example

$$N_{121}(a_1, a_2, a_3) = \begin{pmatrix} 1 & a_1 + a_3 & a_2 a_3 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 \end{pmatrix} = e_1(a_3) e_2(a_2) e_1(a_1)$$

and

$$N_{212}(a'_1, a'_2, a'_3) = \begin{pmatrix} 1 & a'_2 & a'_1 a'_2 \\ 0 & 1 & a'_1 + a'_3 \\ 0 & 0 & 1 \end{pmatrix} = e_2(a'_3) e_1(a'_2) e'_2(a'_1)\,,$$

where $e_1(a) = 1 + a e_{12},\ e_2(a) = 1 + a e_{23}$.

It follows that

$$N_{121}(a_1, a_2, a_3) = N_{212}(a'_1, a'_2, a'_3)\,,$$

[1] The notion of a totally positive matrix first appeared in the works of I. Schoenberg [S] and Gantmacher-Krein [GK].

[2] Anne M. Whitney (1921–2008), a student of Isaac Schoenberg (1903–1990).

provided

$$a_1' = \frac{a_2 a_3}{a_1 + a_3}\,, \quad a_2' = a_1 + a_3\,, \quad a_3' = \frac{a_1 a_2}{a_1 + a_3}\,. \tag{1.3}$$

This is equivalent to

$$a_1 = \frac{a_2' a_3'}{a_1' + a_3'}\,, \quad a_2 = a_1' + a_3'\,, \quad a_3 = \frac{a_1' a_2'}{a_1' + a_3'}\,. \tag{1.4}$$

The transformation (1.3) is *involutive*, that is, its square is the identity.

1.4. **Bethe Ansatz equations.** On the other hand, in the work [MV] it was discovered that the variety N is closely connected with the varieties of critical points of certain *master functions* $\Phi_{\boldsymbol{k}}$.

Namely, for a sequence

$$\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}^r\,,$$

consider a function $\Phi_{\boldsymbol{k}}(\boldsymbol{u})$ depending on $k := \sum_{i=1}^r k_i$ variables subdivided into r groups:

$$\boldsymbol{u} = (u_1^{(1)}, \dots, u_{k_1}^{(1)}; \dots; u_1^{(r)}, \dots, u_{k_r}^{(r)})\,.$$

By definition,

$$\Phi_{\boldsymbol{k}}(\boldsymbol{u}) = \prod_{i=1}^r \prod_{1\leqslant m<l\leqslant k_i} (u_m^{(i)} - u_l^{(i)})^{a_{ii}} \cdot \prod_{1\leqslant i<j\leqslant r} \prod_{m=1}^{k_i} \prod_{l=1}^{k_j} (u_m^{(i)} - u_l^{(j)})^{a_{ij}}\,.$$

Here $A = (a_{ij})$ is the Cartan matrix for the root system of type A_r, in other words,

$$\Phi_{\boldsymbol{k}}(\boldsymbol{u}) = \prod_{i=1}^r \prod_{1\leqslant l<m\leqslant k_i} (u_l^{(i)} - u_m^{(i)})^2 \cdot \prod_{i=1}^{r-1} \prod_{l=1}^{k_i} \prod_{m=1}^{k_{i+1}} (u_l^{(i)} - u_m^{(i+1)})^{-1}\,.$$

Functions of this kind first appeared in [SV] in the study of integral representations for solutions of KZ differential equations.

A point

$$\boldsymbol{t} = (t_1^{(1)}, \dots, t_{k_1}^{(1)}; \dots; t^{(r)}, \dots, t_{k_r}^{(r)})$$

is *critical* for the function $\log \Phi_{\boldsymbol{k}}(\boldsymbol{u})$ if it satisfies the system of k equations

$$\frac{\partial \log \Phi_{\boldsymbol{k}}}{\partial u_m^{(i)}}(\boldsymbol{t}) = \left[\frac{\partial \Phi_{\boldsymbol{k}}}{\partial u_m^{(i)}} \Phi_{\boldsymbol{k}}^{-1}\right](\boldsymbol{t}) = 0\,, \qquad 1 \leqslant i \leqslant r,\ 1 \leqslant m \leqslant k_i\,,$$

or, equivalently,

$$\sum_{l\neq m} \frac{a_{ii}}{t_m^{(i)} - t_l^{(i)}} + \sum_{j\neq i} \sum_{l=1}^{k_j} \frac{a_{ij}}{t_m^{(i)} - t_l^{(j)}} = 0\,, \qquad 1 \leqslant i \leqslant r,\ 1 \leqslant m \leqslant k_i\,. \tag{1.5}$$

This system of critical point equations is also called the system of *Bethe Ansatz equations* in the Gaudin model.

1.5. **Reproduction, or Wronskian evolution (bootstrap).** The following procedure of *reproduction* for constructing critical points has been proposed in [ScV], [MV].

Let us identify the group $\mathbb{Z}^r$ with the root lattice Q of the group G using the base of standard simple roots $\alpha_1, \dots, \alpha_r$. Introduce the usual shifted action of $W = S_{r+1}$ on Q:

$$w * v = w(v - \rho) + \rho \,,$$

where ρ is the half-sum of positive roots.

Let $w \in S_{r+1}$ and let

$$\boldsymbol{h} \;:\; w = s_{i_m} \dots s_{i_1} \tag{1.6}$$

be a reduced decomposition of w. For any $0 \leqslant j \leqslant q$ we define an r-tuple

$$\boldsymbol{k}^{(j)} = s_{i_j} \dots s_{i_1} * (\mathbf{0}) \in \mathbb{N}^r \,,$$

where $\mathbf{0} = (0, \dots, 0) = \boldsymbol{k}^{(0)}$.

Starting from the r-tuple of polynomials

$$\boldsymbol{y}^{(0)} = (1, \dots, 1) \in \mathbb{C}[x]^r \,,$$

one defines inductively a sequence of r-tuples of polynomials

$$\boldsymbol{y}_{\boldsymbol{h}} = (\boldsymbol{y}^{(0)}, \boldsymbol{y}^{(1)}(v_1), \boldsymbol{y}^{(2)}(v_1, v_2), \dots, \boldsymbol{y}^{(m)}(v_1, \dots, v_m)) \,, \tag{1.7}$$

where

$$\boldsymbol{y}^{(j)}(\boldsymbol{v}) = (y_1^{(j)}(\boldsymbol{v}; x), \dots, y_r^{(j)}(\boldsymbol{v}; x)) \in \mathbb{C}[v_1, \dots, v_j; x]^r,$$

$0 \leqslant j \leqslant m$, with

$$\deg \boldsymbol{y}^{(j)}(\boldsymbol{v}) := (\deg y_1^{(j)}(\boldsymbol{v}; x), \dots, \deg y_r^{(j)}(\boldsymbol{v}; x)) = \boldsymbol{k}^{(j)} \,,$$

where deg is the degree with respect to x.

The sequence (1.4.2) is called the *population associated with a reduced word* $\boldsymbol{h}$.

We consider a polynomial $y_i^{(j)}(\boldsymbol{v}; x)$ as a family $y_i^{(j)}(\boldsymbol{c}; x)$ of polynomials of one variable x depending on a parameter $\boldsymbol{c} = \boldsymbol{c}^{(j)} = (c_1, \dots, c_j) \in \mathbb{C}^j$.

Let $\boldsymbol{c} \in \mathbb{C}^j$ and $t_{i,1}, \dots, t_{i,k_i^{(j)}}$ be the roots of $y_i^{(j)}(\boldsymbol{c}; x)$ ordered in any way. Consider the tuple

$$\boldsymbol{t}^{(j)}(\boldsymbol{c}) = (t_{1,1}, \dots, t_{1,k_1^{(j)}}; \dots; t_{r,1}, \dots, t_{r,k_r^{(j)}}) \,.$$

The main property of the sequence $y_{\boldsymbol{h}}$ is:

Theorem 1.2 ([MV])**.** *For each $j = 1, \dots, m$, there exists a Zarisky open dense subspace $U \subset \mathbb{C}^j$ such that for every $\boldsymbol{c} = \boldsymbol{c}^{(j)} \in U$, the tuple of roots $\boldsymbol{t}^{(j)}(\boldsymbol{c})$ is a critical point of the master function $\Phi^{(j)}(\boldsymbol{u}) := \Phi_{\boldsymbol{k}^{(j)}}(\boldsymbol{u})$, i.e. it satisfies the Bethe Ansatz equations* (1.5)*.*

Moreover, if $\Phi_{\boldsymbol{k}}$ is a master function for some index $\boldsymbol{k}$ and $\boldsymbol{t}$ a critical point of $\log \Phi_{\boldsymbol{k}}$ as in Section 1.4, then $\boldsymbol{t}$ appears in this construction for a reduced decomposition $\boldsymbol{h}$ of some element $w \in S_{r+1}$.

The construction of $\boldsymbol{y}_{\boldsymbol{h}}$ for $\boldsymbol{h} \in \mathrm{Red}(w_0)$ see in Section 1.6 below.

1.6. **Wronskian bootstrap: the details.** Starting from the r-tuple of polynomials

$$\boldsymbol{y}^{(0)} = (1, \dots, 1) \in \mathbb{C}[x]^r ,$$

one constructs the sequence

$$\boldsymbol{y}_{\boldsymbol{h}} = (\boldsymbol{y}^{(0)}, \boldsymbol{y}^{(1)}(v_1), \boldsymbol{y}^{(2)}(v_1, v_2), \dots, \boldsymbol{y}^{(m)}(v_1, \dots, v_m)) \qquad (1.5.1)$$

by induction. Assume that the sequence

$$\boldsymbol{y}^{(j)}(\boldsymbol{v}) = (y_1^{(j)}(\boldsymbol{v}; x), \dots, y_r^{(j)}(\boldsymbol{v}; x)) \in \mathbb{C}[v_1, \dots, v_j; x]^r$$

has been constructed. Then the sequence

$$\boldsymbol{y}^{(j+1)}(\boldsymbol{v}) = (y_1^{(j+1)}(\boldsymbol{v}; x), \dots, y_r^{(j+1)}(\boldsymbol{v}; x)) \in \mathbb{C}[v_1, \dots, v_j, v_{j+1}; x]^r$$

is such that

$$y_i^{(j+1)}(\boldsymbol{v}; x) = y_i^{(j)}(\boldsymbol{v}; x), \qquad \forall i \neq i_{j+1} .$$

and $y_{i_{j+1}}^{(j+1)}(v_1, \dots, v_j, v_{j+1}; x)$ is constructed in two steps.

First one shows that there is a unique polynomial $\tilde{y}(v_1, \dots, v_j; x)$ such that

- (i) $\mathrm{Wr}(y_{i_{j+1}}^{(j)}, \tilde{y}) = \mathrm{const}\, y_{i_{j+1}-1}^{(j)} y_{i_{j+1}+1}^{(j)}$, where $\mathrm{Wr}(f, g) = fg' - f'g$ denotes the Wronskian of two functions in x and the constant does not depend on $\boldsymbol{v}, x$;
- (ii) the polynomial $\tilde{y}(v_1, \dots, v_j; x)$ is monic with respect to the variable x;
- (iii) the coefficient in $\tilde{y}(v_1, \dots, v_j; x)$ of the monomial $x^{k_{i_{j+1}}^{(j)}}$ equals zero, where $k_{i_{j+1}}^{(j)}$ is the i_{j+1}-st coordinate of the vector $\boldsymbol{k}^{(j)}$.

Then we define

$$y_{i_{j+1}}^{(j+1)}(v_1, \dots, v_j, v_{j+1}; x) := \tilde{y}(v_1, \dots, v_j; x) + v_{j+1}\, y_{i_{j+1}}^{(j)}(v_1, \dots, v_j; x) .$$

Consider all coordinates of the resulting family

$$\boldsymbol{y}^{(m)}(v_1, \dots, v_m) = (y_1^{(m)}(\boldsymbol{v}; x), \dots, y_r^{(m)}(\boldsymbol{v}; x)) \in \mathbb{C}[v_1, \dots, v_m; x]^r$$

up to multiplication by nonzero numbers. This gives a map

$$F_{\boldsymbol{h}} : \mathbb{C}^m \to \mathbb{P}(\mathbb{C}[x])^r ,$$

where $\mathbb{P}(\mathbb{C}[x])$ is the projective space associated with $\mathbb{C}[x]$. Denote by

$$\mathcal{Z}_{\boldsymbol{h}} = F_{\boldsymbol{h}}(\mathbb{C}^m) \subset \mathbb{P}(\mathbb{C}[x])^r$$

its image.

Let V be an $r+1$-dimensional complex vector space, $X = G/B_-$ the space of all complete flags in V. Let $F_0 \in X$ be a point. The choice of F_0 gives rise to a decomposition of X into $|W| = (r+1)!$ Bruhat cells, which are in bijection with $W = S_{r+1}$:

$$X = \coprod_{w \in W} X_w .$$

We have

$$\dim X_w = \ell(w).$$

For example, for the identity $e \in W$, the cell $X_e = \{F_0\}$ is the zero-dimensional cell, and for the longest element $w_0 \in W$, the cell X_{w_0} is the open cell, the space of all flags in general position with F_0.

Theorem 1.3 ([MV])**.** *The union*

$$\mathcal{Z} = \bigcup_{w,\boldsymbol{h}} \mathcal{Z}_{\boldsymbol{h}} \subset \mathbb{P}(\mathbb{C}[x])^r$$

over all reduced decompositions $\boldsymbol{h}$ of all elements $w \in S_{r+1}$ is an algebraic subvariety of $\mathbb{P}(\mathbb{C}[x])^r$ isomorphic to the variety of complete flags $X = G/B_-$.

The subspace $\mathcal{Z}_{\boldsymbol{h}} \subset \mathcal{Z}$ does not depend on the choice of $\boldsymbol{h} \in Red(w)$, so it may be denoted by $\mathcal{Z}_w$. It is identified with the Bruhat cell $X_w \subset X$. □

In particular, the subset $\mathcal{Z}_{w_0} \cong X_{w_0}$ is isomorphic to the big Bruhat cell $N \subset G/B_-$.

The algebraic subvariety $\mathcal{Z} \subset \mathbb{P}(\mathbb{C}[x])^r$ is what was called in [MV] the *population of critical points originated from $\boldsymbol{y}^{(0)}$*.

The proof of Theorem 1.3 in [MV] identifies the variety $\mathcal{Z}$ with the space of complete flags of a particular $r+1$-dimensional vector space V, where

$$V = V_r = \mathbb{C}[x]_{\leqslant r} \subset \mathbb{C}[x]$$

is the vector space of polynomials of degree $\leqslant r$, and the flag F_0 is the standard complete flag

$$F_0 = (V_0 \subset \ldots \subset V_{r-1} \subset V_r), \qquad V_i = \mathbb{C}[x]_{\leqslant i}\,.$$

1.7. **Example,** [MV, Section 3.5]. For $r = 2$, let us see how the populations give rise to a decomposition of $X = \mathrm{SL}_3(\mathbb{C})/B_-$ into six Bruhat cells.

We have the zero-dimensional Bruhat cell

$$\mathcal{Z}_{\mathrm{id}} = \{(1:1)\} \in \mathbb{P}(\mathbb{C}[x])^2\,.$$

We have two one-dimensional Bruhat cells:

$$\begin{aligned} \mathcal{Z}_{s_1} &= \{(x+c_1 : 1) \mid c_1 \in \mathbb{C}\} \subset \mathbb{P}(\mathbb{C}[x])^2\,,\\ \mathcal{Z}_{s_2} &= \{(1 : x+c_1') \mid c_1' \in \mathbb{C}\} \subset \mathbb{P}(\mathbb{C}[x])^2\,, \end{aligned}$$

two two-dimensional Bruhat cells:

$$\begin{aligned} \mathcal{Z}_{s_2s_1} &= \{(x+c_1 : x^2+2c_1x+c_2) \mid c_1,c_2 \in \mathbb{C}\} \subset \mathbb{P}(\mathbb{C}[x])^2\,,\\ \mathcal{Z}_{s_1s_2} &= \{(x^2+2c_1'x+c_2' : x+c_1') \mid c_1',c_2' \in \mathbb{C}\} \subset \mathbb{P}(\mathbb{C}[x])^2\,. \end{aligned}$$

The longest element $w_0 \in S_3$ has two reduced decompositions $s_1s_2s_1$ and $s_2s_1s_2$, which give two parametrizations of the same three-dimensional Bruhat cell:

$$\begin{aligned} \mathcal{Z}_{s_1s_2s_1} &= \{(x^2+c_3x+c_1c_3-c_2 : x^2+2c_1x+c_2) \mid c_1,c_2,c_3 \in \mathbb{C}\} \subset \mathbb{P}(\mathbb{C}[x])^2\,,\\ \mathcal{Z}_{s_2s_1s_2} &= \{(x^2+2c_1'x+c_2' : x^2+c_3'x+c_1'c_3'-c_2') \mid c_1',c_2',c_3' \in \mathbb{C}\} \subset \mathbb{P}(\mathbb{C}[x])^2\,. \end{aligned}$$

The two coordinate systems are related by the equations

$$c_1' = c_3/2\,, \qquad c_2' = c_1c_3 - c_2\,, \qquad c_3' = 2c_1\,. \tag{1.8}$$

Notice that this transformation is involutive.

The union of the six Bruhat cells gives the subvariety $\mathcal{Z} \subset \mathbb{P}(\mathbb{C}[x])^2$ isomorphic to $\mathrm{SL}_3(\mathbb{C})/B_-$. The subvariety $\mathcal{Z}$ consists of all pairs of quadratic polynomials $(a_2x^2+a_1x+a_0 : b_2x^2+b_1x+b_0)$ such that

$$a_2b_0 - \frac{1}{2}a_1b_1 + a_0b_2 = 0\,. \tag{1.9}$$

Remark. Here are more details on how the description of $\mathcal{Z}_{s_1s_2s_1}$ is obtained. The description of $\mathcal{Z}_{s_2s_1s_2}$ is obtained similarly.

In the formula $\mathcal{Z}_{s_1} = \{(x+c_1\ :\ 1)\mid c_1\in\mathbb{C}\}$, the monic polynomial $x+c_1$ satisfies the equation $\mathrm{Wr}(1,x+c_1)=1$. In the formula $\mathcal{Z}_{s_2s_1} = \{(x+c_1\ :\ x^2+2c_1x+c_2)\mid c_1,c_2\in\mathbb{C}\}$, the monic polynomial $x^2+2c_1x+c_2$ satisfies the equation $\mathrm{Wr}(1,x^2+2c_1x+c_2) = \mathrm{const}\,(x+c_1)$, where $\mathrm{const}=1/2$. In the formula $\mathcal{Z}_{s_1s_2s_1} = \{(x^2+c_3x+c_1c_3-c_2\ :\ x^2+2c_1x+c_2)\mid c_1,c_2,c_3\in\mathbb{C}\}$, the monic polynomial $x^2+c_3x+c_1c_3-c_2$ satisfies the equation

$$\mathrm{Wr}(x+c_1, x^2+c_3x+c_1c_3-c_2) = x^2+2c_1x+c_2,$$

see Section 1.6.

1.8. **Not necessarily reduced words.** One can associate to an arbitrary, not necessarily reduced word $\boldsymbol{h}$ of length m a sequence of m r-tuples (1.3) as well, see [MV].

Namely, using the Wronskian differential equation

$$W(y^{(j)}_{i_{j+1}},\tilde y)\ = \mathrm{const}\cdot y^{(j)}_{i_{j+1}-1}y^{(j)}_{i_{j+1}+1}$$

as in (i) above alone, but without normalizing conditions (ii) and (iii) one gets for each $j=1,\dots,m$ a tuple

$$\boldsymbol{y}^{(j)}(\boldsymbol{v}) = (y^{(j)}_1(\boldsymbol{v};x)\ :\ \dots\ :\ y^{(j)}_r(\boldsymbol{v};x))\ \in P(\mathbb{C}[x])^r\,,$$

where $\boldsymbol{v}$ belongs to a variety of parameters $B^{(j)}$, which is an iterated $\mathbb{C}$-torsor. This means that $B^{(j)}$ is included into a sequence of fibrations

$$B^{(j)}\longrightarrow B^{(j-1)}\longrightarrow\dots\longrightarrow B^{(1)}\cong\mathbb{C}\,,$$

where each step $B^{(p)}\longrightarrow B^{(p-1)}$ is an analytic $\mathbb{C}$-torsor, locally trivial in the usual topology.

But the corresponding cohomology $H^1(\mathbb{C}^p;\mathbb{C})$ vanishes which implies that the torsors are trivial, and this provides a global isomorphism $B^{(j)}\cong\mathbb{C}^j$.

1.9. **Main point.** The main new point of the present work is a definition of a certain *modified* reproduction, which we call the *normalized reproduction*. It provides the varieties of r-tuples of polynomials $\mathcal{Y}^{Bethe}$, to be called the *Bethe cells*, equipped with a system of coordinate charts isomorphic to N equipped with the Whitney-Lusztig charts. We also define the *totally positive subspace* $\mathcal{Y}^{Bethe}_{>0}\subset\mathcal{Y}^{Bethe}$ isomorphic to the subspace $N_{>0}\subset N$ of totally positive upper triangular matrices.

1.10. **Contents of the paper.** Section 2 contains some preparations. In Section 3 a modification of the mutation procedure of Section 1.6 is introduced. Based on it, in Section 4 the *Bethe cell* is defined, and Comparison Theorem 4.4 is proven, which is one of our main results. In Section 5 we first describe in full detail Wronsky evolution for the case of groups SL_3 and SL_4. In particular we compute explicitly the positive part $\mathcal{Y}^{Bethe}_{>0}$ of the Bethe cell, see Theorem 5.2. Afterwards we prove Triangular Theorem 5.3, which establishes an explicit isomorphism between N and the Bethe cell. Section 6 contains a generalization of the previous constructions. Namely, we define the Wronskian evolution, the Bethe cell, etc. associated with a finite-dimensional subspace $V\subset\mathbb{C}[x]$ and a distinguished complete flag F_0 in V. The previous consideration corresponded the subspace $\mathbb{C}[x]_{\leqslant r}$ of polynomials of degree $\leqslant r$. In Section 7 we present a version of the above considerations for the base affine space. Namely, we define a variety $\tilde{\mathcal{Y}}^{Bethe}$, which is related to the previous Bethe variety $\mathcal{Y}^{Bethe}$ in the same way as the big cell in the base affine space G/N_- is related to the big cell in the flag space G/B_-. In Section 8 we interpret the Whitney-Lusztig data in the language of higher Bruhat orders, [MS], in particular give its complete description in the crucial case of SL_4.

We are grateful to E. Mukhin and M. Shapiro for useful discussions.

2. Generalities on Wronskians

2.1. **Wronskian differential equation.** The Wronskian of two functions $f(x), g(x)$ is the function

$$\mathrm{Wr}(f,g) = fg' - f'g = f^2(g/f)'\,.$$

Given $f(x)$ and $h(x)$, the equation

$$\mathrm{Wr}(f,g) = h \tag{2.1}$$

with respect to the function $g(x)$ has a solution

$$g(x) = f(x)\int h(x)f(x)^{-2}dx\,.$$

The general solution is

$$g(x,c) = f(x)\int_{x_0}^{x} h(t)f(t)^{-2}dt + cf(x), \qquad c \in \mathbb{C}\,. \tag{2.2}$$

2.2. **Univaluedness.** Let $f(x), g(x) \in \mathbb{C}[x]$ be polynomials. Then $h(x) := \mathrm{Wr}(f,g)$ is a polynomial. Hence the indefinite integral of a rational function,

$$\int h(x)f(x)^{-2}dx\,,$$

has no logarithmic terms, which is equivalent to the condition:

(U) *The function hf^{-2} has zero residues at its poles.*

2.3. **Wronskian.** Let $f_1(x), \dots, f_n(x)$ be holomorphic functions. Define the Wronskian matrix $(f_b^{(a-1)})_{a,b=1}^n$ and the *Wronskian*

$$\mathrm{Wr}(f_1, \dots, f_n) = \det\,(f_b^{(a-1)})_{a,b=1}^n\,.$$

The Wronskian is a polylinear skew-symmetric function of $f_1, \dots, f_n$.

Example 2.1. We have

$$\mathrm{Wr}\left(1, x, \frac{x^2}{2}, \dots, \frac{x^n}{n!}\right) \;=\; 1\,.$$

More generally,

$$\mathrm{Wr}(x^{d_1}, \dots, x^{d_k}) = \prod_{i<j}(d_j - d_i)\,\cdot\, x^{\sum d_i - k(k+1)/2}, \tag{2.3}$$

see this formula in [MTV].

2.4. **W5 Identity.** Let $f_1(x), f_2(x), \dots$ be a sequence of holomorphic functions. For an ordered finite subset

$$A = \{i_1, \dots, i_a\} \subset \mathbb{N} := \{1, 2, \dots\},$$

we write

$$\mathrm{Wr}(A) = \mathrm{Wr}(f_{i_1}, \dots, f_{i_a})\,.$$

Denote

$$[a] = \{1, 2, \dots, a\}.$$

Proposition 2.1 (W5 Identity, [MV, Section 9])**.** *Let*

$$A=[a+1],\quad B=[a]\cup\{a+2\}.$$

Then

$$\mathrm{Wr}(\mathrm{Wr}(A),\mathrm{Wr}(B))=\mathrm{Wr}(A\cap B)\cdot\mathrm{Wr}(A\cup B). \tag{2.4}$$

Example 2.2. We have

$$\mathrm{Wr}(\mathrm{Wr}(f_1,f_2),\mathrm{Wr}(f_1,f_3))\;=\;f_1\,\mathrm{Wr}(f_1,f_2,f_3), \tag{2.5}$$

as one can easily check.

3. Normalized Wronskian bootstrap

3.1. **Generic and fertile tuples.** Let $\boldsymbol{y}=(y_1(x),\dots,y_r(x))\in\mathbb{C}[x]^r$ be a tuple of polynomials. Define $y_0=y_{r+1}=1$.

We say that the r-tuple $\boldsymbol{y}$ is *fertile*, if for every i the equation

$$\mathrm{Wr}(y_i(x),\hat y_i(x))=\;y_{i-1}(x)\,y_{i+1}(x)$$

with respect to $\hat y_i(x)$ admits a polynomial solution. We say that $\boldsymbol{y}$ is *generic*, if for every i the polynomial $y_i(x)$ has no multiple roots and the polynomials $y_i(x)$ and $y_{i-1}(x)\,y_{i+1}(x)$ have no common roots.

Let $t_{i,1},\dots,t_{i,k_i}$ be the roots of $y_i(x)$ ordered in any way. Consider the tuple

$$\boldsymbol{t}=(t_{1,1},\dots,t_{1,k_1};\dots;t_{r,1},\dots,t_{r,k_r})\,.$$

Lemma 3.1 ([MV])**.** *The tuple $\boldsymbol{y}$ is generic and fertile if and only if the tuple $\boldsymbol{t}$ is a critical point of the master function $\Phi_{\boldsymbol{k}}$, where $\boldsymbol{k}=(\deg y_1,\dots,\deg y_r)$.*

3.2. **Normalized mutations.** Let $\boldsymbol{y}=(y_1,\dots,y_r)\in\mathbb{C}[x]^r$ be an r-tuple of polynomials such

$$y_i(0)=1\,,\qquad i=1,\dots,r\,. \tag{3.1}$$

Lemma 3.2. *There exists a unique solutions $\hat y_i(x)$ of the differential equation*

$$\mathrm{Wr}(y_i(x),\hat y_i(x))\;=\;y_{i-1}(x)\,y_{i+1}(x)\;, \tag{3.2}$$

such that

$$\hat y_i(x)=x+\mathcal{O}(x^2)\quad\text{ as }\quad x\to 0\,. \tag{3.3}$$

Proof. We have

$$\hat y_i(x)\;=\;y_i(x)\int_0^x\frac{y_{i-1}(u)y_{i+1}(u)}{y_i^2(u)}\,du\,. \tag{3.4}$$

□

Assume that the tuple $\boldsymbol{y}$ is generic and fertile, then the function $\hat y_i(x)$ is a polynomial by Lemma 3.1. The polynomial $\hat y_i(x)$ can be determined either by formula (3.4) or by the method of undetermined coefficients, see examples in Section 5.

For $c\in\mathbb{C}$, denote

$$\tilde y_i(c;x)=y_i(x)+c\hat y_i(x)\,. \tag{3.5}$$

Notice that for any c we have $\tilde y_i(c;0)=1$.

Define a new r-tuple of polynomials

$$\nu_i(c)\boldsymbol{y}:=(y_1(x),\dots,y_{i-1}(x),\,\tilde y_i(c;x),\,y_{i+1}(x),\dots,y_r(x)) \tag{3.6}$$

and call it *the i-th normalized mutation* of the fertile generic tuple $\boldsymbol{y}$.

Equation (3.2) with the normalizing condition (3.3) will be called the *normalized Wronskian evolution*, or *bootstrap, equation.*

3.3. **Normalized population related to a word.** Let

$$\boldsymbol{h} = s_{i_m} \dots s_{i_1} \tag{3.7}$$

be *any* word in S_{r+1}. Sometimes we will write for brevity simply

$$\boldsymbol{h} = (i_m \dots i_1) \tag{3.8}$$

instead of (3.7).

Now we proceed as in Section 1.5, but will use the normalized mutations ν_i. Namely, we start with

$$\boldsymbol{y}^{(0)} = (1, \dots, 1) \ \in \mathbb{C}[x]^r$$

and for each $\boldsymbol{c} = (c_1, \dots, c_m) \in \mathbb{C}^m$ define an r-tuples of polynomials by the formula:

$$\boldsymbol{y}_{\boldsymbol{h}}(\boldsymbol{c}) = (y_{\boldsymbol{h},1}(\boldsymbol{c}; x), \dots, y_{\boldsymbol{h},r}(\boldsymbol{c}; x)) \ := \ \nu_{i_m}(c_m) \dots \nu_{i_1}(c_1) \boldsymbol{y}^{(0)} \,. \tag{3.9}$$

Notice that for any $i = 1, \dots, r$ and $\boldsymbol{c} \in \mathbb{C}^m$ we have

$$y_{\boldsymbol{h},i}(\boldsymbol{c}; 0) \ = \ 1 \,. \tag{3.10}$$

Theorem 3.3 ([MV])**.** *For any $\boldsymbol{h}$ and $\boldsymbol{c} \in \mathbb{C}^m$, the tuple $\boldsymbol{y}_{\boldsymbol{h}}(\boldsymbol{c})$ is a tuple of polynomials. Moreover, there exists a Zarisky open dense subspace $U \subset \mathbb{C}^m$ such that for every $\boldsymbol{c} \in U$, the tuple of roots $\boldsymbol{t}_{\boldsymbol{h}}(\boldsymbol{c})$ of the polynomials $(y_{\boldsymbol{h},1}(\boldsymbol{c}; x), \dots, y_{\boldsymbol{h},r}(\boldsymbol{c}; x))$ is a critical point of the corresponding master function $\Phi_{\boldsymbol{k}}$.*

4. Whitney-Lusztig charts and the comparison theorem

4.1. **Birational isomorphisms.** Recall the group N from Section 1.1.

Let $e_{ij} \in \mathfrak{gl}_{r+1}(\mathbb{C})$ denote the elementary matrix,

$$(e_{ij})_{ab} = \delta_{ia} \delta_{jb} \,.$$

Define the matrices

$$e_i(c) = 1 + c e_{i,i+1} \in N \,, \qquad i = 1, \dots, r \,, \quad c \in \mathbb{C} \,. \tag{4.1}$$

Given a word $\boldsymbol{h} = s_{i_m} \dots s_{i_1}$, define a map

$$\mathcal{L}_{\boldsymbol{h}} : \ \mathbb{C}^m \longrightarrow N \,, \qquad (c_1, \dots, c_m) \mapsto e_{i_m}(c_m) \dots e_{i_1}(c_1) \,. \tag{4.2}$$

Suppose that the word $\boldsymbol{h}$ is a reduced decomposition of the longest element w_0 as in (1.1). Then $\boldsymbol{h}$ is of length $q = r(r+1)/2$. The corresponding map

$$\mathcal{L}_{\boldsymbol{h}} \ : \ \mathbb{C}^q \longrightarrow N \tag{4.3}$$

is called the *Whitney-Lusztig chart* corresponding to $\boldsymbol{h}$. The map $\mathcal{L}_{\boldsymbol{h}}$ is a birational isomorphism.

Remark. The map $\mathcal{L}_{\boldsymbol{h}}$ is *not* epimorphic, and N is not even equal to the union of the images of $\mathcal{L}_{\boldsymbol{h}}$ for $\boldsymbol{h} \in \mathrm{Red}(w_0)$. For example for $r = 2$ the set $\mathrm{Red}(w_0)$ has two elements, the words (121) and (212). It is easy to see that the matrices of the form $1 + a e_{13}, a \neq 0$, are *inaccessible*,

$$1 + a e_{13} \ \notin \ \mathcal{L}_{121}(\mathbb{C}^3) \cup \mathcal{L}_{212}(\mathbb{C}^3) \,.$$

4.2. **Comparison Theorem.** Denote $M = \mathbb{C}[x]^{r+1}$. We will write the elements of M as $r+1$-sequences $(f_1, \dots, f_{r+1})$ with $f_i \in \mathbb{C}[x]$, but will think of them as column $r+1$-vectors with coordinates $f_1, \dots, f_{r+1}$. Then the algebra $\mathfrak{gl}_{r+1}(\mathbb{C})$ of $(r+1)\times(r+1)$-matrices acts on M from the left in the standard way.

We introduce a distinguished element

$$\boldsymbol{b}^{(0)} = \Big(1, x, \frac{x^2}{2}, \dots, \frac{x^r}{r!}\Big). \tag{4.4}$$

For a word $\boldsymbol{h} = s_{i_m} \dots s_{i_1}$ in S_{r+1} and $\boldsymbol{c} = (c_1, \dots, c_m) \in \mathbb{C}^m$, define an element

$$\boldsymbol{b}_{\boldsymbol{h}}(\boldsymbol{c}) \in M\,,$$

by the formula

$$\boldsymbol{b}_{\boldsymbol{h}}(\boldsymbol{c}) := \mathcal{L}_{\boldsymbol{h}}(c_1, \dots c_m)\boldsymbol{b}^{(0)}\,. \tag{4.5}$$

Let

$$\boldsymbol{b}_{\boldsymbol{h}}(\boldsymbol{c}) = (b_{\boldsymbol{h},1}(\boldsymbol{c}; x), \dots, b_{\boldsymbol{h},r+1}(\boldsymbol{c}; x))\,,$$

where $b_{\boldsymbol{h},1}(\boldsymbol{c}; x) \in \mathbb{C}[x]$ be coordinates of $\boldsymbol{b}_{\boldsymbol{h}}(\boldsymbol{c})$.

Recall the r-tuple of polynomials

$$\boldsymbol{y}_{\boldsymbol{h}}(\boldsymbol{c}) = (y_{\boldsymbol{h},1}(\boldsymbol{c}; x), \dots, y_{\boldsymbol{h},r}(\boldsymbol{c}; x)) = \nu_{i_m}(c_m) \dots \nu_{i_1}(c_1)\boldsymbol{y}^{(0)}\,, \tag{4.6}$$

defined in (3.9).

Theorem 4.1 (Comparison Theorem)**.** *For any $j = 1, \dots, r$, we have*

$$\mathrm{Wr}(b_{\boldsymbol{h},1}(\boldsymbol{c}; x), \dots, b_{\boldsymbol{h},j}(\boldsymbol{c}; x)) \;=\; y_{\boldsymbol{h},j}(\boldsymbol{c}; x)\,.$$

Proof. The theorem is a consequence of a general statement, see Theorem 4.3 below. □

4.3. **Cell $\mathcal{N}$.** Denote

$$\mathcal{N} = \mathcal{N}_r \;:=\; N\boldsymbol{b}^{(0)} \subset M\,,$$

the orbit of the element $\boldsymbol{b}^{(0)}$ under the action of N.

An element $\boldsymbol{b} = (b_1, \dots, b_{r+1}) \in M$ belongs to $\mathcal{N}$ if and only if

$$b_i \;=\; \frac{x^{i-1}}{(i-1)!} + \sum_{j=i}^{r} b_{ij}x^j, \qquad i = 1, \dots, r+1, \tag{4.7}$$

for some $b_{ij} \in \mathbb{C}$.

For any $\boldsymbol{b} = (b_1, \dots, b_{r+1}) \in \mathcal{N}$ we have

$$\mathrm{Wr}(b_1, \dots, b_{r+1}) \;=\; 1\,. \tag{4.8}$$

We define the *totally positive subvariety* $\mathcal{N}_{>0} \subset \mathcal{N}$ as

$$\mathcal{N}_{>0} = N_{>0}\boldsymbol{b}^{(0)}\,. \tag{4.9}$$

4.4. **$\mathcal{N}$-$\mathcal{Y}$ correspondence.** Define the *Wronski map*

$$W:\mathcal{N}\longrightarrow \mathbb{C}[x]^r\,,\quad \boldsymbol{b}\mapsto (b_1,\mathrm{Wr}(b_1,b_2),\mathrm{Wr}(b_1,b_2,b_3),\dots,\mathrm{Wr}(b_1,\dots,b_r))\,. \tag{4.10}$$

Lemma 4.2. *If $\boldsymbol{b}=(b_1,\dots,b_{r+1})\in\mathcal{N}$ and $\boldsymbol{y}=(y_1,\dots,y_r)=W(\boldsymbol{b})$, then*

$$y_i=\mathrm{Wr}(b_1,\dots,b_i) \;=\; 1+\mathcal{O}(x)\,,\qquad i=1,\dots,r\,, \tag{4.11}$$

$$\mathrm{Wr}(b_1,\dots,b_{i-1},b_{i+1}) \;=\; x+\mathcal{O}(x^2)\,,\qquad i=1,\dots,r-1\,, \tag{4.12}$$

as $x\to 0$. □

We define the *Bethe cell* or *variety of Bethe r-tuples* as

$$\mathcal{Y}^{Bethe}:=W(\mathcal{N})\subset\mathbb{C}[x]^r\,, \tag{4.13}$$

the image of the Wronski map, and the *totally positive Bethe subvariety* or *positive population* as

$$\mathcal{Y}^{Bethe}_{>0}:=W(\mathcal{N}_{>0})\subset W(\mathcal{N})=\mathcal{Y}^{Bethe}\,.$$

Theorem 4.3. *The Wronski map induces an isomorphism*

$$W:\ \mathcal{N}\xrightarrow{\sim}\mathcal{Y}^{Bethe}\,,\qquad \mathcal{N}_{>0}\xrightarrow{\sim}\mathcal{Y}^{Bethe}_{>0}\,.$$

This theorem is a consequence of the Triangular Theorem below, see Section 5.3.

Theorem 4.4. *The Lusztig's mutations, i.e. multiplications by $e_i(c)$ on the left, are translated by W to the Wronskian mutations $\nu_i(c)$ on the right:*

$$W(e_i(c)\boldsymbol{b})=\nu_i(c)W(\boldsymbol{b}),\qquad i=1,\dots,r\,. \tag{4.8.1}$$

Proof. The theorem follows from the W5 Identity. Let us treat the case $r=3$. We have

$$\boldsymbol{b}=(b_1,b_2,b_3,b_4),\quad b_i=\frac{x^{i-1}}{(i-1)!}+\dots,\quad \mathrm{Wr}(\boldsymbol{b})=1\,,\quad W(\boldsymbol{b})=\boldsymbol{y}=(y_1,y_2,y_3,1)\,.$$

The $i=1$ case. We have

$$\begin{aligned} e_1(c)\boldsymbol{b} &= (b_1+cb_2,b_2,b_3,b_4),\qquad y_1=b_1\,,\\ \mathrm{Wr}(y_1,b_2) &= \mathrm{Wr}(b_1,b_2)=y_2\,. \end{aligned}$$

Hence $\hat{y}_1=b_2$ is the solution of the normalized Wronskian equations (3.2), (3.3), and

$$\nu_1(c)\boldsymbol{y}=(y_1+c\hat{y}_1,y_2,y_3)\;=\;W(e_1(c)\boldsymbol{y})\,,$$

cf. formula (3.6).

The $i=2$ case. We have

$$e_2(c)\boldsymbol{b}=(b_1,b_2+cb_3,b_3,b_4).$$

Denote $\hat{y}_2:=\mathrm{Wr}(b_1,b_3)$. Then

$$\hat{y}_2(x)=x+\mathcal{O}(x^2)\quad\text{as}\quad x\to 0\,,$$

by formula (2.3), and

$$\mathrm{Wr}(y_2,\hat{y}_2)=\mathrm{Wr}(\mathrm{Wr}(b_1,b_2),\mathrm{Wr}(b_1,b_3))=\mathrm{Wr}(b_1)\,\mathrm{Wr}(b_1,b_2,b_3)=y_1y_3\,,$$

by the W5 Identity. Hence $\hat{y}_2$ is the solution of the normalized Wronskian equations (3.2), (3.3), and

$$\nu_2(c)\boldsymbol{y}=(y_1,y_2+c\hat{y}_2,y_3)\;=\;W(e_2(c)\boldsymbol{y})\,.$$

The $i=3$ case. We have

$$e_3(c)\boldsymbol{b}=(b_1,b_2,b_3+cb_4,b_4)$$

Denote $\hat{y}_3 := \mathrm{Wr}(b_1, b_2, b_4)$. Then

$$\hat{y}_3(x) = x + \mathcal{O}(x^2) \quad \text{as} \quad x \to 0\,,$$

by formula (2.3), and

$$\mathrm{Wr}(y_3, \hat{y}_3) = \mathrm{Wr}(\mathrm{Wr}(b_1, b_2, b_3), \mathrm{Wr}(b_1, b_2, b_4)) = \mathrm{Wr}(b_1, b_2)\,\mathrm{Wr}(b_1, b_2, b_3, b_4) = y_2\,,$$

by the W5 Identity and formula (4.8). Hence $\hat{y}_3$ is the solution of the normalized Wronskian equations (3.2), (3.3), and

$$\nu_3(c)\boldsymbol{y} = (y_1, y_2, y_3 + c\hat{y}_3) \;=\; W(e_3(c)\boldsymbol{y})\,.$$

The case of arbitrary r is similar. □

5. Triangular coordinates on the Bethe cell

5.1. Example of evolutions, group SL_3.

5.1.1. *Wronskian evolution.* We start with $\boldsymbol{y}^{(0)} = (1, 1)$ and a reduced decomposition of the longest element in S_3,

$$\boldsymbol{h} \;=\; (121) \;:\; w_0 \;=\; s_1 s_2 s_1\,.$$

The pair $\boldsymbol{y}^{(0)}$ evolves by means of the normalized mutations:

$$\begin{aligned} (1,1) &\xrightarrow{\nu_1(b_1)} (1 + b_1 x, 1) \xrightarrow{\nu_2(b_2)} (1 + b_1 x, 1 + b_2(x + b_1 x^2/2)) \\ &\xrightarrow{\nu_1(b_3)} (1 + b_1 x + b_3(x + b_2 x^2/2), 1 + b_2(x + b_1 x^2/2))\,. \end{aligned} \tag{5.1}$$

The last pair is $\nu_1(b_3)\nu_2(b_2)\nu_1(b_1)\,\boldsymbol{y}^{(0)}$.

The second reduced word, $\boldsymbol{h}' = (212)$, gives rise to another evolution:

$$\begin{aligned} (1,1) &\xrightarrow{\nu_2(c_1)} (1, 1 + c_1 x) \xrightarrow{\nu_1(c_2)} (1 + c_2(x + c_1 x^2/2), 1 + c_1 x) \\ &\xrightarrow{\nu_2(c_3)} (1 + c_2(x + c_1 x^2/2), 1 + c_1 x + c_3(x + c_2 x^2/2))\,. \end{aligned} \tag{5.2}$$

The change of coordinates on $\mathcal{Y}_3^{Bethe}$ from (b_1, b_2, b_3) to (c_1, c_2, c_3) is

$$c_2 = b_1 + b_3, \quad c_1 c_2 = b_2 b_3, \qquad c_1 + c_3 = b_2, \quad c_2 c_3 = b_1 b_2\,,$$

whence

$$c_1 = \frac{b_2 b_3}{b_1 + b_3}\,, \qquad c_2 = b_1 + b_3, \qquad c_3 = \frac{b_1 b_2}{b_1 + b_3}\,, \tag{5.3}$$

cf. [L, Proposition 2.5].

5.1.2. *Bethe cell and positive population.* Analyzing formulas (5.1) and (5.2) we observe that the Bethe cell $\mathcal{Y}_3^{Bethe}$ consists of the polynomials $(1 + e_1 x + e_2 x^2/2,\, 1 + f_1 x + f_2 x^2/2)$ such that

$$e_2 + f_2 = e_1 f_1\,, \tag{5.4}$$

cf. (1.9), and the positive population $\mathcal{Y}_{3,\,>0}^{Bethe} \subset \mathcal{Y}_3^{Bethe}$ is cut from $\mathcal{Y}_3^{Bethe}$ by the inequalities

$$e_1,\, e_2,\, f_1,\, f_2 \;>\; 0\,. \tag{5.5}$$

5.1.3. *Whitney-Lusztig evolution.* We start with

$$\boldsymbol{b}^{(0)} = (b_1, b_2, b_3) = (1, x, x^2/2)$$

and act on it by the elementary unipotent matrices

$$e_i(t) = 1 + te_{i,i+1}, \qquad i = 1, 2\,,$$

in the order dictated by the reduced word $\boldsymbol{h} = (121)$:

$$\begin{aligned}(1, x, x^2/2) \quad &\xrightarrow{e_1(t_1)} \quad (1 + t_1x, x, x^2/2) \xrightarrow{e_2(t_2)} (1 + t_1x, x + t_2x^2/2, x^2/2)\\ &\xrightarrow{e_1(t_3)} \quad (1 + t_1x + t_3(x + t_2x^2/2), x + t_2x^2/2, x^2/2)\,.\end{aligned}$$

The resulting triple is

$$\begin{aligned}\boldsymbol{b_h}(\boldsymbol{t}) \quad &= \quad (b_{\boldsymbol{h},1}(\boldsymbol{t}; x), b_{\boldsymbol{h},2}(\boldsymbol{t}; x), b_{\boldsymbol{h},3}(\boldsymbol{t}; x))\\ &:= \quad (1 + t_1x + t_3(x + t_2x^2/2), x + t_2x^2/2, x^2/2)\,,\end{aligned} \tag{5.6}$$

cf. Section 4.2. One easily checks that applying the Wronski map W to the evolution (5.6) we obtain the evolution (5.1).

5.1.4. *Remark.* Note a useful formula

$$\mathrm{Wr}(1 + t_1x, x + t_2x^2/2) = 1 + t_2(x + t_1x^2/2)\,, \tag{5.7}$$

which could be written symbolically as

$$\mathrm{Wr}(e_1(t_1), e_2(t_2)) = e_1(t_2)e_2(t_1)\,.$$

In general let

$$\boldsymbol{b} = (1 + ax + bx^2/2, x + cx^2/2, x^2/2)\,.$$

Using the formulas $(\mathrm{Wr}(1, x), \mathrm{Wr}(1, x^2/2), \mathrm{Wr}(x, x^2/2)) = (1, x, x^2/2)$, we get

$$\boldsymbol{y} := W(\boldsymbol{b}) = (1 + ax + bx^2/2,\ 1 + cx + (ac - b)x^2/2)\,.$$

Thus, the 2×2-matrix of nontrivial coefficients of $\boldsymbol{y}$ has the form

$$\begin{pmatrix} a & b \\ c & ac - b \end{pmatrix}.$$

The variety $\mathcal{Y}_3^{Bethe}$ may be identified with

$$\{(a_{ij}) \in \mathfrak{gl}_2(\mathbb{C}) \mid a_{11}a_{21} = a_{12} + a_{22}\} \subset \mathfrak{gl}_2(\mathbb{C}),$$

an exchange relation familiar in the theory of cluster algebras. Cf. Example 7.4 below.

5.2. **Example of evolutions, group SL_4.**

5.2.1. *Wronskian evolution.* There are 16 distinct reduced decompositions of the longest element $w_0 \in S_4$. We choose one of them:

$$\boldsymbol{h} = (121321)\,.$$

We start with $\boldsymbol{y}^{(0)} = (1, 1, 1)$ and perform the sequence of normalized mutations corresponding to the reduced decomposition $\boldsymbol{h}$:

$$\begin{aligned}(1, 1, 1) &\xrightarrow{\nu_1(a_1)} (1 + a_1x, 1, 1) \xrightarrow{\nu_2(a_2)} (1 + a_1x, 1 + a_2(x + a_1x^2/2), 1)\\ &\xrightarrow{\nu_3(a_3)} (1 + a_1x, 1 + a_2(x + a_1x^2/2), 1 + a_3(x + a_2(x^2/2 + a_1x^3/6)))\\ &\xrightarrow{\nu_1(a_4)} (1 + a_1x + a_4(x + a_2x^2/2), 1 + a_2(x + a_1x^2/2), 1 + a_3(x + a_2(x^2/2 + a_1x^3/6)))\,.\end{aligned}$$

To find the next normalized mutation, $\nu_2(a_5)$, we have to solve an equation

$$\mathrm{Wr}(1+a_2(x+a_1x^2/2, x+b_2x^2/2+b_3x^3/6+b_4x^4/24) = \big(1+a_1x+a_4(x+a_2x^2/2)\big)\cdot\big(1+a_3(x+a_2(x^2+a_1x^3/6))\big) \tag{5.8}$$

with respect to b_2, b_3, b_4. We calculate inductively the coefficients of x, x^2, x^3 in (5.8) and obtain:

$$b_2 = a_1+a_3+a_4, \quad b_3 = 2(a_1+a_4)a_3, \quad b_4 = 2a_2a_3a_4\,.$$

Note that the coefficients of x^4 and x^5 in the left-hand and right-hand sides of (5.8) should be equal as well, but the corresponding additional equations on b_2, b_3, b_4 will be satisfied identically – these are incarnations of the "Bethe equations".

Thus,

$$\begin{aligned}&\nu_2(a_5)\nu_1(a_4)\nu_3(a_3)\nu_2(a_2)\nu_1(a_1)\boldsymbol{y}^{(0)} = (1+a_1x+a_4(x+a_2x^2/2),\\ &\quad 1+a_2(x+a_1x^2/2)+a_5(x+(a_1+a_3+a_4)x^2/2+2(a_1+a_4)a_3x^3/6+2a_2a_3a_4x^4/24),\\ &\qquad 1+a_3(x+a_2(x^2/2+a_1x^3/6)))\,.\end{aligned}$$

Finally, to apply $\nu_1(a_6)$ to this 3-tuple, we have to find c_2, c_3 from the equation

$$\begin{aligned}&\mathrm{Wr}(1+a_1x+a_4(x+a_2x^2/2), x+c_2x^2/2+c_3x^6/6)\\ &\quad = 1+a_2(x+a_1x^2/2)+a_5(x+(a_1+a_3+a_4)x^2/2+2(a_1+a_4)a_3x^3/6+2a_2a_3a_4x^4/24\,.\end{aligned}$$

We find c_2, c_3 by equating the coefficients of x, x^2:

$$c_2 = a_2+a_5, \qquad c_3 = a_3a_5\,.$$

Then the coefficients of x^3, x^4 will be equal as well by the "Bethe equations". Thus,

$$\begin{aligned}&\boldsymbol{y_h}(\boldsymbol{a}) = (y_{\boldsymbol{h},1}(\boldsymbol{a};x), y_{\boldsymbol{h},2}(\boldsymbol{a};x), y_{\boldsymbol{h},3}(\boldsymbol{a};x)) = \nu_1(a_6)\nu_2(a_5)\nu_1(a_4)\nu_3(a_3)\nu_2(a_2)\nu_1(a_1)\boldsymbol{y}^{(0)}\\ &= (1+a_1x+a_4(x+a_2x^2/2)+a_6(x+(a_2+a_5)x^2/2+a_3a_5x^3/6),\\ &1+a_2(x+a_1x^2/2)+a_5(x+(a_1+a_3+a_4)x^2/2+2(a_1+a_4)a_3x^3/6+2a_2a_3a_4x^4/24),\\ &1+a_3(x+a_2(x^2/2+a_1x^3/6)))\,,\end{aligned}$$

cf. formula (3.9).

5.2.2. *Lusztig evolution.* Consider the Lusztig evolution corresponding to the same word $\boldsymbol{h} = (121321)$:

$$\begin{aligned}&\boldsymbol{b}^{(0)} = (1, x, x^2/2, x^3/6) \overset{e_1(a_1)}{\longrightarrow} (1+a_1x, x, x^2/2, x^3/6)\\ &\overset{e_2(a_2)}{\longrightarrow} (1+a_1x, x+a_2x^2/2, x^2/2, x^3/6) \overset{e_3(a_3)}{\longrightarrow} (1+a_1x, x+a_2x^2/2, x^2/2+a_3x^3/6, x^3/6)\\ &\overset{e_1(a_4)}{\longrightarrow} (1+a_1x+a_4(x+a_2x^2/2), x^2/2+a_3x^3/6, x^3/6)\\ &\overset{e_2(a_5)}{\longrightarrow} (1+a_1x+a_4(1+a_2x^2/2), x+a_2x^2/2+a_5(x^2/2+a_3x^3/6), x^2/2+a_3x^3/6, x^3/6)\\ &\overset{e_1(a_6)}{\longrightarrow} (1+a_1x+a_4(1+a_2x^2/2)+a_6(x+a_2x^2/2+a_5(x^2/2+a_3x^3/6)),\\ &\qquad x+a_2x^2/2+a_5(x^2/2+a_3x^3/6), x^2/2+a_3x^3/6, x^3/6)\\ &= \quad \boldsymbol{b_h}(\boldsymbol{a}) = (b_{\boldsymbol{h},1}(\boldsymbol{a};x), b_{\boldsymbol{h},2}(\boldsymbol{a};x), b_{\boldsymbol{h},3}(\boldsymbol{a};x), b_{\boldsymbol{h},4}(\boldsymbol{a};x))\,,\end{aligned}$$

cf. formula (4.5).

5.2.3. *Comparison.* We have

$$\boldsymbol{y}_{\boldsymbol{h}}(\boldsymbol{a}) = W(\boldsymbol{b}_{\boldsymbol{h}}(\boldsymbol{a}))\,.$$

Namely,

$$\begin{aligned} y_{\boldsymbol{h},1}(\boldsymbol{a};x) &= b_{\boldsymbol{h},1}(\boldsymbol{a};x)\,,\\ y_{\boldsymbol{h},2}(\boldsymbol{a};x) &= \mathrm{Wr}(b_{\boldsymbol{h},1}(\boldsymbol{a};x), b_{\boldsymbol{h},2}(\boldsymbol{a};x))\,,\\ y_{\boldsymbol{h},3}(\boldsymbol{a};x) &= \mathrm{Wr}(b_{\boldsymbol{h},1}(\boldsymbol{a};x), b_{\boldsymbol{h},2}(\boldsymbol{a};x), b_{\boldsymbol{h},3}(\boldsymbol{a};x))\,. \end{aligned}$$

5.2.4. *Wronskian map and the minors of g.* Let

$$\boldsymbol{b} = (1 + a_1x + a_2x^2/2 + a_3x^3/6, x + b_2x^2/2 + b_3x^3/6, x^2/2 + c_3x^3/6) \in \mathcal{N} \tag{5.9}$$

and

$$g = \begin{pmatrix} 1 & a_1 & a_2 & a_3\\ 0 & 1 & b_2 & b_3\\ 0 & 0 & 1 & c_3\\ 0 & 0 & 0 & 1 \end{pmatrix} \in N\,. \tag{5.10}$$

Then $\boldsymbol{b} = g\boldsymbol{b}^{(0)}$.

Let us compute $\boldsymbol{y} = (y_1, y_2, y_3) = W(\boldsymbol{b})$. Clearly,

$$\begin{aligned} y_1 &= 1 + a_1x + a_2x^2/2 + a_3x^3/6\,,\\ y_2 &= 1 + \begin{vmatrix} 1 & a_2\\ 0 & b_2 \end{vmatrix} x + \left(\begin{vmatrix} 1 & a_3\\ 0 & b_3 \end{vmatrix} + \begin{vmatrix} a_1 & a_2\\ 1 & b_2 \end{vmatrix} \right) x^2/2\\ &\quad + \begin{vmatrix} a_1 & a_3\\ 1 & b_3 \end{vmatrix} x^3/3 + \begin{vmatrix} a_2 & a_3\\ b_2 & b_3 \end{vmatrix} x^4/12\,,\\ y_3 &= 1 + \begin{vmatrix} 1 & a_1 & a_3\\ 0 & 1 & b_3\\ 0 & 0 & c_3 \end{vmatrix} x + \begin{vmatrix} 1 & a_2 & a_3\\ 0 & b_2 & b_3\\ 0 & 1 & c_3 \end{vmatrix} x^2/2 + \begin{vmatrix} a_1 & a_2 & a_3\\ 1 & b_2 & b_3\\ 0 & 1 & c_3 \end{vmatrix} x^3/6\,. \end{aligned} \tag{5.11}$$

Notice that all these determinants are minors of the matrix g. In particular, if the matrix g is totally positive, then all these determinants are positive.

5.2.5. *Wronskian map in matrix form.* By formula (5.11):

$$\begin{aligned} y_1 &= 1 + a_1x + a_2x^2/2 + a_3x^3/6\,,\\ y_2 &= 1 + b_2x + (b_3 + a_1b_2 - a_2)x^2/2 + (a_1b_3 - a_3)x^3/3 + (a_2b_3 - a_3b_2)x^4/12\,,\\ y_3 &= 1 + c_3x + (b_2c_3 - b_3)x^2/2 + (a_1(b_2c_3 - b_3) - (a_2c_3 - a_3))x^3/6\,. \end{aligned}$$

These formulas show that the inverse map

$$W^{-1}:\ \mathcal{Y}^{Bethe} \xrightarrow{\sim} \mathcal{N}$$

assigns to a triple $\boldsymbol{y} = (y_1, y_2, y_3)$ with

$$\begin{aligned} y_1 &= 1 + \alpha_1x + \alpha_2x^2/2 + \alpha_3x^3/6\,,\\ y_2 &= 1 + \beta_2x + \beta_3x^2/2 + \beta_4x^3/6 + \beta_5x^4/24\,,\\ y_3 &= 1 + \gamma_3x + \gamma_4x^2/2 + \gamma_5x^3/6\,, \end{aligned}$$

the triple $\boldsymbol{b} = W^{-1}(\boldsymbol{y})$, as in (5.9), with

$$\begin{aligned} a_1 = \alpha_1,\qquad a_2 &= \alpha_2,\qquad a_3 = \alpha_3,\\ b_2 &= \beta_2,\qquad b_3 = \beta_3 - (\alpha_1\beta_2 - \alpha_2),\\ &\qquad\qquad c_3 = \gamma_3\,. \end{aligned} \tag{5.12}$$

In a matrix form, the map W is given by the formula

$$\begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 & 0 & 0 \\ 0 & 1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \\ 0 & 0 & 1 & \gamma_3 & \gamma_4 & \gamma_5 \end{pmatrix} \tag{5.13}$$
$$= \begin{pmatrix} 1 & a_1 & a_2 & a_3 & 0 & 0 \\ 0 & 1 & b_2 & b_3 + (a_1b_2 - a_2) & 2(a_1b_3 - a_3) & 2(a_2b_3 - a_3b_2) \\ 0 & 0 & 1 & c_3 & b_2c_3 - b_3 & a_1(b_2c_3 - b_3) - (a_2c_3 - a_3) \end{pmatrix},$$

and the inverse map W^{-1} is given by the formula

$$\begin{pmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & b_2 & b_3 \\ 0 & 0 & 1 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & 1 & \beta_2 & \beta_3 - (\alpha_1\beta_2 - \alpha_2) \\ 0 & 0 & 1 & \gamma_3 \end{pmatrix}. \tag{5.14}$$

Theorem 5.1. *The coefficients $\alpha_1, \alpha_2, \alpha_3, \beta_2, \beta_3, \gamma_3$ of the polynomials y_1, y_2, y_3 serve as global coordinates on the Bethe cell $\mathcal{Y}_4^{Bethe}$, the other coefficients $\beta_4, \beta_5, \gamma_4$ are polynomial functions of the global coordinates due to formulas* (5.13) *and* (5.14). □

Theorem 5.2. *The positive population $\mathcal{Y}_{4,>0}^{Bethe} \subset \mathcal{Y}_4^{Bethe}$ is cut from $\mathcal{Y}_4^{Bethe}$ by the inequalities*

$$\text{all} \quad \alpha_i, \beta_i, \gamma_i > 0 \qquad \text{and} \qquad \beta_3 > \alpha_1\beta_2 - \alpha_2 > 0\,. \tag{5.15}$$

Proof. The proof follows from (5.11), (5.13), (5.14). □

5.3. **Triangular coordinates.** For an arbitrary r, consider an element $\boldsymbol{b} = (b_1, \dots, b_{r+1}) \in \mathcal{N}$, where

$$b_i \;=\; x^{i-1}/(i-1)! + \sum_{j=i}^{r} b_{ij}x^j, \qquad i = 1, \dots, r+1\,.$$

Denote by

$$M(\boldsymbol{b}) = (b_{ij})_{1\leqslant i\leqslant j\leqslant r}$$

the triangular array of nontrivial coefficients.

Let $\boldsymbol{y}$ be the corresponding Bethe tuple,

$$W(\boldsymbol{b}) = \boldsymbol{y} = (y_1, \dots, y_r) \in \mathcal{Y}^{Bethe},$$

with

$$y_i(x) = 1 + \sum_{j\geqslant 1} a_{ij}x^j/j!\,.$$

Define *the triangular part $\boldsymbol{y}^{\triangle}$ of $\boldsymbol{y}$*,

$$\boldsymbol{y}^{\triangle} = (y_1^{\leqslant r}, y_2^{\leqslant r-1}, \dots, y_r^{\leqslant 1})\,.$$

Here for a polynomial $f(x) = \sum_{i\geqslant 0} a_ix^i/i! \in \mathbb{C}[x]$ we use the notation

$$f^{\leqslant n}(x) = \sum_{i=0}^{n} a_ix^i/i!\,.$$

Denote

$$A(\boldsymbol{y}^{\triangle}) = (a_{ij})_{1\leqslant i\leqslant r;1\leqslant j\leqslant r+1-i}\,,$$

the triangular array of the nontrivial coefficients of $\boldsymbol{y}^{\triangle}$; thus we take all r nontrivial coefficients of $y_1(x)$; the first $r-1$ nontrivial coefficients of $y_2(x)$, etc.

The following statement describes the relationship between the two arrays $M(\boldsymbol{b})$ and $A(\boldsymbol{y})$.

Theorem 5.3 (Triangular Theorem)**.**

(i) *For all* $1 \leqslant i \leqslant j \leqslant r$*, we have*

$$a_{i,j-i+1} = b_{ij} + \varphi_{ij}((b_{kl})_{k<i})\,, \tag{5.16}$$

where φ_{ij} *is a polynomial with all monomials of degree at least 2.*

(ii) *Conversely, for all* $1 \leqslant i \leqslant j \leqslant r$*, we have*

$$b_{ij} = a_{i,j-i+1} + \psi_{ij}((a_{kl})_{k<i})\,, \tag{5.17}$$

where ψ_{ij} *is a polynomial with all monomials of degree at least 2.*

(iii) *The map*

$$\mathcal{Y}^{Bethe} \longrightarrow \mathbb{C}^q, \quad \boldsymbol{y} \mapsto A(\boldsymbol{y}^{\triangle})\,,$$

is an isomorphism.

(iv) *The Wronski map*

$$W : \mathcal{N} \longrightarrow \mathcal{Y}^{Bethe}$$

is an isomorphism.

Proof. All statements are corollaries of statement (i). We leave a proof of (i) to the reader, cf. formulas (5.13) and (5.14). □

Corollary 5.4. *The coefficients* $(a_{ij})_{1\leqslant i\leqslant r;1\leqslant j\leqslant r+1-i}$ *of the polynomials* $\boldsymbol{y} = (y_1, \dots, y_r)$ *serve as global coordinates on the Bethe cell* $\mathcal{Y}_r^{Bethe}$*, the other coefficients of* $\boldsymbol{y}$ *are polynomial functions of the global coordinates due to formulas* (5.16) *and* (5.17)*.*

5.4. **Polynomials** φ_{ij}**.** The next statement gives information on polynomials $\varphi_{ij}(\boldsymbol{b})$. Let $\boldsymbol{b}^{(0)}$ be given by (4.4). An arbitrary $\boldsymbol{b} \in \mathcal{B}$ has the form $\boldsymbol{b} = g\boldsymbol{b}^{(0)}$ for some unique $g \in N$.

Theorem 5.5. *The polynomials* $\varphi_{ij}(\boldsymbol{b})$ *are linear combinations, with strictly positive coefficients, of some minors of the matrix* g*. Consequently, if* g *is totally positive, then all the polynomials* y_i *of the tuple* $W(\boldsymbol{b}) = (y_1, \dots, y_r)$ *have strictly positive coefficients.*

Proof. The proof follows from formula (2.3), cf. formula (5.11). □

6. Bethe cells from subspaces of $\mathbb{C}[x]$ and from critical points

In this section we describe a generalization of the previous correspondence.

6.1. **From a vector space of polynomials to a population** $\mathcal{Z}_V$**, see** [MV]**.** Let $V \subset \mathbb{C}[x]$ be an $r+1$-dimensional vector space of polynomials in x. We assume that V has not *base points*, that is for any $z \in \mathbb{C}$ there is $f(x) \in V$ such that $f(z) \neq 0$.

For any $z \in \mathbb{C}$ there exists a unique $r+1$-tuple of integers $\boldsymbol{\lambda} = (\lambda_0 = 0 < \lambda_1 < \dots < \lambda_r)$ such that for any $i = 0, \dots, r$, there exists $f(x) \in V$ with the property:

$$\frac{d^{\lambda_i} f}{dx^{\lambda_i}}(z) \neq 0, \qquad \frac{d^j f}{dx^j}(z)(z) = 0, \qquad j < \lambda_i\,.$$

The tuple $\boldsymbol{\lambda}$ is called the *tuple of exponents* of V at z.

Having $\boldsymbol{\lambda}$ introduce an r-tuple of nonnegative integers $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$ by the formula

$$\mu_1 + \dots + \mu_i + i = \lambda_i, \qquad i = 1, \dots, r\,.$$

The tuple μ is called the $\mathfrak{sl}_{r+1}$ *weight* of V at z.

A point z is *regular* for V if $\boldsymbol{\lambda} = (0, 1, \dots, r)$ and hence, $\mu = (0, \dots, 0)$, otherwise the point z is *singular*. Denote by Σ_V the set of singular points. The set Σ_V is finite. Denote

$$\Sigma_V = \{z_1, \dots, z_n\} \subset \mathbb{C}\,.$$

Denote $\boldsymbol{\mu}^{(a)} = (\mu_1^{(a)}, \dots, \mu_r^{(a)})$ the weight vector at a singular point z_a.

Introduce an r-tuple of polynomials $\boldsymbol{T} = (T_1, \dots, T_r)$,

$$T_i(x) = \prod_{a=1}^{n} (x - z_a)^{\mu_i^{(a)}} . \tag{6.1}$$

Let $\boldsymbol{b} = (b_1, \dots, b_{r+1})$ be a basis of V, then the Wronskian $\mathrm{Wr}(b_1, \dots, b_{r+1})$ does not depend on the choice of the basis up to multiplication by a nonzero constant. Moreover,

$$\mathrm{Wr}(b_1, \dots, b_{r+1}) \,=\, \mathrm{const}\; T_1^r T_2^{r-1} \dots T_{r-1}^2 T_r \,.$$

This formula shows that the set Σ_V of singular points of V is the set of zeros of the Wronskian of a basis of V.

For any $i = 2, \dots, r$ and $b_1, \dots, b_i \in V$ introduce the *reduced Wronskian*

$$\mathrm{Wr}^\dagger_V(b_1, \dots, b_i) = \mathrm{Wr}(b_1, \dots, b_i)\, T_1^{1-i} T_2^{2-i} \dots T_{i-1}^{-1} \,.$$

For any $i = 1, \dots, r$ and $b_1, \dots, b_i \in V$, the reduced Wronskian is a polynomial.

Introduce the *reduced Wronski map* $W^\dagger_V$, which maps the variety of bases of V to the space of r-tuples of polynomials. If $\boldsymbol{b} = (b_1, \dots, b_{r+1})$ is a basis of V, then

$$W^\dagger_V : \boldsymbol{b} \mapsto \boldsymbol{y} = (y_1, \dots, y_r) := (b_1, \mathrm{Wr}^\dagger_V(b_1, b_2), \dots, \mathrm{Wr}^\dagger_V(b_1, \dots, b_r)) \,, \tag{6.2}$$

cf. (4.10). We set $y_0 = y_{r+1} = 1$.

Denote

$$\hat{y}_i = \mathrm{Wr}^\dagger_V(b_1, b_2, \dots, b_{i-1} b_{i+1}) \,, \qquad i = 1, \dots, r \,. \tag{6.3}$$

Then

$$\mathrm{Wr}(y_i, \hat{y}_i) \,=\, \mathrm{const}\; T_i\, y_{i-1}\, y_{i+1} \,, \qquad i = 1, \dots, r \,. \tag{6.4}$$

The generalized Wronski map $W^\dagger_V$ induces a map of the variety of bases of V to the direct product of projective spaces $\mathbb{P}(\mathbb{C}[x])^r$. The bases defining the same complete flag of V are mapped to the same point of $\mathbb{P}(\mathbb{C}[x])^r$. Hence the generalized Wronski map induces a map,

$$W^\dagger_V : X_V \to \mathbb{P}(\mathbb{C}[x])^r \,,$$

of the variety X_V of complete flags of V to $\mathbb{P}(\mathbb{C}[x])^r$.

Theorem 6.1 ([MV])**.** *This map is an embedding. The image*

$$\mathcal{Z}_V \subset \mathbb{P}(\mathbb{C}[x])^r \tag{6.5}$$

of this map is isomorphic to the variety of complete flags of V.

The variety $\mathcal{Z}_V$ is called the *population* associated with V, see [MV].

See in Sections 1.5-1.7 the example of this construction corresponding to the case, where $V = \mathbb{C}[x]_{\leqslant r}$ is the space of all the polynomials of degree $\leqslant r$. In that case the set of singular points of V is empty and $T_1 = \dots = T_r = 1$.

A tuple $\boldsymbol{y} = (y_1, \dots, y_r) \in \mathbb{C}[x]^r$ is called *fertile* with respect to $T_1, \dots, T_r$, if for any $i = 1, \dots, r$ the equation

$$\mathrm{Wr}(y_i, \hat{y}_i) \,=\, T_i\, y_{i-1}\, y_{i+1} \tag{6.6}$$

with respect to $\hat{y}_i(x)$ admits a polynomial solution. For example, all tuples $(y_1 : \dots : y_r) \in \mathcal{Z}_V$ are fertile due to (6.4).

A tuple $\boldsymbol{y} = (y_1, \dots, y_r) \in \mathbb{C}[x]^r$ is called *generic* with respect to $T_1, \dots, T_r$, if for any i the polynomial $y_i(x)$ has no multiple roots and the polynomials $y_i(x)$ and $T_i(x) y_{i-1}(x)\, y_{i+1}(x)$ have no common roots.

Lemma 6.2 ([MV])**.** *All $\boldsymbol{y} \in \mathcal{Z}_V$ are fertile. Also there exists a Zariski open subset $U \subset \mathcal{Z}_V$ such that any $\boldsymbol{y} \in U$ is generic.*

Let $\boldsymbol{y} = (y_1, \dots, y_r)$ be a tuple of polynomials. Denote $\boldsymbol{k} = (k_1, \dots, k_r) := (\deg y_1, \dots, y_r)$. Let

$$y_i(x) = \prod_{j=1}^{k_i} (x - t_j^{(i)}) \,, \qquad i = 1, \dots, r \,,$$

where $t_j^{(i)}$ are the roots of y_i. Denote $\boldsymbol{t} = (t_1^{(1)}, \dots, t_{k_1}^{(1)}; \dots; t_1^{(r)}, \dots, t_{k_r}^{(r)})$, the tuple of roots of $\boldsymbol{y}$ ordered in any way.

Theorem 6.3 ([MV])**.** *A tuple $\boldsymbol{y}$ is generic and fertile with respect to $T_1, \dots, T_r$ if and only if the tuple of roots $\boldsymbol{t}$ is a critical points of the master function*

$$\begin{aligned} \Phi_{\boldsymbol{k}}(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\mu}) \;&=\; \prod_{i=1}^{r} \prod_{j=1}^{k_i} \prod_{a=1}^{n} (u_j^{(i)} - z_a)^{-\mu_i^{(a)}} \\ &\times \prod_{i=1}^{r} \prod_{1 \leqslant l < m \leqslant k_i} (u_l^{(i)} - u_m^{(i)})^2 \cdot \prod_{i=1}^{r-1} \prod_{l=1}^{k_i} \prod_{m=1}^{k_{i+1}} (u_l^{(i)} - u_m^{(i+1)})^{-1} \,. \end{aligned} \tag{6.7}$$

These master functions appear in the integral representations of the KZ equations associated with $\mathfrak{gl}_{r+1}$, see [SV].

The following statement is converse to the statement of Theorem 6.3.

Given a finite set $\boldsymbol{z} = (z_1, \dots, z_n)$, a collection of $\boldsymbol{\mu}^{(a)} = (\mu_1^{(a)}, \dots, \mu_r^{(a)}) \in \mathbb{Z}^r_{\geqslant 0}$, $a = 1, \dots, n$, a vector $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{\geqslant 0}$, we define the master function $\Phi_{\boldsymbol{k}}(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\mu})$ by formula (6.7).

Theorem 6.4 ([MV])**.** *If $\boldsymbol{t}$ is a critical point of a master function $\Phi_{\boldsymbol{k}}(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\mu})$ with respect to the $\boldsymbol{u}$-variables, then there exists a unique $r+1$-dimensional vector space $V \subset \mathbb{C}[x]$, a basis $\boldsymbol{b}$ of V, such that the roots of the r-tuple of polynomials $W_V^{\dagger}(\boldsymbol{b})$ give $\boldsymbol{t}$.*

Cf. Theorem 1.2.

6.2. **From a critical point to a population $\mathcal{Z}_V$,** [MV]**.** Given $\boldsymbol{z} = (z_1, \dots, z_n)$, a collection of $\boldsymbol{\mu}^{(a)} = (\mu_1^{(a)}, \dots, \mu_r^{(a)}) \in \mathbb{Z}^r_{\geqslant 0}$, $a = 1, \dots, n$, a vector $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{Z}^r_{\geqslant 0}$, let $\boldsymbol{t} = (t_1^{(1)}, \dots, t_{k_1}^{(1)}; \dots; t_1^{(r)}, \dots, t_{k_r}^{(r)})$, be a critical point of the master function $\Phi_{\boldsymbol{k}}(\boldsymbol{u}, \boldsymbol{z}, \boldsymbol{\mu})$. Define

$$T_i(x) \;=\; \prod_{a=1}^{n} (x - z_a)^{\mu_i^{(a)}} \,, \qquad i = 1, \dots, r \,, \tag{6.8}$$

$$y_i(x) \;=\; \prod_{j=1}^{k_i} (x - t_j^{(i)}) \,, \qquad i = 1, \dots, r \,, \tag{6.9}$$

and the tuple

$$\boldsymbol{y} = (y_1 : \dots : y_r) \;\in \mathbb{P}(\mathbb{C}[x])^r \,.$$

The tuple $\boldsymbol{y}$ is generic and fertile with respect to $T_1, \dots, T_r$. Hence for every i, there exists a polynomial $\hat{y}_i$ satisfying the equation

$$\mathrm{Wr}(y_i, \hat{y}_i) = \text{ const } T_i \, y_{i-1} \, y_{i+1} \,.$$

Choose one solution $\hat{y}_i$ of this equation, denote

$$\tilde{y}_i(c; x) = y_i(x) + c\hat{y}_i(x) \,, \qquad c \in \mathbb{C} \,,$$

and define a curve in $\mathbb{P}(\mathbb{C}[x])^r$ by the formula

$$\boldsymbol{y}^{(i)}(c; x) = (y_1(x) : \dots : \tilde{y}_i(c; x) : \dots : y_r(x)) \,\in\, \mathbb{P}(\mathbb{C}[x])^r \,. \tag{6.10}$$

This curve is called the *generation* from $\boldsymbol{y}$ in the i-th direction.

Thus starting from the point $\boldsymbol{y}$ in $\mathbb{P}(\mathbb{C}[x])^r$ we have constructed r curves in $\mathbb{P}(\mathbb{C}[x])^r$. Now starting from any point of the constructed curves we may repeat this procedure and generate new r curves in $\mathbb{P}(\mathbb{C}[x])^r$ in any of the r directions. Repeating this procedure in all possible direction in any number of steps we obtain a subset $\mathcal{Z} \subset \mathbb{P}(\mathbb{C}[x])^r$ of all points appearing in this way. The subset $\mathcal{Z}$ is called the *population* generated from the critical point $\boldsymbol{t}$

Theorem 6.5 ([MV])**.** *The set $\mathcal{Z}$ is an algebraic variety isomorphic to the variety X of complete flags in an $r+1$-dimensional vector space. Moreover, starting with given $\boldsymbol{t}$ one can also determine uniquely an $r+1$-dimensional vector space V and a basis $\boldsymbol{b}$ of V, such that $\boldsymbol{y} = W^{\dagger}_V(\boldsymbol{b})$ and $\mathcal{Z} = \mathcal{Z}_V$.*

6.3. **Bethe cells associated with** $(V; z)$**.** Let V be an $r+1$-dimensional vector space as in Section 6.1. Let $\Sigma_V = \{z_1, \dots, z_n\} \subset \mathbb{C}$ be the set of singular points of V. Fix a complex number $z \notin \Sigma_V$, a regular point of V.

We say that a basis $\boldsymbol{b} = (b_1, \dots, b_{r+1})$ of V is a *unipotent basis* of V with respect to z, if for any $i = 1, \dots, r+1$, we have

$$b_i \,=\, \frac{(x-z)^{i-1}}{(i-1)!} \,+\, \mathcal{O}((x-z)^i) \qquad \text{as} \qquad x \to z \,. \tag{6.11}$$

Denote by $\mathcal{N}(V; z)$ the set of all unipotent bases of V at z.

If we consider each basis $\boldsymbol{b}$ of V as an $r+1$-column vector, then the group N freely acts on $\mathcal{N}(V; z)$ from the left with one orbit. We call $\mathcal{N}(V; z)$ *the cell of bases of V unipotent at z.*

For any $i = 2, \dots, r$ and $b_1, \dots, b_i \in V$ introduce the *reduced Wronskian normalized at z* by the formula

$$\begin{aligned}\mathrm{Wr}^{\dagger}_{V,z}(b_1, \dots, b_i) \;&:=\; \mathrm{Wr}^{\dagger}_V(b_1, \dots, b_i) \, T_1^{i-1}(z) T_2^{i-2}(z) \dots T_{i-1}(z) \\ &=\; \mathrm{Wr}(b_1, \dots, b_i) \, \frac{T_1^{i-1}(z) T_2^{i-2}(z) \dots T_{i-1}(z)}{T_1^{i-1}(x) T_2^{i-2}(x) \dots T_{i-1}(x)} \,.\end{aligned}$$

For any $i = 1, \dots, r$ and $b_1, \dots, b_i \in V$, this is a polynomial.

Introduce the *reduced Wronski map* $W^{\dagger}_{V,z}$, which maps the variety of bases of V to the space of r-tuples of polynomials. If $\boldsymbol{b} = (b_1, \dots, b_{r+1})$ is a basis of V, then

$$W^{\dagger}_V : \boldsymbol{b} \mapsto \boldsymbol{y} = (y_1, \dots, y_r) := (b_1, \mathrm{Wr}^{\dagger}_{V,z}(b_1, b_2), \dots, \mathrm{Wr}^{\dagger}_{V,z}(b_1, \dots, b_r)) \,, \tag{6.12}$$

cf. (4.10). We set $y_0 = y_{r+1} = 1$.

If $\boldsymbol{b} = (b_1, \dots, b_{r+1}) \in \mathcal{N}(V; z)$, then

$$y_i(z) = 1 \,, \qquad i = 1, \dots, r \,. \tag{6.13}$$

Introduce the *Bethe cell* $\mathcal{Y}^{Bethe}(V;z)$ as the image of the cell $\mathcal{N}(V;z)$ under the reduced Wronski map $W^{\dagger}_{V,z}$,

$$\mathcal{Y}^{Bethe}(V;z) := W^{\dagger}_{V,z}(\mathcal{N}(V;z)) \subset \mathbb{C}[x]^r . \tag{6.14}$$

Theorem 6.6. *The reduced Wronski map* $W^{\dagger}_{V,z}$ *induces an isomorphism*

$$W^{\dagger}_{V,z} : \mathcal{N}(V;z) \to \mathcal{Y}^{Bethe}(V;z) . \tag{6.15}$$

Proof. The theorem is deduced from Theorem 6.1, or it can be proved along the lines of the proof of Triangular Theorem 5.3, which corresponds to the subspace $V = \mathbb{C}[x]_{\leqslant r}$ of polynomials of degree $\leqslant r$ and $z = 0$. □

Corollary 6.7. *The action of* N *on* $\mathcal{N}(V;z)$ *and the Wronski map* $W^{\dagger}_{V,z}$ *induce an action of* N *on the Bethe cell* $\mathcal{Y}^{Bethe}(V;z)$.

Lemma 6.8. *There exists a Zariski open subset* $U \subset \mathcal{Y}^{Bethe}(V;z)$, *such that any* $\boldsymbol{y} \in U$ *is generic and fertile.*

Proof. The theorem follows from Lemma 6.2. □

Recall that by Theorem 6.3, if $\boldsymbol{y} = (y_1, \dots, y_r)$ is generic and fertile, then roots of these polynomials determine a critical point of a suitable master function.

6.4. **Normalized mutations and $\mathcal{N}$-$\mathcal{Y}$ correspondence.** Let $\boldsymbol{y} = (y_1, \dots, y_r) \in \mathcal{Y}^{Bethe}(V;z)$ and $i = 1, \dots, r$. Define the *normalized mutation* of $\boldsymbol{y}$ in the i-th direction. Consider the differential equation

$$\mathrm{Wr}(y_i(x), \hat{y}_i(x)) = \frac{T_i(x)}{T_i(z)}\, y_{i-1}(x)\, y_{i+1}(x), \tag{6.16}$$

with respect to $\hat{y}_i$ with initial condition

$$\hat{y}_i(z) = 0, \qquad \frac{d\hat{y}_i}{dx}(z) = 1. \tag{6.17}$$

It has the unique solution

$$\hat{y}_i(x) = y_i(x) \int_z^x \frac{T_i(u) y_{i-1}(u) y_{i+1}(u)}{T_i(z) y_i(u)^2}\, du . \tag{6.18}$$

This solution is a polynomial, cf. equation (6.4). For $c \in \mathbb{C}$, denote

$$\tilde{y}_i(c;x) = y_i(x) + c\hat{y}_i(x) . \tag{6.19}$$

Notice that $\tilde{y}_i(c;z) = 1$.

Define a new r-tuple of polynomials

$$\nu_i(c)\boldsymbol{y} := (y_1(x), \dots, y_{i-1}(x),\, \tilde{y}_i(c;x),\, y_{i+1}(x), \dots, y_r(x)) . \tag{6.20}$$

We call it *the i-th normalized mutation* of the tuple $\boldsymbol{y} \in \mathcal{N}(V;z)$.

Lemma 6.9. *For any* i, c *the tuple* $\nu_i(c)\boldsymbol{y}$ *lies in* $\mathcal{Y}^{Bethe}(V;z)$.

Proof. The lemma follows from formula (6.4). □

By this lemma we have a map

$$\nu_i(c) : \mathcal{Y}^{Bethe}(V;z) \to \mathcal{Y}^{Bethe}(V;z) .$$

Recall the unipotent matrices $e_i(c)$, $i = 1, \dots, r$, $c \in \mathbb{C}$, introduced in (4.1).

Theorem 6.10 (Comparison Theorem)**.** *Let $\boldsymbol{b} \in \mathcal{N}(V;z)$, $i = 1,\dots,r$, $c \in \mathbb{C}$. Then*

$$W^{\dagger}_{V,z}(e_i(c)\boldsymbol{b}) \,=\, \mu_i(c) W^{\dagger}_{V,z}(\boldsymbol{b})\,. \tag{6.21}$$

Proof. The proof follows from formula (6.4). Cf. the proof of Theorem 4.4. □

Corollary 6.11. *By Corollary 6.7 the group N acts on the Bethe cell $\mathcal{Y}^{Bethe}(V;z)$. By Theorem 6.10 a unipotent matrix $e_i(c)$ acts on the Bethe cell $\mathcal{Y}^{Bethe}(V;z)$ as the normalized mutation $\mu_i(c)$.*

6.5. **Positive populations.** Fix any point $\boldsymbol{b}^{(0)} \in \mathcal{N}(V;z)$, that is any basis of V unipotent at z. Then

$$N\boldsymbol{b}^{(0)} = \mathcal{N}(V,z)\,.$$

Define the *totally positive part* of $\mathcal{N}(V,z)$ as

$$\mathcal{N}_{>0}(V;z;\boldsymbol{b}^{(0)}) \,:=\, N_{>0}\boldsymbol{b}^{(0)}\,.$$

The totally positive part depends on the choice of $\boldsymbol{b}^{(0)}$.

Denote

$$\boldsymbol{y}^{(0)} \,:=\, W^{\dagger}_{V,z}(\boldsymbol{b}^{(0)}) \,\in\, \mathcal{Y}^{Bethe}(V;z)\,.$$

Since $\boldsymbol{b}^{(0)}$ is any point of $\mathcal{N}(V;z)$, the point $\boldsymbol{y}^{(0)}$ could be an any point of the Bethe cell $\mathcal{Y}^{Bethe}(V;z)$.

Define the *totally positive Bethe subvariety* or *positive population* as

$$\mathcal{Y}^{Bethe}_{>0}(V;z;\boldsymbol{b}^{(0)}) := N_{>0}(\boldsymbol{y}^{(0)})) \subset \mathcal{Y}^{Bethe}(V;z)\,. \tag{6.22}$$

We also have

$$\mathcal{Y}^{Bethe}_{>0}(V;z;\boldsymbol{b}^{(0)}) = W^{\dagger}_{V,z}(\mathcal{N}_{>0}(V;z;\boldsymbol{b}^{(0)})\,. \tag{6.23}$$

6.6. **Coordinates on the Bethe cell.** Let $\boldsymbol{b}^{(0)}$ be a point of the Bethe cell $\mathcal{Y}^{Bethe}(V;z)$. Let $\boldsymbol{h} = s_{i_q}\dots s_{i_1} \in \mathrm{Red}(w_0)$ be a reduced decomposition of the longest element $w_0 \in S_{r+1}$.

We call the map

$$\nu_{\boldsymbol{h}} \,:\, \mathbb{C}^q \longrightarrow \mathcal{Y}(V;z)^{Bethe}, \qquad (c_q,\dots,c_1) \mapsto \nu_{i_q}(c_q)\dots\nu_{i_1}(c_1)\boldsymbol{y}^{(0)}\,,$$

the *Wronskian chart* corresponding to $\boldsymbol{h}$. Its image is Zariski open. The map $\nu_{\boldsymbol{h}}$ is a birational isomorphism.

For any two reduced words $\boldsymbol{h},\boldsymbol{h}' \in \mathrm{Red}(w_0)$ we have the transition function $\nu^{-1}_{\boldsymbol{h}'} \circ \nu_{\boldsymbol{h}}$, which defines an automorphism

$$\mathcal{R}_{\boldsymbol{h},\boldsymbol{h}'} \,:\, \mathbb{F} \xrightarrow{\sim} \mathbb{F} \tag{6.24}$$

of the field $\mathbb{F} := \mathbb{C}(c_1,\dots,c_q)$.

Recall the Whitney-Lusztig charts on N, see (4.3),

$$\mathcal{L}_{\boldsymbol{h}} \,:\, \mathbb{C}^q \longrightarrow N$$

and transition function automorphisms

$$R_{\boldsymbol{h},\boldsymbol{h}'} \,:\, \mathbb{F} \xrightarrow{\sim} \mathbb{F}\,,$$

defined for any two words $\boldsymbol{h},\boldsymbol{h}' \in \mathrm{Red}(w_0)$, see Section 1.3.

Theorem 6.12. *For any $\boldsymbol{h},\boldsymbol{h}' \in \mathrm{Red}(w_0)$ we have*

$$\mathcal{R}_{\boldsymbol{h},\boldsymbol{h}'} \,=\, R_{\boldsymbol{h},\boldsymbol{h}'}\,. \tag{6.25}$$

Proof. The theorem follows from Comparison Theorem 6.10. □

Corollary 6.13. *For any $\boldsymbol{h}, \boldsymbol{h}' \in Red(w_0)$, the map $\nu_{\boldsymbol{h}'} \circ \nu_{\boldsymbol{h}}^{-1}$ is well defined on the positive population $\mathcal{Y}^{Bethe}_{>0}(V; z; \boldsymbol{b}^{(0)})$ and defines an isomorphism*

$$\nu_{\boldsymbol{h}'} \circ \nu_{\boldsymbol{h}}^{-1} \,:\, \mathcal{Y}^{Bethe}_{>0}(V; z; \boldsymbol{b}^{(0)}) \,\to\, \mathcal{Y}^{Bethe}_{>0}(V; z; \boldsymbol{b}^{(0)})\,.$$

Proof. The corollary follows from Theorem 1.1 and Comparison Theorem 6.10. □

7. Base affine space and fat populations

7.1. **Base affine space.** Let $N, N_- \subset G := \mathrm{SL}_{r+1}(\mathbb{C})$ be the subgroups of the upper and lower triangular matrices with 1's on the diagonal. Let $T \subset G$ be the subgroup of diagonal matrices. Let $B_- = N_- T$ be the subgroup of lower triangular matrices and $B = NT$ the subgroup of upper triangular matrices

The quotient G/N_- is called the *base affine space* of G. It is fibered over the flag space G/B_- with fiber T.

The image $\mathcal{B}$ of B in G/N_- is called the *big cell* of the base affine space G/N_- .

7.2. **Fat population.** Let V be an $r+1$-dimensional vector space as in Section 6.1.

The vector space V has a *volume form.* The *volume* of a basis $\boldsymbol{b} = (b_1, \dots, b_{r+1})$ of V is defined to be the number

$$\mathrm{Wr}^{\dagger}_V(b_1, \dots, b_{r+1})\,. \tag{7.1}$$

Denote by B_V the set of all bases of V of volume 1. We consider every basis vector as an $r+1$-column vector. Then the group SL_{r+1} acts on B_V on the left freely with one orbit. The quotient B_V/N_- is isomorphic to the base affine space of SL_{r+1}.

The reduced Wronski map $W^{\dagger}_V$ defined in (6.2) induces a map

$$W^{\dagger}_V \,:\, B_V/N_- \,\to\, \mathbb{C}[x]^r\,. \tag{7.2}$$

Theorem 7.1. *The reduced Wronski map $W^{\dagger}_V \,:\, B_V/N_- \,\to\, \mathbb{C}[x]^r$ is an embedding. The image of the map, denoted by* $\mathrm{Fat}(\mathcal{Z}_V)$*, is isomorphic to the affine base space G/N_- of the group* SL_{r+1}.

The image $\mathrm{Fat}(\mathcal{Z}_V)$ will be called the *fat population* associated with V.

Proof. The proof is parallel to the proof of Theorem 6.1 in [MV]. □

We have another description of the fat population as a bundle

$$\mathrm{Fat}(\mathcal{Z}_V) \,\to\, \mathcal{Z}_V \tag{7.3}$$

over the population $\mathcal{Z}_V$, defined in Theorem 6.1, with fiber isomorphic to $(\mathbb{C}^{\times})^r$. Namely, if $\boldsymbol{y} = (y_1 : \dots : y_r) \in \mathcal{Z}_V \subset \mathbb{P}(\mathbb{C}[x])^r$ and $(y_1, \dots, y_r) \in \mathbb{C}[x]^r$ is a representative of $\boldsymbol{y}$, then the fiber over $\boldsymbol{y}$ consist of the following r-tuples of polynomials

$$\{(d_1 y_1, \dots, d_r y_r) \in \mathbb{C}[x]^r \mid (d_1, \dots, d_r) \in (\mathbb{C}^{\times})^r\}\,.$$

7.3. **Fat Bethe cell.** Let $\Sigma_V = \{z_1, \dots, z_n\} \subset \mathbb{C}$ be the set of singular points of V. Fix a complex number $z \notin \Sigma_V$, a regular point of V. Let $\mathcal{Y}^{Bethe}(V; z) \subset \mathbb{C}[x]^r$ be the Bethe cell defined in (6.14). Define the *fat Bethe cell* $\mathrm{Fat}(\mathcal{Y}^{Bethe}(V; z))$ by the formula

$$\mathrm{Fat}(\mathcal{Y}^{Bethe}(V;z)) = \\ = \{(d_1y_1, \dots, d_ry_r) \in \mathbb{C}[x]^r \mid (y_1, \dots, y_r) \in \mathcal{Y}^{Bethe}(V; z)\,,\ (d_1, \dots, d_r) \in (\mathbb{C}^\times)^r\}\,. \tag{7.4}$$

Recall $\mathcal{N}(V; z)$, the cell of bases of V unipotent at z. Define the fat cell $\mathrm{Fat}(\mathcal{N}(V; z))$ by the formula

$$\mathrm{Fat}(\mathcal{N}(V;z)) = \{(b_1d_1, b_2d_2/d_1, \dots, b_rd_r/d_{r-1}, b_{r+1}/d_r) \in \mathbb{C}[x]^r \mid \\ \mid (b_1, \dots, b_{r+1}) \in \mathcal{N}(V; z)\,,\ (d_1, \dots, d_r) \in (\mathbb{C}^\times)^r\}\,. \tag{7.5}$$

Clearly the fat cell is isomorphic to the big cell $\mathcal{B}$ of the base affine space G/N_-.

Theorem 7.2. *The reduced Wronski map induces an isomorphism*

$$W_V^\dagger\ :\ \mathrm{Fat}(\mathcal{N}(V;z))\ \to\ \mathrm{Fat}(\mathcal{Y}^{Bethe}(V;z))\,. \tag{7.6}$$

Hence the fat Bathe cell $\mathrm{Fat}(\mathcal{Y}^{Bethe}(V; z))$ *is isomorphic to the big cell* $\mathcal{B}$ *of the base affine space* G/N_-.

Proof. This theorem is a corollary of Theorem 6.6. □

7.4. **Example of a cluster structure on a fat Bethe cell.** Consider the example of the 3-dimensional vector space $V = \mathbb{C}[x]_{\leqslant 2}$ of quadratic polynomials. In this case the set Σ_V of singular points of V is empty. We choose $z = 0$, a regular point for V, and consider the corresponding fat Bethe cell $\mathrm{Fat}(\mathcal{Y}^{Bethe}(V; z))$. It consist of pairs of polynomials

$$(\alpha_0 + \alpha_1 x + \alpha_2 x^2/2, \beta_0 + \beta_1 x + \beta_2 x^2/2) \tag{7.7}$$

such that

$$\alpha_0 \neq 0, \qquad \beta_0 \neq 0$$

and such that the Plücker equation holds,

$$\alpha_1\beta_1 = \alpha_0\beta_2 + \alpha_2\beta_0\,.$$

This is a familiar relation in the cluster algebra structure of type A_1 on the ring $\mathbb{C}[\mathrm{SL}_3/N_-]$, where the cluster variables are α_1, β_1, cf. [Z, Section 3.1] and [FZ].

In fact, in this case the coefficients of the polynomials in (7.7) are nothing else but the Plücker coordinates on $\mathbb{C}[\mathrm{SL}_3/N_-]$.

8. Appendix: Fourteen- and Eightfold Ways

8.1. **Group SL_3 and a 2-category.** One can reformulate the Whitney-Lusztig data from Section 1.3 in the language of [MS].

Namely, consider a 2-*category*[3] $\mathcal{S}_3$ whose objects are in bijection with S_3; the 1-arrows correspond to the *weak Bruhat order* on this group: there are 6 elementary arrows:

$$(123) \xrightarrow{\tau_{12}} (213) \xrightarrow{\tau_{23}} (231) \xrightarrow{\tau'_{12}} (321) \tag{8.1}$$

and

$$(123) \xrightarrow{\tau'_{23}} (132) \xrightarrow{\tau''_{12}} (312) \xrightarrow{\tau''_{23}} (321). \tag{8.2}$$

[3]for a definition of (globular) n-categories see [St]

Finally, there is one nontrivial 2-arrow between two compositions

$$h : \tau'_{12}\tau_{23}\tau_{12} \longrightarrow \tau''_{23}\tau''_{12}\tau'_{23}\,. \tag{8.3}$$

This structure is conveniently visualized in a hexagon, to be denoted $\mathcal{P}_3$: its 6 vertices correspond to objects of $\mathcal{S}_3$, 6 oriented edges - to elementary 1-morphisms τ, and the unique 2 cell - to the 2-morphism h.

The Whitney-Lusztig data described above may be called a 2-*representation* ρ of $\mathcal{S}_3$: to each object we assign the same vector space $V_{(ijk)} = \mathbb{F}^3$ with a fixed standard base. To the elementary arrows $\tau_{ij}^{(\prime,\prime\prime)}$ we assign elementary matrices:

$$\rho(\tau_{12}) = e_1(a_1), \qquad \rho(\tau_{23}) = e_2(a_2), \tag{8.4}$$

etc. To products of elementary arrows we assign the products of the corresponding matrices.

Finally, $\rho(h)$ will be an automorphism of $\mathbb{F}$ given by

$$\rho(h) = R_{121;212} =: R\,, \tag{8.5}$$

see formulas (1.3) and (1.4). Note that

$$R^2 = \mathrm{Id}_{\mathbb{F}}, \tag{8.6}$$

see Section 1.3.

Thus, the matrix $\rho(\tau''_{23}\tau''_{12}\tau'_{23})$ is obtained from the matrix $\rho(\tau'_{12}\tau_{23}\tau_{12})$ by applying the field automorphism $\rho(h)$:

$$\rho(\tau''_{23}\tau''_{12}\tau'_{23}) = \rho(h)\big\{\rho(\tau'_{12}\tau_{23}\tau_{12})\big\}. \tag{8.7}$$

Similarly, for any r one defines in [MS] an r-*category* $\mathcal{S}_r$, whose 1-coskeleton is a usual 1-category corresponding to the symmetric group S_{r+1} with the weak Bruhat order. Informally speaking, $\mathcal{S}_r$ is an r-category structure on the $r+1$-th *permutohedron* $\mathcal{P}_{r+1}$, the r-dimensional polyhedron in $\mathbb{R}^{r+1}$, the convex hull of a generic S_{r+1}-orbit of a point $\boldsymbol{x} \in \mathbb{R}^{r+1}$.

In particular for each $w \in S_{r+1}$ we have an $r-1$-category $\mathcal{H}om_{\mathcal{S}_r}(e, w)$; its 1-skeleton

$$\mathcal{H}om^{\leqslant 1}_{\mathcal{S}_r}(e, w) := \mathrm{Sk}_1\, \mathcal{H}om_{\mathcal{S}_r}(e, w)$$

is a usual (1-)category; the set of its objects is by definition the set $\overline{\mathrm{Red}(w)}$ which is obtained from $\mathrm{Red}(w)$ by identifying any two words $\boldsymbol{h}$ and $\boldsymbol{h}'$ if their only distinction is a couple ij in $\boldsymbol{h}$ vs ji in $\boldsymbol{h}'$ somewhere in the middle, with $|i-j| > 1$.

According to [MS], $\overline{\mathrm{Red}(w)}$ is equipped with a partial order, *the* 2*nd Bruhat order* $\leqslant_2$. The category $\mathcal{H}om^{\leqslant 1}_{\mathcal{S}_r}(e, w)$ corresponds to this order, i.e. two words $\boldsymbol{h}, \boldsymbol{h}' \in \overline{\mathrm{Red}(w)}$ are connected by a unique arrow if and only if $\boldsymbol{h} \leqslant_2 \boldsymbol{h}'$.

8.2. Group SL_4 and a tetrahedron equation.

8.2.1. *The Forteenfold Way.* [4] Consider the case $r = 3$. The 3-category $\mathcal{S}_4$ is related to a polyhedron $\mathcal{P}_4$ which looks as follows. We take a regular octahedron and cut from it 6 little square pyramids at its vertices; we get a polyhedron with $6 \times 4 = 24$ vertices. It has 8 hexagons and 6 squares as its 2-faces.

The vertices of $\mathcal{P}_4$ are in bijection with S_4. An oriented edge $x \to y$ connects elements $x, y \in S_4$ such that $x \leqslant y$ and $\ell(y) = \ell(x) + 1$ where $\leqslant$ is the weak Bruhat order, and $\ell(x)$ is the usual length (the number of factors in a reduced decomposition).

Its eight hexagonal 2-faces correspond to identities of the form

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}\,, \tag{8.8}$$

[4] Cf. [GN].

which give rise to eight "elementary" 2-morphisms of type (1.3).

Its six square edges correspond to identities of the form

$$s_i s_j = s_j s_i, \qquad |i-j| > 1. \tag{8.9}$$

which give rise to the *identity* 2-morphisms in $\mathcal{S}_4$.

It is convenient to imagine the vertices e and w_0 as the "North" and "South" poles of $\mathcal{P}_4$. The set $\mathrm{Red}(w_0)$ consists of 16 longest paths on $\mathcal{P}_4$, of length 6, going downstairs, which connect e and w_0.

For example, the path ℓ_1 is

$$(1234) \xrightarrow{\tau_1} (2134) \xrightarrow{\tau_2} (2314) \xrightarrow{\tau_1'} (3214) \xrightarrow{\tau_3} (3241) \xrightarrow{\tau_2'} (3421) \xrightarrow{\tau_1''} (4321),$$

whereas the path ℓ_5 is

$$(1234) \xrightarrow{\tau_3'} (1243) \xrightarrow{\tau_2''} (1423) \xrightarrow{\tau_3''} (1432) \xrightarrow{\tau_1'''} (4132) \xrightarrow{\tau_2'''} (4312) \xrightarrow{\tau_3'''} (4321).$$

Fourteen paths are depicted below, it is a 14-*fold Way*:

$$\begin{array}{ccccc} 212321 & \longrightarrow & 213231 = 231231 = 231213 & \longrightarrow & 232123 \\ \uparrow & & & & \downarrow \\ 121321 & & & & 323123 \\ || & & & & || \\ 123121 & & & & 321323 \\ \downarrow & & & & \uparrow \\ 123212 & \longrightarrow & 132312 = 132132 = 312132 & \longrightarrow & 321232 \end{array} . \tag{8.10}$$

The elementary paths which are equal as 1-morphisms are connected by the signs =. They correspond to 6 mutations of type (8.9) and are in bijection with the square 2-faces of $\mathcal{P}_4$. If we identify the paths related by =, we are left with the set

$$Hom^{(1)}(e, w_0) = \overline{\mathrm{Red}(w_0)}\,,$$

which contains 8 elements.

Let us number the elements of $\overline{\mathrm{Red}(w_0)}$ as $\ell_c,\ c \in \mathbb{Z}/8\mathbb{Z}$. These elements are connected by 8 mutations of type (8.8), which are geometrically given by hexagons:

$$\begin{array}{ccccccccc} \ell_0 & \xrightarrow{h_1(1)} & \ell_1 & \xrightarrow{h_2(3)} & \ell_2 & \xrightarrow{h_1(3)} & \ell_3 & \xrightarrow{h_2(1)} & \ell_4 \\ || & & & & & & & & || \\ \ell_0 & \xrightarrow{h_1(4)} & \ell_{-1} & \xrightarrow{h_2(2)} & \ell_{-2} & \xrightarrow{h_1(2)} & \ell_{-3} & \xrightarrow{h_2(4)} & \ell_{-4} \end{array} . \tag{8.11}$$

Thus, the 2-nd Bruhat order on the set $\overline{\mathrm{Red}(w_0)}$ converts it to an octagon.

Finally we have one 3-morphism (homotopy)

$$k:\ h_2(1)h_1(3)h_2(3)h_1(1) \longrightarrow h_2(4)h_1(2)h_2(2)h_1(4)$$

which corresponds to the single 3-cell of $\mathcal{P}_4$, its body.

8.2.2. *Representation of the* 3-*category.* The Whitney - Lusztig data give rise to a 2-*representation* of the 3-category $\mathcal{S}_4$. It is visualized on the permutohedron $\mathcal{P}_4$ as follows.

Our base field will be a purely transcendental extension of $\mathbb{C}$,

$$\mathbb{F} = \mathbb{C}(\boldsymbol{a}) = \mathbb{C}(a_1, a_2, a_3, a_4, a_5, a_6) \cong \mathbb{C}(N_4).$$

At each vertex we put the based $\mathbb{F}$-vector space $V = \mathbb{F}^6$. At the edges we put the elementary matrices $e_i(a_j) \in N_3(\mathbb{F}),\ 1 \leqslant i \leqslant 3, 1 \leqslant j \leqslant 6$. For example, at the 3 edges going down from e

we put the matrices $e_1(a_1), e_2(a_1), e_3(a_1)$, and to the last three edges coming to w_0 we put the matrices $e_1(a_6), e_2(a_6), e_3(a_6)$. To any path we assign the product of the corresponding matrices.

For example

$$\rho(\ell_1) = e_1(a_6)e_2(a_5)e_3(a_4)e_1(a_3)e_2(a_2)e_1(a_1),$$

whereas

$$\rho(\ell_2) = e_3(a_6)e_2(a_5)e_1(a_4)e_3(a_3)e_2(a_2)e_3(a_1),$$

On 2-faces of $\mathcal{P}_4$ we put certain automorphisms of the base field $\mathbb{F}$.

Namely, let us introduce involutive operators $L(i) : \mathbb{F} \xrightarrow{\sim} \mathbb{F},\ 1 \leqslant i \leqslant 5$, by

$$\begin{aligned} L(i)(a_i) &= a_{i+1}, \qquad L(i)(a_{i+1}) = a_i, \\ L(i)(a_k) &= a_k \qquad \text{if } k \neq i, i+1, \end{aligned} \tag{8.12}$$

and $R(j) :\ \mathbb{F} \xrightarrow{\sim} \mathbb{F},\ 1 \leqslant j \leqslant 4$, given by

$$\begin{aligned} R(j)(a_j) &= a_{j+1}a_{j+2}/(a_j + a_{j+2}), \\ R(j)(a_{j+1}) &= a_j + a_{j+2}, \\ R(j)(a_{j+2}) &= a_j a_{j+1}/(a_j + a_{j+2}), \\ R(j)(a_k) &= a_k \quad \text{if } k \neq i, i+1, \end{aligned} \tag{8.13}$$

cf. formula (1.3).

On eight hexagons (resp. on six squares) we put the involutions R (resp. L) according to the picture:

(8.14)

$$\begin{array}{ccccccccc} \mathbb{F}(212321) & \xrightarrow{R(3)} & \mathbb{F}(213231) & \xrightarrow{L(2)} & \mathbb{F}(231231) & \xrightarrow{L(5)} & \mathbb{F}(231213) & \xrightarrow{R(3)} & \mathbb{F}(232123) \\ R(1)\uparrow & & & & & & & & \downarrow R(1) \\ \mathbb{F}(121321) & & & & & & & & \mathbb{F}(323123) \\ L(3)\downarrow & & & & & & & & \downarrow L(3) \\ \mathbb{F}(123121) & & & & & & & & \mathbb{F}(321323) \\ R(4)\downarrow & & & & & & & & \uparrow R(4) \\ \mathbb{F}(123212) & \xrightarrow{R(2)} & \mathbb{F}(132312) & \xrightarrow{L(4)} & \mathbb{F}(132132) & \xrightarrow{L(1)} & \mathbb{F}(312132) & \xrightarrow{R(2)} & \mathbb{F}(321232) \end{array}.$$

Here $\mathbb{F}(\boldsymbol{h}), \boldsymbol{h} \in \mathrm{Red}(w_0)$, is a copy of the field $\mathbb{F}$.

This way we have associated to any path $\ell = \ell(\boldsymbol{h})$, $\boldsymbol{h} \in \mathrm{Red}(w_0)$, an upper triangular matrix $\rho(\ell) \in N(\mathbb{F})$, and to every homotopy $\ell \xrightarrow{h} \ell'$ an automorphism

$$R(h) :\ \mathbb{F} \xrightarrow{\sim} \mathbb{F}$$

such that

$$\rho(\ell') = R(h)\{\rho(\ell)\} \tag{8.15}$$

Theorem 8.1. *The diagram* (8.14) *is commutative, i.e.,*

$$L(3)R(1)R(3)L(2)L(5)R(3)R(1) = R(4)R(2)L(1)L(4)R(2)R(4)L(3)\,.$$

8.2.3. *Eightfold Way.* We can rewrite this assertion as follows.

Define 8 automorphisms (involutions or compositions of two involutions):

$$\begin{aligned} R_1(1) &= R(1), & R_2(1) &= L(3)R(1),\\ R_1(2) &= L(4)R(2), & R_2(2) &= R(2)L(1),\\ R_1(3) &= L(5)R(3), & R_2(3) &= R(3)\boldsymbol{L}(2),\\ R_1(4) &= R(4)L(3), & R_2(4) &= R(4). \end{aligned}$$

Then we have a *tetrahedron equation*

$$R_2(1)R_2(3)R_1(3)R_1(1) = R_2(4)R_2(2)R_1(2)R_1(4)\,. \tag{8.16}$$

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