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# THE GENERIC RANK OF THE BAUM-BOTT MAP FOR DEGREE-2 FOLIATIONS ON EVEN-DIMENSIONAL PROJECTIVE SPACES

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Abstract. The Baum-Bott map associates to a one-dimensional foliation on a complex manifold its Baum-Bott indexes at each singular point. The generic rank of this map on the space of foliations on the projective plane is known. In this work, we give an upper bound of the generic rank of the map for higher-dimensional projective spaces. We also determine the generic rank for degree-2 foliations on higher even-dimensional projective spaces. Additionally, we study the rank at the Jouanolou foliation.

## 1. Introduction

A one-dimensional holomorphic foliation on a complex manifold is given by a section of the tensor product of the tangent bundle and a line bundle. Considering foliations with only isolated singularities, Baum and Bott [2] showed that the Chern classes of the tensor product can be calculated by some local invariants of the foliation around its singularities. These local invariants are the Baum-Bott indexes of the holomorphic foliation. Therefore, if we know the Baum-Bott indexes of a holomorphic foliation, we can know the Chern classes of the tensor product. Fixing a line bundle, there are some universal relations among the Baum-Bott indexes, because each relation does not depend on the chosen foliation. A general problem is to determine the number of universal relations among these indexes.

For foliations on the projective plane $\mathbb{P}^2$ with fixed degree greather than 1, from [2] we obtain one universal relation among the Baum-Bott indexes. Gómez-Mont and Luengo asked in [8] if there are other hidden relations. Lins-Neto and Pereira [11] gave an answer. They defined the Baum-Bott map for foliations on $\mathbb{P}^2$, which associates to a foliation with only isolated singularities its Baum-Bott indexes. The question is equivalent to finding the generic rank of this map. They found the generic rank of the map. As a consequense, they showed that the unique relation among these indexes is the one given in [2]. They also determined the rank of the local Baum-Bott map at the Jouanolou foliation. Later, Lins-Neto [10] studied the generic fiber of this map for degree-2 foliations. In this case, the dimension of a generic fiber coincides with the dimension of the automorphism group $\mathrm{Aut}(\mathbb{P}^2)$. A generic fiber is a finite union of orbits of the action of $\mathrm{Aut}(\mathbb{P}^2)$ on the space of degree-2 foliations. He found the exact number of orbits. A similar problem for quadric vector fields on $\mathbb{C}^2$ is to determine the universal relations between the spectra of finite singularities and the characteristic numbers at infinity for a generic quadratic vector field, which has been studied by Ramirez in [13].

The generic rank of the Baum-Bott map for foliations on the projective space $\mathbb{P}^3$ is computed in [9], but only for some specific degrees.

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We extend the known results given for foliations on $\mathbb{P}^2$ and $\mathbb{P}^3$. We consider the Baum-Bott map on the space of foliations on the projective space $\mathbb{P}^n$ with fixed degree $d$. We give un upper bound for the generic rank of the map in terms of $n$ and $d$. We compute the rank of the local Baum-Bott map at the Jouanolou foliation on $\mathbb{P}^n$. This give us a lower bound for the generic rank. For $n$ even and $d = 2$, we use this foliation to prove that the generic rank is the given upper bound. In those cases, the generic fiber of the map is a finite union of orbits of the action of $\mathrm{Aut}(\mathbb{P}^n)$ on the space of foliations, and we determine the dimension of the generic fiber.

The paper is organized as follows. In Section 2 we define the Baum-Bott indexes for one-dimensional holomorphic foliations on a complex compact manifold. Next, we determine the Baum-Bott indexes of the Jouanolou foliation and finally, we estimate an upper bound for the generic rank of the Baum-Bott map for foliations on the projective space.

Section 3 is devoted to the study of how the space of vector fields on the complex space, which generate foliations on the projective space, is decomposed in terms of the automorphisms of the Jouanolou foliation. This will be important for computing the rank of the local Baum-Bott map at this foliation.

In the last section, we prove that the rank of the Baum-Bott map at the Jouanolou foliation can be given in terms of $n$, $d$ and some linear transformations. From this, we obtain the generic rank for degree-2 foliations on even-dimensional projective spaces. We also explicitly calculate its rank at the Jouanolou foliation on $\mathbb{P}^3$.

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## 2. Definitions and Results

**2.1. Foliation and Baum-Bott indexes.** Let $M$ be a complex manifold. A one-dimensional singular holomorphic foliation on $M$ is locally defined by a covering $(U_\alpha)_{\alpha\in A}$ of $M$ by open subsets, a collection $(X_\alpha)_{\alpha\in A}$ of holomorphic vector fields, where each $X_\alpha$ is defined on $U_\alpha$, and a multiplicative cocycle $(f_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\emptyset}$, such that $X_\alpha = f_{\alpha\beta}X_\beta$ on $U_\alpha \cap U_\beta \neq 0$. The cocycle $(f_{\alpha\beta})$ induces a holomorphic line bundle on $M$. This line bundle we denote it by $T^*_{\mathcal{F}}$ and we call it the cotangent bundle of $\mathcal{F}$. And $T_{\mathcal{F}} = (T^*_{\mathcal{F}})^*$ is the tangent bundle of $\mathcal{F}$. A point $p \in U_\alpha$ is a singular point of $\mathcal{F}$ if $X_\alpha(p) = 0$. The singular set of $\mathcal{F}$ is the set of singular points of $\mathcal{F}$ and we denote it by $\mathrm{Sing}(\mathcal{F})$. A singular point $p \in U_\alpha$ is non-degenerate if $\det DX_\alpha(p) \neq 0$.

Analogously, a codimension 1 singular holomorphic foliation $\mathcal{F}$ on $M$ can be defined locally by a covering $(U_\alpha)_{\alpha\in A}$ of $M$ by open sets, a collection of holomorphic 1-forms $(\omega_\alpha)_{\alpha\in A}$, where each $\omega_\alpha$ is defined on $U_\alpha$, satisfying $d\omega_\alpha \wedge \omega_\alpha = 0$, and a multiplicative cocycle $(g_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\emptyset}$ such that $\omega_\alpha = g_{\alpha\beta}\omega_\beta$. The cocycle $(g_{\alpha\beta})$ induces a holomorphic line bundle on $M$, the normal bundle of $\mathcal{F}$, we denote it by $N\mathcal{F}$. A point $p \in U_\alpha$ is a singular point of $\mathcal{F}$ if $w_\alpha(p) = 0$. We denote by $\mathrm{Sing}(\mathcal{F})$ the set of singular points of $\mathcal{F}$.

Let $M$ be a compact complex manifold of dimension $n$. Let us denote by $TM$ the holomorphic tangent bundle of $M$. We write $c_i$ for the $i$-th Chern class of a vector bundle on $M$. Let $C_i$ be the $i$-th elementary symmetric function of the eigenvalues of an $n \times n$ matrix and for any multi-index $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n_{\geq 0}$ set $C^\nu = C_1^{\nu_1} C_2^{\nu_2} \dots C_n^{\nu_n}$, see [15].

Baum and Bott [2] proved that the Chern classes of the tensor product of the tangent bundle with any holomorphic line bundle on $M$ are the sum of suitable indexes, which are called the Baum-Bott indexes:

**Theorem 2.1** ([2, 14])**.** *Let $L$ be a holomorphic line bundle on $M$ and $\xi$ be a holomorphic section of $TM \otimes L$ with isolated zeros. Consider the Chern classes:*

$$c^\nu(TM \otimes L) = c_1^{\nu_1}(TM \otimes L) \dots c_n^{\nu_n}(TM \otimes L),$$

$$\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n_{\geq 0} \text{ and } \nu_1 + 2\nu_2 + \dots + n\nu_n = n.$$

*Then*

$$\int_M c^\nu(TM \otimes L) = \sum_{p:\ \xi(p)=0} \operatorname{Res}_p \left\{ \frac{C^\nu(J\xi) dz_1 \wedge \dots \wedge dz_n}{\xi_1 \dots \xi_n} \right\},$$

*where* $J\xi = \left( \dfrac{\partial \xi_i}{\partial z_j} \right)$ *is the Jacobian matrix and the Grothendieck residue symbol* $\operatorname{Res}_p \left\{ \dfrac{C^\nu(J\xi) dz_1 \wedge \dots \wedge dz_n}{\xi_1 \dots \xi_n} \right\}$ *is the Baum-Bott index $C^\nu$ of $\xi$ at $p$.*

Regarding one-dimensional foliations on $M$ as sections of $TM \otimes L$, where $L$ is a holomorphic line bundle on $M$, allows us to apply the Baum-Bott theorem to foliations with only isolated singularities.

In what follows the set of foliations on $M$ with cotangent bundle $T^*_{\mathcal{F}} = L$ will be denoted by $\mathcal{F}\mathrm{ol}(M, L) := \mathbb{P}H^0(M, TM \otimes L)$. Let

$$\mathcal{F}\mathrm{ol}_{\mathrm{red}}(M, L) := \{\mathcal{F} \in \mathcal{F}\mathrm{ol}(M, L) \mid \text{ all the singularities of } \mathcal{F} \text{ are non-degenerate}\}.$$

It is known that all the foliations in $\mathcal{F}\mathrm{ol}_{\mathrm{red}}(M, L)$ have the same number of singularities

$$N = N(L) = c_n(TM \otimes L).$$

Moreover, there is an explicit formula for the Baum-Bott indexes at non-degenerate singular points in terms of the eigenvalues of the linear part of a germ of a vector field. Indeeed, let $p(\mathcal{F})$ be a non-degenerate singularity of a foliation $\mathcal{F}$ on $M$ and $X_{\mathcal{F}}$ be a germ of vector field which defines $\mathcal{F}$ around $p(\mathcal{F})$. Let $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}^n_{\geq 0}$ with $\nu_1 + 2\nu_2 + \dots + n\nu_n = n$. Then the Baum-Bott index $C^\nu$ of $\mathcal{F}$ at $p(\mathcal{F})$ is

$$BB_\nu(\mathcal{F}, p(\mathcal{F})) = \frac{C^\nu(DX_{\mathcal{F}}(p(\mathcal{F})))}{\det(DX_{\mathcal{F}}(p(\mathcal{F}))}.$$

**Remark 2.2.** The projectivization of the $n$-tuple of eigenvalues of $DX_{\mathcal{F}}(p(\mathcal{F}))$ allows us to compute the Baum-Bott indexes of $\mathcal{F}$ at $p(\mathcal{F})$. Conversely, if $\mathcal{F}$ is sufficiently generic, then the projectivization of the $n$-tuple of eigenvalues associated to $\mathcal{F}$ at $p(\mathcal{F})$ can be calculated in terms of

$$BB_1(\mathcal{F}, p(\mathcal{F})), \dots, BB_{n-1}(\mathcal{F}, p(\mathcal{F})),$$

where

$$BB_1 = BB_{(n,0,0,\dots,0)}, \qquad BB_2 = BB_{(n-2,1,0,\dots,0)}, \dots,$$
$$BB_{n-2} = BB_{(2,0,\dots,0,1,0,0)}, \qquad BB_{n-1} = BB_{(1,0,\dots,0,1,0)}.$$

From now on we focus on the case where $M$ is the projective space. A one-dimensional foliation on the projective space $\mathbb{P}^n$ has cotangent bundle $\mathcal{O}(d-1)$, for some non-negative integer $d$. This number is called the degree of the foliation. The set of degree-$d$ foliations on $\mathbb{P}^n$ will be denoted by

$$\mathcal{F}\mathrm{ol}(n, d) := \mathbb{P}H^0(\mathbb{P}^n, T\mathbb{P}^n \otimes \mathcal{O}(d-1)).$$

A foliation in $\mathcal{F}\mathrm{ol}(n,d)$ can be defined in an affine coordinate system $(\mathbb{C}^n,(x_1,\dots,x_n))$ by a polynomial vector field of the form

$$X=\sum_{j=1}^{n} Q_j(x)\partial_j+G\mathcal{R},$$

where $Q_1,\dots,Q_n$ are polynomials of degree less than or equal to $d$, $G$ is a homogeneous polynomial of degree $d$, and $\mathcal{R}=\sum_{j=1}^{n} x_j\partial_j$ is the radial vector field (see [15, Theorem 6.4.1]). If $n\geq 2$ and $d\geq 1$, then we have

$$\dim \mathcal{F}\mathrm{ol}(n,d)=(n+1)\binom{n+d}{n}-\binom{n+d-1}{n}-1.$$

We denote by

$$\mathcal{F}\mathrm{ol}_{\mathrm{red}}(n,d):=\{\mathcal{F}\in\mathcal{F}\mathrm{ol}(n,d)\mid \text{ all the singularities of } \mathcal{F} \text{ are non-degenerate}\}.$$

**Remark 2.3.** Note that $\mathcal{F}\mathrm{ol}_{\mathrm{red}}(n,d)$ is a Zariski open and dense subset of $\mathcal{F}\mathrm{ol}(n,d)$. Moreover, if a foliation $\mathcal{F}_0\in\mathcal{F}\mathrm{ol}_{\mathrm{red}}(n,d)$ then it has exactly $N=d^n+d^{n-1}+\ldots+d+1$ singularities.

**2.2. The Baum-Bott indexes of the Jouanolou foliation.** In this section we explicitly calculate the Baum-Bott indexes of the Jouanolou foliation.

Let $d\geq 2$. The degree-$d$ Jouanolou foliation $\mathcal{J}_d$ on $\mathbb{P}^n$ can be defined in homogeneous coordinates by the radial vector field in $\mathbb{C}^{n+1}$ and the homogeneous vector field

$$\mathbb{X}_{\mathcal{J}_d}=(x_2^d,x_3^d,\dots,x_{n+1}^d,x_1^d).$$

The Jouanolou foliation is defined in the affine coordinates $(x_1,\dots,x_n,1)$ by the vector field

$$X_{\mathcal{J}_d}=\sum_{i=1}^{n-1}(x_{i+1}^d-x_ix_1^d)\partial_i+(1-x_nx_1^d)\partial_n,$$

see [12].

Consider $\xi$ a primitive $N$-th root of unity. Let $\mathbf{A}:\mathbb{C}^n\to\mathbb{C}^n$ be the linear transformation given by

$$\mathbf{A}(x_1,\dots,x_n)=(\alpha_1x_1,\dots,\alpha_{n-1}x_{n-1},\alpha_nx_n),$$

where $\alpha_j=\xi^{-(d^{n+1-j}+d^{n-j}+\ldots+d)}$ for $j=1,\dots,n$. Observe that $p_i=\mathbf{A}^{i-1}(1,\dots,1)$ for $i=1,\dots,N$ give all the singular points of the foliation $\mathcal{J}_d$.

Let us calculate the Baum-Bott indexes of the Jouanolou foliation. Since $\mathbf{A}^*(\mathrm{X}_{\mathcal{J}_d})=\xi^d\mathrm{X}_{\mathcal{J}_d}$, we have:

$$BB_j(\mathrm{X}_{\mathcal{J}_d},p_i)=BB_j(\mathbf{A}^*(\mathrm{X}_{\mathcal{J}_d}),\mathbf{A}^{-1}(p_i))=BB_j(\mathrm{X}_{\mathcal{J}_d},p_1).$$

Then, it is enough to calculate the indexes at $(1,\dots,1)\in\mathbb{C}^n$. We now must find the elementary symmetric functions of the eigenvalues of the matrix $DX_{\mathcal{J}_d}(1,\dots,1)$. We have that

$$DX_{\mathcal{J}_d}(1,\dots,1)=-\mathrm{Id}+dB,$$

where

$$B=\begin{bmatrix} -1 & 1 & 0 & \dots & 0 & 0\\ -1 & 0 & 1 & \dots & 0 & 0\\ -1 & 0 & 0 & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ -1 & 0 & 0 & \dots & 0 & 1\\ -1 & 0 & 0 & \dots & 0 & 0\end{bmatrix}.$$

Observe that $B^{n+1} = \mathrm{Id}$ and its characteristic polynomial is

$$P_B(\lambda) = \lambda^n + \lambda^{n-1} + \ldots + 1.$$

Then the characteristic polynomial of $DX_{\mathcal{J}_d}(1,\ldots,1)$ is

$$\det(\lambda I + DX_{\mathcal{J}_d}(1,\ldots,1)) = \sum_{j=0}^{n} C_{n-j}(DX_{\mathcal{J}_d}(1,\ldots,1))\lambda^j = \sum_{j=0}^{n} d^{n-j}(\lambda+1)^j.$$

It follows that

$$C_j(DX_{\mathcal{J}_d}(1,\ldots,1)) = (-1)^j \sum_{k=0}^{j} \binom{n-j+k}{k} d^{j-k}. \tag{2.1}$$

In particular, the Baum-Bott indexes at $p_i$ are determined by

$$\begin{aligned} BB_j(\mathcal{J}_d, p_i) =& \frac{C_1^{n-j}(DX_{\mathcal{J}_d}(1,\ldots,1))C_j(DX_{\mathcal{J}_d}(1,\ldots,1))}{C_n(DX_{\mathcal{J}_d}(1,\ldots,1))} \\ =& \frac{(d+n)^{n-j}\left(\sum\limits_{k=0}^{j}\binom{n-j+k}{k}d^{j-k}\right)}{d^n + d^{n-1} + \ldots + d + 1}, \text{ for } j = 1,\ldots,n-1. \end{aligned}$$

Applying Theorem 2.1 we obtain the following proposition.

**Proposition 2.4.** *Let $\mathcal{F} \in \mathcal{F}\mathrm{ol}(n,d)$ with only isolated singularities. If $d \geq 2$, then*

$$\sum_{p(\mathcal{F})\in \mathrm{Sing}(\mathcal{F})} BB_j(\mathcal{F}, p(\mathcal{F})) = (d+n)^{n-j}\left(\sum_{k=0}^{j}\binom{n-j+k}{k}d^{j-k}\right).$$

**2.3. The Baum-Bott map.** We associate to each foliation its Baum-Bott indexes in the following way. Set

$$\Delta = \left\{(\nu_1,\ldots,\nu_n) \in \mathbb{Z}^n_{\geq 0} \mid \ \nu_1 + 2\nu_2 + \ldots + n\nu_n = n \text{ and } \nu_n \neq 1\right\} = \{N_1,\ldots,N_k\}.$$

Let $\mathcal{F}_0 \in \mathcal{F}\mathrm{ol}_{\mathrm{red}}(M,L)$ with singular set $\mathrm{Sing}(\mathcal{F}_0) = \{p_1^0,\ldots,p_N^0\}$. Then there is an open neighborhood $U \subset \mathcal{F}\mathrm{ol}_{\mathrm{red}}(M,L)$ of $\mathcal{F}_0$, and there are holomorphic maps $\gamma_1,\ldots,\gamma_N : U \to M$ such that $\mathrm{Sing}(\mathcal{F}) = \{\gamma_1(\mathcal{F}),\ldots,\gamma_N(\mathcal{F})\}$ and $\gamma_j(\mathcal{F}_0) = p_j^0$, for $j = 1,\ldots,N$. The *local Baum-Bott map* $BB : U \to (\mathbb{C}^k)^N$ is defined by:

$$\begin{aligned} \mathcal{F} \mapsto (& BB_{N_1}(\mathcal{F},\gamma_1(\mathcal{F})),\ldots,BB_{N_k}(\mathcal{F},\gamma_1(\mathcal{F})),\ldots, \\ & BB_{N_1}(\mathcal{F},\gamma_N(\mathcal{F})),\ldots,BB_{N_k}(\mathcal{F},\gamma_N(\mathcal{F}))). \end{aligned}$$

We extend the domain of the local Baum-Bott map to $\mathcal{F}\mathrm{ol}_{\mathrm{red}}(M,L)$ by symmetry. More specifically, let $S_N$ be the group of permutations of $N$ elements. We denote by $(\mathbb{C}^k)^N/S_N$ the quotient of $(\mathbb{C}^k)^N$ by the equivalence relation which identifies the points $(z_1,\ldots,z_N)$ and $(z_{\sigma(1)},\ldots,z_{\sigma(N)})$, where $z_i \in \mathbb{C}^k$ and $\sigma \in S_N$. In this way we have the map $BB : \mathcal{F}\mathrm{ol}_{\mathrm{red}}(M,L) \to (\mathbb{C}^k)^N/S_N$:

$$\begin{aligned} \mathcal{F} \mapsto [& BB_{N_1}(\mathcal{F},\gamma_1(\mathcal{F})),\ldots,BB_{N_k}(\mathcal{F},\gamma_1(\mathcal{F})),\ldots, \\ & BB_{N_1}(\mathcal{F},\gamma_N(\mathcal{F})),\ldots,BB_{N_k}(\mathcal{F},\gamma_N(\mathcal{F}))], \end{aligned}$$

where $[z_1,\ldots,z_N]$ denotes the class of $(z_1,\ldots,z_N)$ in $(\mathbb{C}^k)^N/S_N$.

The *global Baum-Bott map* $BB : \mathcal{F}\mathrm{ol}(M,L) \dashrightarrow (\mathbb{P}^k)^N/S_N$ is the rational map which extends the Baum-Bott map given above.

**2.4. An upper bound for the generic rank of the Baum-Bott map.** We see that the Baum-Bott map is not dominant, which is clear from Theorem 2.1.

In the case of compact surfaces, the Baum-Bott Theorem 2.1 give us an explicit relation between the indexes. Let $\mathcal{F}$ be a holomorphic foliation on a compact complex surface $M$, and let $N_{\mathcal{F}}$ denote the normal bundle of $\mathcal{F}$.

**Theorem 2.5** ([1, 3])**.** *Let $M$ be a compact complex surface. If the foliation $\mathcal{F}$ has only isolated singularities, then*

$$\sum_{p\in \operatorname{Sing}(\mathcal{F})} BB(\mathcal{F},p) = N_{\mathcal{F}}.N_{\mathcal{F}}.$$

*Moreover, if $\mathcal{F}$ is a degree-d foliation on $\mathbb{P}^2$, then we get*

$$\sum_{p\in \operatorname{Sing}(\mathcal{F})} BB(\mathcal{F},p) = (d+2)^2.$$

In the above statement the symbol $BB(\mathcal{F},p)$ corresponds to the Baum-Bott index associated to $C_1^2$ of $\mathcal{F}$ at $p$.

Lins-Neto and Pereira [11] proved the following theorem.

**Theorem 2.6** ([11, Theorem 1])**.** *Let $d \geq 2$. Then the maximal rank of the Baum-Bott map for degree-d foliations on $\mathbb{P}^2$ is $d^2+d$. In particular, the dimension of a generic fiber of the map is $3d+2$.*

This means that the only relation between the Baum-Bott indexes for foliations on $\mathbb{P}^2$ is the Baum-Bott relation given in Theorem 2.5.

Similarly, we can ask if there are other hidden relations between the Baum-Bott indexes of a degree-d foliation on $\mathbb{P}^n$. The question translates into finding the generic rank of the Baum-Bott map. We denote by $\operatorname{gr}(n,d)$ the generic rank of the Baum-Bott map on $\mathcal{F}\mathrm{ol}(n,d)$. In particular, $\operatorname{gr}(n,0)=0$. Indeed, any foliation of degree zero has only one singularity and in some affine coordinate system $\mathbb{C}^n \subset \mathbb{P}^n$ the foliation is defined by the radial vector field. For degree greather than zero we have the following proposition.

**Proposition 2.7.** *Let $d \geq 2$. Then:*

- (a) $\operatorname{gr}(n,1) = n-1$*;*
- (b) $\operatorname{gr}(2,d) = d^2+d$*;*
- (c) *if $n \geq 3$, then* $\operatorname{gr}(n,d) \leq \dim \mathcal{F}\mathrm{ol}(n,d) - \dim \operatorname{Aut}(\mathbb{P}^n)$*.*

*Proof.* We will find two upper bounds of $\operatorname{gr}(n,d)$. The minimum of these will imply $(c)$. Finally, we will see that for $(a)$ and $(b)$ the minimum is sharp.

We claim that $\operatorname{gr}(n,d) \leq (N-1)(n-1)$. Indeed, the Baum-Bott indexes of a generic foliation depend on $BB_1,\dots,BB_{n-1}$. By Theorem 2.1 we have at least $(n-1)$ relations among these indexes. Therefore

$$\operatorname{gr}(n,d) \leq (n-1)N-(n-1) = (N-1)(n-1). \tag{2.2}$$

We now estimate the second upper bound. Note that the Baum-Bott map is invariant by the action of the automorphism group $\operatorname{Aut}(\mathbb{P}^n)$. Let $\mathcal{O}\mathrm{rb}(\mathcal{F})$ be the orbit of the foliation $\mathcal{F}$ and $\operatorname{Stab}(\mathcal{F})$ be its stabilizer. Considering a generic foliation $\mathcal{F}$ and a generic fiber $F$ of the Baum-Bott map and applying the fiber dimension theorem leads to

$$\dim \mathcal{O}\mathrm{rb}(\mathcal{F}) = \dim \frac{\operatorname{Aut}(\mathbb{P}^n)}{\operatorname{Stab}(\mathcal{F})} \leq \dim F = \dim \mathcal{F}\mathrm{ol}(n,d) - \operatorname{gr}(n,d). \tag{2.3}$$

In order to get the upper bound, it is necessary to compute $\dim \operatorname{Stab}(\mathcal{F})$. First, consider $d\geq 2$. We know that an automorphism is uniquely determined by $n+2$ points in generic position.

The singular set of a generic foliation has more than $n+2$ points and it is invariant by every automorphism in the stabilizer of the foliation. Hence the stabilizer contains only the identity automorphism and (2.3) becomes

$$\mathrm{gr}(n,d) \leq \dim \mathcal{F}\mathrm{ol}(n,d) - \dim \mathrm{Aut}(\mathbb{P}^n). \tag{2.4}$$

Consider $d=1$. A generic foliation is generated by a homogeneous vector field in $\mathbb{C}^{n+1}$ of the form:

$$X = \lambda_1 x_1 \partial_1 + \ldots + \lambda_{n+1} x_{n+1} \partial_{n+1}, \text{ where } \lambda_1, \ldots, \lambda_{n+1} \in \mathbb{C}^*. \tag{2.5}$$

Let $A$ be an $(n+1)\times(n+1)$ matrix that determines an automorphism in the stabilizer of the foliation. Then $A^{-1}\cdot X \circ A = \mu X$, for some $\mu \in \mathbb{C}^*$. Since the foliation is generic, we can choose $\lambda_1, \ldots, \lambda_{n+1}$ to be generic. It follows that $A$ is a diagonal matrix. Replacing in (2.3) we obtain

$$\mathrm{gr}(n,1) \leq n-1. \tag{2.6}$$

Let us conclude the proof. For $d \geq 2$ we take the minimum to the right sides of (2.2) and (2.4). If $n=2$, then the minimum is $d^2+d$, and by Theorem 2.6 we obtain $(b)$. If $n \geq 3$, then we get $(c)$. For $d=1$ the minimum of the right sides of (2.2) and (2.6) is $n-1$. Showing that there is a foliation such that its rank at the Baum-Bott map is $n-1$ yields $\mathrm{gr}(n,1)=n-1$. Since the map

$$[\lambda = (\lambda_1, \ldots, \lambda_n)]_{\mathbb{P}} \mapsto [C_1^n(\lambda), \ldots, C_1^{n-i}(\lambda)C_i(\lambda), \ldots, C_n(\lambda)]_{\mathbb{P}},$$

defined on some open subset of $\mathbb{P}^{n-1}$, is a bijection and a generic foliation is generated by a vector field of the form (2.5), we obtain $(a)$. $\square$

In particular, we have:

**Corollary 2.8.** *For degree-1 foliations on $\mathbb{P}^n$, a generic fiber of the Baum-Bott map has dimension $n(n+1)$. It is a finite union of orbits of the action of $\mathrm{Aut}(\mathbb{P}^n)$ on $\mathcal{F}\mathrm{ol}(n,1)$.*

**Remark 2.9.** For $d=2,\ldots,9$ and $n=3$, the upper bound given in $(c)$ is sharp; see [9, Theorem 2.3.1].

**Remark 2.10.** We have $\dim \mathcal{F}\mathrm{ol}(2,2) - \dim \mathrm{Aut}(\mathbb{P}^2) = \mathrm{gr}(2,2)$.

**Remark 2.11.** Since $\mathrm{gr}(2,1)=1$, it follows that the Camacho-Sad index over an invariant line gives the extra condition among the Baum-Bott indexes, see [4, p. 592].

**Remark 2.12.** Let $n \geq 2$ and $d \geq 2$. The dimensions of a generic fiber of the Baum-Bott map and $\mathrm{Aut}(\mathbb{P}^n)$ coincide if and only if

$$\mathrm{gr}(n,d) = \dim \mathcal{F}\mathrm{ol}(n,d) - \dim \mathrm{Aut}(\mathbb{P}^n).$$

**2.5. Statements of main results.** For simplicity of notation, let $BB$ stand for the Baum-Bott map restricted to the Baum-Bott indexes $BB_1, \ldots, BB_{n-1}$.

In general, finding the rank of the Baum-Bott map at a random foliation is difficult. Lins-Neto and Pereira [11] calculate the rank of the Baum-Bott map at the Jouanolou foliation on $\mathbb{P}^2$.

**Theorem 2.13** ([11, Theorem 2])**.** *For any $d \geq 2$, the rank of the local Baum-Bott map at $\mathcal{J}_d$ on $\mathbb{P}^2$ is*

$$\frac{d^2+7d-6}{2}.$$

*In particular, if $d=2,3$, then* $\mathrm{rank}(BB, \mathcal{J}_d) = d^2+d$, *and if $d \geq 4$, then* $\mathrm{rank}(BB, \mathcal{J}_d) < d^2+d$.

For higher dimensional projective spaces, we compute the rank at the Jouanolou foliation in terms of some linear transformations. Let $J = \{j_1, \ldots, j_r\}$ be an ordered set. Let $V_j$, for $j \in J$, be vectors of same dimension. We denote $[V_j]_{j\in J} = [V_{j_1} \ldots V_{j_r}]$, the matrix whose ordered column vectors are $V_{j_1}, \ldots, V_{j_r}$. Let $n, d \in \mathbb{Z}$, and $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$, we define the linear transformation $M_{d,I} : \mathbb{C}^n \to \mathbb{C}^{n-1}$ given by the matrix:

$$M_{d,I} = \begin{bmatrix} (i_{j+1} - i_j)d + (i_{j+2} - i_{j+1})(d+1) \\ i_{j+3} - i_{j+2} \\ \vdots \\ i_{j+n} - i_{j+n-1} \end{bmatrix}_{1\leq j\leq n},$$

where $i_{n+1} = d - 1 - (i_1 + \ldots + i_n)$. We are identifying $i_j = i_{j \mod (n+1)}$. Consider the $\mathbb{C}$-vector space

$$\mathbf{V}_d := H^0(\mathbb{P}^n, T\mathbb{P}^n \otimes \mathcal{O}_{\mathbb{P}^n}(d-1)).$$

**Theorem 2.14.** *Let $n \geq 3$ and $d \geq 2$. The rank of the local Baum-Bott map $BB : \mathcal{F}\mathrm{ol}_{red}(n,d) \to (\mathbb{C}^{n-1})^N$ at the degree-$d$ Jouanolou foliation is*

$$\dim \mathbf{V}_d - n(n+1) - \sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|\leq d-1}} \dim\ker(M_{d,I}) \qquad , \textit{if } n \textit{ is even;}$$

$$\dim \mathbf{V}_d - n(n+1) - \sum_{\substack{I\in\mathbb{Z}^n_{\geq 0}\\ |I|\leq d-1}} \dim\ker(M_{d,I}) - \frac{(n+1)}{2} \quad , \textit{if } n \textit{ is odd.}$$

Observe that the Jouanolou foliation give us a lower bound for the generic rank of the Baum-Bott map.

For degree-2 foliations on $\mathbb{P}^n$, with $n$ even, we obtain:

**Theorem 2.15.** *Let $n \geq 2$ be an even number. The generic rank of the Baum-Bott map for degree-2 foliations on $\mathbb{P}^n$ is*

$$\dim \mathcal{F}\mathrm{ol}(n,2) - \dim \mathrm{Aut}(\mathbb{P}^n) = (n-1)\frac{(n+1)(n+2)}{2}.$$

As a consequence, we get

**Corollary 2.16.** *Let $n \geq 2$ be an even number. A generic fiber of the global Baum-Bott map defined on $\mathcal{F}\mathrm{ol}(n,2)$ has dimension $\dim \mathrm{Aut}(\mathbb{P}^n)$. The generic fiber is a finite union of orbits of the action of $\mathrm{Aut}(\mathbb{P}^n)$ on $\mathcal{F}\mathrm{ol}(n,2)$.*

In the other cases, the rank at the Jouanolou foliation is stricly less than the upper bound given in Proposition 2.7.

**Proposition 2.17.** *If $n \geq 3$, then the rank of the local Baum-Bott map*

$$BB : \mathcal{F}\mathrm{ol}_{red}(n,d) \to (\mathbb{C}^{n-1})^N$$

*at the degree-$d$ Jouanolou foliation, for degree $d$ greater than two, is strictly less than the upper bound given in Proposition 2.7. The same holds for degree $d = 2$ with odd dimension $n$.*

*Proof.* Observe that $\operatorname{rank} M_{d,I} \leq n-1$. Then the result follows from Theorem 2.14. $\square$

We can explicitly estimate the rank of the local Baum-Bott map at the Jouanolou foliation on $\mathbb{P}^3$.

**Theorem 2.18.** *Let $d \geq 2$. The rank of the local Baum-Bott map at the degree-$d$ Jouanolou foliation $\mathcal{J}_d$ on the projective space $\mathbb{P}^3$ is*

$$\begin{array}{ll} \dim \mathbf{V}_d - 16 - \left(\binom{d+2}{3} - 2\right), & \text{if } d = 0 \mod (2); \\ \dim \mathbf{V}_d - 16 - \left(\binom{d+2}{3} + \frac{d-3}{2}\right), & \text{if } d = -1 \mod (4); \\ \dim \mathbf{V}_d - 16 - \left(\binom{d+2}{3} + \frac{d-1}{2}\right), & \text{if } d = 1 \mod (4). \end{array}$$

One may conjecture that

**Conjecture.** *For $n, d \geq 2$, the generic rank of the Baum-Bott map*

$$BB : \mathcal{F}\mathrm{ol}(n,d) \dashrightarrow (\mathbb{P}^k)^N / S_N$$

*is*

$$\min\{\dim \mathcal{F}\mathrm{ol}(n,d) - \dim \mathrm{Aut}(\mathbb{P}^n), (N-1)(n-1)\}.$$

## 3. Eigenspaces Associated to the Jouanolou Foliation

The goal of this section is to decompose the space of vector fields in $\mathbb{C}^n$, which generate foliations on $\mathbb{P}^n$ of a given degree, into the eigenspaces of a certain operator which is derived from an automorphism leaving the Jouanolou foliation invariant. Moreover, we identify the generators of the eigenspaces. This will help us to find the rank of the Baum-Bott map at the Jouanolou foliation, we will see this in Section 4.

From now on, consider $n \geq 3$ and $d \geq 2$. We use the notation of subsection 2.2.

Let $\mathbf{S} : \mathbb{C}^n \dashrightarrow \mathbb{C}^n$ be the map given by

$$\mathbf{S}(x_1, \ldots, x_n) = \frac{1}{x_n}(1, x_1, \ldots, x_{n-1}).$$

The maps $\mathbf{A}$ and $\mathbf{S}$ define automorphisms on $\mathbb{P}^n$ that leave invariant the Jouanolou foliation. The pull-back maps associated to $\mathbf{A}$ and $\mathbf{S}$ are

$$\mathbf{A}^* : \mathbf{V}_d \to \mathbf{V}_d \text{ and } \mathbf{S}^* : \mathbf{V}_d \to \mathbf{S}^*(\mathbf{V}_d),$$

respectively. They are defined by

$$\mathbf{A}^*(X) = D\mathbf{A}^{-1}.X \circ \mathbf{A} \text{ and } \mathbf{S}^*(X) = D\mathbf{S}^{-1}.X \circ \mathbf{S}.$$

Given $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, we set $x^I = x_1^{i_1} \ldots x_n^{i_n}$ and $|I| = i_1 + \ldots + i_n$. Note that the set

$$\mathbf{B}_d = \{x^I \partial_k,\ x^R \mathcal{R} \mid\ I,\ R \in \mathbb{Z}^n_{\geq 0},\ |I| \leq d,\ |R| = d, \text{ and } k = 1, \ldots, n\}$$

is a basis of $\mathbf{V}_d$. It is also a basis of eigenvectors of $\mathbf{A}^*$. The eigenvalues are $N$-th roots of unity.

Let

$$E_j = \{V \in \mathbf{V}_d \mid\ \mathbf{A}^* V = \xi^j V\}$$

be the eigenspace of the operator $\mathbf{A}^*$ associated to the eigenvalue $\xi^j$. Set $\alpha = (\alpha_1, \ldots, \alpha_n)$. We have the following lemma.

**Lemma 3.1.** *Every vector in $\mathbf{B}_d$ is an eigenvector of $\mathbf{A}^* : \mathbf{V}_d \to \mathbf{V}_d$ and $\mathbf{V}_d = \bigoplus_{j=1}^{N} E_j$. Moreover, if $x^I \partial_k$ and $x^R \mathcal{R} \in \mathbf{B}_d$, then*

$$\mathbf{A}^*(x^I \partial_k) = \alpha_k^{-1} \alpha^I x^I \partial_k \quad \textit{and} \quad \mathbf{A}^*(x^R \mathcal{R}) = \alpha^R x^R \mathcal{R}.$$

Now, our goal is to write the generators of each eigenspace in terms of $\mathbf{S}^*$. For this purpose, we need to study how the map $\mathbf{S}^*$ acts in $\mathbf{V}_d$. In general, $\mathbf{S}^*$ applied to a vector in $\mathbf{V}_d$ has $x_n^{d-1}$ as a pole. Therefore, we define

$$\widetilde{\mathbf{S}^*} : \mathbf{V}_d \to \mathbf{V}_d, \quad \widetilde{\mathbf{S}^*} := x_n^{d-1}\mathbf{S}^*.$$

**Lemma 3.2.** *Let $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n_{\geq 0}$ such that $|I| \leq d$, and set $i_{n+1} = d - |I|$. Then*

$$\begin{aligned}
\widetilde{\mathbf{S}^*}(x^I\partial_1) &= -x_1^{i_2}\ldots x_{n-1}^{i_n}x_n^{i_{n+1}}\mathcal{R},\\
\widetilde{\mathbf{S}^*}(x^I\partial_k) &= x_1^{i_2}\ldots x_{n-1}^{i_n}x_n^{i_{n+1}}\partial_{k-1} \text{ for } k = 2, \ldots, n, \text{ and}\\
\widetilde{\mathbf{S}^*}(x^I\mathcal{R}) &= -x_1^{i_2}\ldots x_{n-1}^{i_n}x_n^{i_{n+1}}\partial_n.
\end{aligned}$$

This operator maps eigenspaces into eigenspaces.

**Lemma 3.3.** *Let $f(k) = (dk - d^2 + d) \mod (N)$. Then*

(1) $\widetilde{\mathbf{S}^*}(E_k) = E_{f(k)}$, *and*

(2) $\mathrm{X}_{\mathcal{J}_d} = \sum_{j=0}^{n} (\widetilde{\mathbf{S}^*})^j(\partial_n)$.

*Proof.* To obtain (1), it suffices to show that $\mathbf{A}^*\circ\widetilde{\mathbf{S}^*} = \xi^{-d(d-1)}\widetilde{\mathbf{S}^*}\circ(\mathbf{A}^*)^d$. From $\mathbf{S}^{-1}\circ\mathbf{A}\circ\mathbf{S} = \mathbf{A}^{d^n}$ we have $\mathbf{S}^{-1}\circ\mathbf{A}^d\circ\mathbf{S} = \mathbf{A}^{d^{n+1}} = \mathbf{A}$. Hence $\mathbf{S}\circ\mathbf{A} = \mathbf{A}^d\circ\mathbf{S}$ and we deduce that $\mathbf{A}^*\circ\mathbf{S}^* = \mathbf{S}^*\circ(\mathbf{A}^*)^d$. It follows that

$$x_n^{d-1}\mathbf{A}^* \circ \mathbf{S}^* = \widetilde{\mathbf{S}^*} \circ (\mathbf{A}^*)^d. \tag{3.1}$$

Since $\mathbf{A}^* \circ (x_n\mathbf{S}^*) = \xi^{-d}x_n\mathbf{A}^* \circ \mathbf{S}^*$, we have

$$x_n^{d-1}\mathbf{A}^* \circ \mathbf{S}^* = \xi^{d(d-1)}\mathbf{A}^* \circ \widetilde{\mathbf{S}^*}. \tag{3.2}$$

From (3.1) and (3.2), we conclude that $\mathbf{A}^* \circ \widetilde{\mathbf{S}^*} = \xi^{-d(d-1)}\widetilde{\mathbf{S}^*} \circ (\mathbf{A}^*)^d$.

By Lemma 3.2 we obtain item (2). □

Set $\alpha_{n+1} = 1$. The automorphisms $\mathbf{A}$ and $\mathbf{S}$ are given in homogeneous coordinates by $\mathbb{A}$ and $\mathbb{S}$, respectively, where

$$\begin{aligned}
\mathbb{A}(x_1, \ldots, x_{n+1}) &= (\alpha_1x_1, \ldots, \alpha_nx_n, \alpha_{n+1}x_{n+1})\\
\mathbb{S}(x_1, \ldots, x_{n+1}) &= (x_{n+1}, x_1, \ldots, x_n)
\end{aligned}$$

The automorphism $\mathbb{A}$ generates a subgroup of order $N$ and $\mathbb{S}$ generates a cyclic subgroup of order $(n+1)$ of the automorphism group $\mathrm{Aut}(\mathbb{P}^n)$. Analogously, the automorphisms $\mathbb{A}$ and $\mathbb{S}$ induce operators $\mathbb{A}^*, \mathbb{S}^* : \mathbb{V}_d \to \mathbb{V}_d$, where $\mathbb{V}_d$ is the space of homogeneous vector fields in $\mathbb{C}^{n+1}$ of degree $d$.

Let us fix some notations. Given $\mathbb{I} = (i_1, \ldots, i_{n+1}) \in \mathbb{Z}^{n+1}_{\geq 0}$, we set $x^{\mathbb{I}} = x_1^{i_1}\ldots x_{n+1}^{i_{n+1}}$ and $|\mathbb{I}| = i_1 + \ldots + i_{n+1}$. Let

$$\mathbb{B}_d = \{x^{\mathbb{I}}\partial_k \in \mathbb{V}_d \mid \ |\mathbb{I}| = d, \text{ and } k = 1, \ldots, n+1\}.$$

Denote

$$\mathbb{E}_j = \{V \in \mathbb{V}_d \mid \mathbb{A}^*V = \xi^jV\}$$

the eigenspace of $\mathbb{A}^*$ associated to $\xi^j$.

Let $\mathbb{I} = (i_1, \ldots, i_{n+1}) \in \mathbb{Z}^{n+1}_{\geq 0}$ such that $|\mathbb{I}| = d$, we write $I = (i_1, \ldots, i_n)$. Observe that the eigenvalue of the operator $\mathbb{A}^*$ on the vector $x^{\mathbb{I}}\partial_k$ is the same as the eigenvalue of the operator $\mathbf{A}^*$ on the vector $x^I\partial_k$, if $k \leq n$, and $-x^I\mathcal{R}$, if $k = n+1$. We also see that $\mathbb{S}^*$ evaluated in $x^{\mathbb{I}}\partial_k$ has the same image in $\mathbf{V}_d$ as $\widetilde{\mathbf{S}^*}$ at $x^I\partial_k$, if $k \leq n$, and $-x^I\partial_k\mathcal{R}$, if $k = n+1$. Let $\alpha^{\mathbb{I}} = \alpha_1^{i_1}\ldots\alpha_{n+1}^{i_{n+1}}$. We obtain similar lemmas for $\mathbb{A}$ and $\mathbb{S}$.

**Lemma 3.4.** *Every vector in $\mathbb{B}_d$ is an eigenvector of $\mathbb{A}^* : \mathbb{V}_d \to \mathbb{V}_d$ with eigenvalue some $N$th rooth of unity. Moreover, $\mathbb{V}_d = \bigoplus_{j=1}^{N} \mathbb{E}_j$ and if $x^{\mathbb{I}}\partial_k \in \mathbb{B}_d$, then*

$$\mathbb{A}^*(x^{\mathbb{I}}\partial_k) = \alpha_k^{-1}\alpha^{\mathbb{I}}x^{\mathbb{I}}\partial_k. \tag{3.3}$$

Identifying $\partial_k = \partial_{k \mod (n+1)}$, we have:

**Lemma 3.5.** *The operator $\mathbb{S}^* : \mathbb{V}_d \to \mathbb{V}_d$ maps eigenspaces of $\mathbb{A}^*$ into eigenspaces of $\mathbb{A}^*$. Furthermore, if $x^{\mathbb{I}}\partial_k \in \mathbb{B}_d$, then $\mathbb{S}^*(x^{\mathbb{I}}\partial_k) = x^{\mathbb{S}^{-1}(\mathbb{I})}\partial_{k-1}$.*

We can identify the monomials of degree $d$ in $n+1$ variables with the monomials in $n$ variables of degree at most $d$ with the following correspondence:

$$x_1^{i_1}\dots x_n^{i_n}x_{n+1}^{i_{n+1}} \mapsto x_1^{i_1}\dots x_n^{i_n}, \text{ where } i_1+\dots+i_n+i_{n+1} = d.$$

In the following lemma we extend the identifications to vector fields.

**Lemma 3.6.** *Let $\mathbb{I} = (i_1,\dots,i_n,i_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1}$ such that $|\mathbb{I}| = d$, and set $I = (i_1,\dots,i_n)$. If we identify $x^{\mathbb{I}}\partial_k$ with $x^I\partial_k$ for $k = 1,\dots,n$, and $x^{\mathbb{I}}\partial_{n+1}$ with $-x^I\mathcal{R}$, then*

$$\mathbf{A}^*(x^I\partial_k) = \mathbb{A}^*(x^{\mathbb{I}}\partial_k), \qquad \mathbf{A}^*(-x^I\mathcal{R}) = \mathbb{A}^*(x^{\mathbb{I}}\partial_{n+1}).$$
$$\widetilde{\mathbf{S}^*}(x^I\partial_k) = \mathbb{S}^*(x^{\mathbb{I}}\partial_k), \qquad \widetilde{\mathbf{S}^*}(-x^I\mathcal{R}) = \mathbb{S}^*(x^{\mathbb{I}}\partial_{n+1}).$$

Consequently, we can identify an eigenspace of $\mathbf{A}^*$ with a subspace of an eigenspace of $\mathbb{A}^*$. We can subsequently apply $\mathbb{S}^*$ to a fixed eigenspace of $\mathbb{A}^*$ until one eigenvector is of the form $x^{\mathbb{I}}\partial_1$. It follows that it suffices to study the eigenspaces that contain an eigenvector $x^{\mathbb{I}}\partial_1 \in \mathbb{B}_d$. We identity $x_k = x_{k \mod (n+1)}$.

**Lemma 3.7.** *Let $x^{\mathbb{I}}\partial_1$ and $x^{\mathbb{J}}\partial_k$ be vector fields in $\mathbb{B}_d$ contained in the same eigenspace of $\mathbb{A}^*$.*
1. *If $k = 1$, then $\mathbb{I} = \mathbb{J}$.*
2. *If $i_1 > 0$, then there exists $\tilde{\mathbb{I}} \in \mathbb{Z}_{\geq 0}^{n+1}$ with $|\tilde{\mathbb{I}}| = d-1$ such that $x^{\mathbb{I}} = x_1x^{\tilde{\mathbb{I}}}$ and $x^{\mathbb{J}} = x_kx^{\tilde{\mathbb{I}}}$.*
3. *If $i_1 = 0$ and $i_2 = d$, then $x^{\mathbb{I}} = x_2^d$ and $x^{\mathbb{J}} = x_{k+1}^d$.*
4. *If $i_1 = 0$ and $i_2 < d$,*
   (a) *if $k = 2$, then there exists $r \in \{3,\dots,n+1\}$ such that $x^{\mathbb{I}} = x_2^{d-1}x_r$ and $x^{\mathbb{J}} = x_{r+1}^d$.*
   (b) *if $3 \leq k \leq n$, then $x^{\mathbb{I}} = x_k^{d-1}x_{n+1}$ and $x^{\mathbb{J}} = x_1^{d-1}x_{k-1}$.*
   (c) *if $k = n+1$, then there exists $r \in \{2,\dots,n\}$ such that $x^{\mathbb{I}} = x_{r+1}^d$ and $x^{\mathbb{J}} = x_1^{d-1}x_r$.*

*Proof.* (1) Suppose, contrary to our claim, that $\mathbb{I} \neq \mathbb{J}$. By equation (3.3) there exists $K \in \mathbb{Z}$ such that

$$\begin{aligned} KN =&(i_1-j_1)d^n + (i_1+i_2-(j_1+j_2))d^{n-1}+\dots+\\ &(i_1+\dots+i_{n-1}-(j_1+\dots+j_{n-1}))d^2+\\ &(i_1+\dots+i_n-(j_1+\dots+j_n))d. \end{aligned} \tag{3.4}$$

The above relation and $N = 1 \mod (d)$ imply that $d$ divides $K$. We claim that $K = 0$. In effect, since $i_1+\dots+i_r \leq d$ and $j_1+\dots+j_r \leq d$ for $r = 1,\dots,n$, we have $-dN < KN < dN$. It follows that $-d < K < d$. Since $d$ divides $K$ we have $K = 0$. Now (3.4) becomes

$$\begin{aligned} 0 =&(i_1-j_1)d^{n-1} + (i_1+i_2-(j_1+j_2))d^{n-2}+\dots+\\ &(i_1+\dots+i_{n-1}-(j_1+\dots+j_{n-1}))d+\\ &(i_1+\dots+i_n-(j_1+\dots+j_n)). \end{aligned} \tag{3.5}$$

Hence $(i_1+\dots+i_n-(j_1+\dots+j_n)) = 0\text{mod}(d)$. We have three cases.

*Case 1.* If $i_1+\ldots+i_n=(j_1+\ldots+j_n)+d$, then $j_1=\ldots=j_n=0$. From this and (3.5) we obtain $i_1d^{n-1}+(i_1+i_2)d^{n-2}+\ldots+(i_1+\ldots+i_{n-1})d+(i_1+\ldots+i_n)=0$. This gives $i_1=\ldots=i_n=0$, which is a contradiction.

*Case 2.* If $i_1+\ldots+i_n=j_1+\ldots+j_n-d$, then as in the proof of Case 1, we obtain a contradiction.

*Case 3.* If $i_1+\ldots+i_n=j_1+\ldots+j_n$, then we have

$$\begin{aligned}0=&(i_1-j_1)d^{n-2}+(i_1+i_2-(j_1+j_2))d^{n-3}+\ldots+\\&(i_1+\ldots+i_{n-2}-(j_1+\ldots+j_{n-2}))d+\\&(i_1+\ldots+i_{n-1}-(j_1+\ldots+j_{n-1})).\end{aligned}$$

Hence $((i_1+\ldots+i_{n-1})-(j_1+\ldots+j_{n-1}))=0 \mod (d)$. It follows that

$$i_1+\ldots+i_{n-1}=j_1+\ldots+j_{n-1}.$$

Continuing this proccess, we get $\mathbb{I}=\mathbb{J}$. This contradicts our initial assumption.

(2) Let $i_1>0$. Then there exists $\tilde{\mathbb{I}}\in\mathbb{Z}^{n+1}_{\geq 0}$ such that $x^{\mathbb{I}}=x_1x^{\tilde{\mathbb{I}}}$. From Lemma 3.4 we deduce that $x_1x^{\tilde{\mathbb{I}}}\partial_1$ and $x_kx^{\tilde{\mathbb{I}}}\partial_k$ belong to the same eigenspace. Combining Lemma 3.5 with item (1) we conclude that $x^{\mathbb{J}}=x_kx^{\tilde{\mathbb{I}}}$.

Similar arguments apply to the proofs of items (3) and (4). □

Now we are able to count the number of eigenspaces with same dimension. Let $\lfloor\cdot\rfloor$ be the floor function. Set

$$\mathcal{E}^2_j=< x_2^{d-1}x_j\partial_1, x_{1+j}^d\partial_2>,\ \mathcal{E}^j=< x_j^{d-1}x_{n+1}\partial_1, x_1^{d-1}x_{j-1}\partial_j> .$$

We define

$$\mathsf{D}=\{x_1x^{\tilde{\mathbb{I}}}, x_2^{d-1}x_j, x_j^{d-1}x_{n+1}, x_j^d \mid \tilde{\mathbb{I}}\in\mathbb{Z}^{n+1}_{\geq 0} \text{ with } |\tilde{\mathbb{I}}|=d-1, \text{ and } 2\leq j\leq n+1\}.$$

**Theorem 3.8.** *The operator $\mathbf{A}^*:\mathbf{V}_d\to\mathbf{V}_d$ has $\binom{n+d}{n}+n\binom{n+d-1}{n-1}-(n-1)\frac{3n+4}{2}$ non-trivial eigenspaces. More precisely:*

(1) *There is one eigenspace of dimension $n+1$, which is*

$$E_d=< x_2^d\partial_1,\ldots,x_n^d\partial_{n-1},\partial_n,x_1^d\mathcal{R}> .$$

(2) *There are $\binom{n+d-1}{n}$ eigenspaces of dimension $n$. They are of the form*

$$< x_1x^{\mathbb{I}}\partial_1,\ldots,x_nx^{\mathbb{I}}\partial_n>,\ \textit{where } |\mathbb{I}|=d-1.$$

(3) *There are $\frac{(3n-4)(n+1)}{2}$ eigenspaces of dimension two. These are*
   (a) *$(\mathbb{S}^*)^k(\mathcal{E}^2_j)$ for $j=3,\ldots,n+1$ and $k=0,\ldots,n$;*
   (b) *$(\mathbb{S}^*)^k(\mathcal{E}^j)$ for $j=3,\ldots,\lfloor\frac{n+2}{2}\rfloor$ and $k=0,\ldots,n$. If $n$ is an odd number, then we also have $j=\frac{n+3}{2}$ with $k=0,\ldots,\frac{n-1}{2}$.*

(4) *There are $(n+1)\left[\binom{n+d-1}{n-1}-3(n-1)\right]$ eigenspaces of dimension 1. These are generated by $(\mathbb{S}^*)^k(x^{\mathbb{I}}\partial_1)$ for $k=0,\ldots,n$ such that $x^{\mathbb{I}}\notin\mathsf{D}$.*

*The list is complete and each eigenspace of $\mathbf{A}^*$ corresponds to exactly one of the spaces given in the above items.*

*Proof.* From Lemma 3.1 we see that every element in $\mathbf{B}_d$ is an eigenvector of $\mathbf{A}^*$. Similarly, Lemma 3.4 shows that the same being true for $\mathbb{B}_d$ and $\mathbb{A}^*$. Let $E$ be an eigenspace of $\mathbf{A}^*$. By Lemma 3.6, $E$ is included in an eigenspace $\mathbb{E}$ of $\mathbb{A}^*$ and these eigenspaces are equal if no $x^{\mathbb{I}}\partial_{n+1}\in\mathbb{B}_d$ with $i_{n+1}>0$ belongs to $\mathbb{E}$. Therefore, we will focus on the eigenspaces of $\mathbb{A}^*$.

Fix $k \in \{1, \ldots, n+1\}$. Combining Lemma 3.5 and item 1 of Lemma 3.7 we conclude that each eigenspace of $\mathbb{A}^*$ contains at most one eigenvector of the form $x^{\mathbb{I}}\partial_k \in \mathbb{B}_d$.

(1) According to item 3 of Lemma 3.7, there is one eigenspace of $\mathbb{A}^*$ generated by $x_{j+1}\partial_j$, for $j = 1, \ldots, n+1$. The corresponding eigenspace of $\mathbf{A}^*$ is $E_d$.

(2) Let $\mathbb{I} \in \mathbb{Z}^{n+1}_{\geq 0}$ with $|\mathbb{I}| = d-1$. By item 2 of Lemma 3.7 the vectors $x_1x^{\mathbb{I}}\partial_1, \ldots, x_{n+1}x^{\mathbb{I}}\partial_{n+1}$ belong to the same eigenspace of $\mathbb{A}^*$. The corresponding eigenspace of $\mathbf{A}^*$ is $< x_1x^{\mathbb{I}}\partial_1, \ldots, x_nx^{\mathbb{I}}\partial_n >$. There are $\binom{n+d-1}{n}$ eigenspaces of this kind.

(3) Let $E$ be an eigenspace of $\mathbf{A}^*$ of dimension 2. By Lemmas 3.2 and 3.3 there is some $k$ such that $(\widetilde{\mathbf{S}^*})^k(E)$ can be identified with the eigenspace $< x^{\mathbb{I}}\partial_1, x^{\mathbb{J}}\partial_{1+j} >$ of $\mathbb{A}^*$, for some $\mathbb{I}, \mathbb{J} \in \mathbb{Z}^{n+1}_{\geq 0}$ and $0 < j < n$. Lemma 3.5 implies that $E$ corresponds to the eigenspace

$$(\mathbb{S}^*)^{n+1-k}(< x^{\mathbb{I}}\partial_1, x^{\mathbb{J}}\partial_{1+j} >)$$

of dimension 2 of $\mathbb{A}^*$. By item 4 of Lemma 3.7 the eigenspaces of dimension 2 of $\mathbb{A}^*$ are:

$$(\mathbb{S}^*)^k(< x_2^{d-1}x_j\partial_1, x_{1+j}^d\partial_2 >) \text{ for } j = 3, \ldots, n+1 \text{ and } k = 0, \ldots, n,$$

$$(\mathbb{S}^*)^k(< x_{2+j}^{d-1}x_{n+1}\partial_1, x_1^{d-1}x_{1+j}\partial_{2+j} >) \text{ for } j = 1, \ldots, n-2 \text{ and } k = 0, \ldots, n.$$

We now want to exclude the repeated eigenspaces. We have two cases:

*Case 1.* Suppose

$$(\mathbb{S}^*)^k(< x_2^{d-1}x_j\partial_1, x_{1+j}^d\partial_2 >) =< x_2^{d-1}x_r\partial_1, x_{1+r}^d\partial_2 >)$$

for some $j, r, k \in \mathbb{Z}$ such that $3 \leq j < r \leq n+1$ and $1 \leq k \leq n$. Then

$$< x_{2-k}^{d-2}x_{j-k}\partial_{1-k}, x_{1+j-k}\partial_{2-k} >=< x_2^{d-1}x_r\partial_1, x_{1+r}^d\partial_2 >,$$

which is impossible. We conclude item $(a)$.

*Case 2.* Let

$$(\mathbb{S}^*)^k(< x_j^{d-1}x_{n+1}\partial_1, x_1^{d-1}x_{j-1}\partial_j >) =< x_r^{d-1}x_{n+1}\partial_1, x_1^{d-1}x_{r-1}\partial_r >$$

for some $j, r, k \in \mathbb{Z}$ such that $3 \leq j < r \leq n$ and $k = 1, \ldots, n$. Hence

$$x_{j-k}^{d-1}x_{n+1-k}\partial_{n+2-k} = x_1^{d-1}x_{r-1}\partial_r$$

and $x_{n+2-k}^{d-1}x_{n+j-k}\partial_{j-k} = x_r^{d-1}x_{n+1}\partial_1$. It follows that $k = j-1$ and $r = n+3-j$. Therefore $(\mathbb{S}^*)^{j-1}(\mathcal{E}^j) = \mathcal{E}^{n+3-j}$. We deduce that $(\mathbb{S}^*)^k(\mathcal{E}^j)$ generates different eigenspaces in the following cases:

*if n is even:* $j = 3, \ldots, \frac{n+2}{2}$ and $k = 0, \ldots, n$.

*if n is odd:* $j = 3, \ldots, \frac{n+1}{2}$ and $k = 0, \ldots, n$. We also have the case $j = \frac{n+3}{2}$ with $k = 0, \ldots, \frac{n+3}{2}$.

We conclude item $(b)$.

(4) Using the same arguments of the above proof we obtain that the eigenspaces of dimension 1 of $\mathbf{A}^*$ correspond to eigenspaces of $\mathbb{A}^*$ generated by $(\mathbb{S}^*)^k(x^{\mathbb{I}}\partial_1)$, for $k = 0, \ldots, n$. We use Lemma 3.7 to exclude the vector $x^{\mathbb{I}}\partial_1$ which belong to eigenspaces of dimension greather than 1. Thus we conclude the proof. □

## 4. The Rank at the Jouanolou Foliation

We state some lemmas which will helps us estimate the rank of the local Baum-Bott map at the Jouanolou foliation. We will show that the rank can be computed by using only one singular point of the foliation. The computation relies on the eigenspaces of the space $\mathbf{V}_d$ of the previous section.

Since the singularities of $\mathrm{X}_{\mathcal{J}_d}$ are non-degenerate, there exists a neighborhood $U$ of $\mathrm{X}_{\mathcal{J}_d}$ in $\mathbf{V}_d$ and holomorphic maps $\gamma_j : U \to \mathbb{C}^n$, for $j = 1, \ldots, N$, such that $\gamma_j(\mathrm{X}_{\mathcal{J}_d}) = p_j$ and

$\mathrm{Sing}(X) = \{\gamma_1(X), \dots, \gamma_N(X)\}$, for all $X \in U$. The local Baum-Bott map can be written on $U$ as

$$\begin{array}{rcllll} BB(X) & = & (BB_1(X,\gamma_1(X)), & \dots & , B_{n-1}(X,\gamma_1(X)), \\ & & \vdots & \dots & \vdots \\ & & BB_1(X,\gamma_N(X)), & \dots & , BB_{n-1}(X,\gamma_N(X))). \end{array}$$

In this way, we only need to compute the rank of the linear map

$$DBB(\mathrm{X}_{\mathcal{J}_d}) : \mathbf{V}_d \to \mathbb{C}^{(n-1)N}.$$

For simplicity we denote by

$$T_j := (DBB_1(\mathrm{X}_{\mathcal{J}_d}, p_j), \dots, DBB_{n-1}(\mathrm{X}_{\mathcal{J}_d}, p_j)) : \mathbf{V}_d \to \mathbb{C}^{n-1}$$

for $1 \leq j \leq N$, and

$$DBB(\mathrm{X}_{\mathcal{J}_d}) = T := (T_1, \dots, T_N).$$

We will prove that $T$ can be factorized into a product of a non-singular matrix and a block diagonal matrix. More specifically, we have the following lemma:

**Lemma 4.1.** *The rank of the Baum-Bott map at the Jouanolou foliation is*

$$\mathrm{rank}\, T = \sum_{i=1}^{N} \mathrm{rank}\, T_1|_{E_i}.$$

*Proof.* Set $p_{N+1} = p_1$. We fix $k$ and $j$ such that $1 \leq k \leq n-1$ and $1 \leq j \leq N$. Let $V \in \mathbf{V}_d$. We claim that

$$DBB_k(\mathrm{X}_{\mathcal{J}_d}, p_{j+1}).V = \xi^{-d} DBB_k(\mathrm{X}_{\mathcal{J}_d}, p_j).\mathbf{A}^*(V). \tag{4.1}$$

Indeed, we know that if $X$ is a vector field in $\mathbb{C}^n$ with a non-degenerate singular point $p$, then $BB_k(\mathbf{A}^* X, p) = BB_k(X, \mathbf{A}^{-1}(p))$. Consider $r > 0$ such that $\mathrm{X}_{\mathcal{J}_d} + tV \in U$ for all $t \in \mathbb{C}$ with $|t| < r$. Set $\rho_j(t) := \gamma_j(\mathrm{X}_{\mathcal{J}_d} + tV)$. Since $\mathbf{A}^* \mathrm{X}_{\mathcal{J}_d} = \xi^d \mathrm{X}_{\mathcal{J}_d}$, it follows that

$$\begin{aligned} BB_k(\mathrm{X}_{\mathcal{J}_d} + tV, \rho_{j+1}(t)) &= BB_k(\xi^d \mathrm{X}_{\mathcal{J}_d} + t\mathbf{A}^*(V), \mathbf{A}^{-1}(\rho_{j+1}(t))) \\ &= BB_k(\mathrm{X}_{\mathcal{J}_d} + t\xi^{-d}\mathbf{A}^*(V), \mathbf{A}^{-1}(\rho_{j+1}(t))). \end{aligned}$$

Taking the derivative with respect to $t$ at $t = 0$, we get

$$\begin{aligned} DBB_k(\mathrm{X}_{\mathcal{J}_d}, p_{j+1}).V &= DBB_k(\mathrm{X}_{\mathcal{J}_d}, p_j).\xi^{-d}\mathbf{A}^*(V) \\ &= \xi^{-d} DBB_k(\mathrm{X}_{\mathcal{J}_d}, p_j).\mathbf{A}^*(V). \end{aligned}$$

Let $V$ be an eigenvector of $\mathbf{A}^*$ associated to the eigenvalue $\xi^i$. Equation (4.1) implies

$$T_j(V) = \xi^{(j-1)(i-d)} T_1(V).$$

Choosing a basis for each eigenspace $E_i$ associated to the eigenvalue $\xi^i$ leads to

$$\begin{aligned}
T &= \begin{bmatrix} T_1|_{E_1} & T_1|_{E_2} & \dots & T_1|_{E_N} \\ T_2|_{E_1} & T_2|_{E_2} & \dots & T_2|_{E_N} \\ \vdots & \vdots & \vdots & \vdots \\ T_N|_{E_1} & T_N|_{E_2} & \dots & T_N|_{E_N} \end{bmatrix} \\
&= \begin{bmatrix} T_1|_{E_1} & T_1|_{E_2} & \dots & T_1|_{E_N} \\ \xi^{(1-d)}T_1|_{E_1} & \xi^{2-d}T_1|_{E_2} & \dots & \xi^{N-d}T_1|_{E_N} \\ \vdots & \vdots & \vdots & \vdots \\ \xi^{(N-1)(1-d)}T_1|_{E_1} & \xi^{(N-1)(2-d)}T_1|_{E_2} & \dots & \xi^{(N-1)(N-d)}T_1|_{E_N} \end{bmatrix} \\
&= M \begin{bmatrix} T_1|_{E_1} & 0 & \dots & 0 \\ 0 & T_1|_{E_2} & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & T_1|_{E_N} \end{bmatrix},
\end{aligned}$$

where $M \in M(\mathbb{C}, (n-1)N)$ given by

$$M = \begin{bmatrix} \mathrm{Id} & \mathrm{Id} & \dots & \mathrm{Id} \\ \xi^{(1-d)}\mathrm{Id} & \xi^{2-d}\mathrm{Id} & \dots & \xi^{N-d}\mathrm{Id} \\ \vdots & \vdots & \vdots & \vdots \\ \xi^{(N-1)(1-d)}\mathrm{Id} & \xi^{(N-1)(2-d)}\mathrm{Id} & \dots & \xi^{(N-1)(N-d)}\mathrm{Id} \end{bmatrix}$$

and Id is the $(n-1) \times (n-1)$ identity matrix. Since $\det(M) \neq 0$, the lemma follows. □

In order to compute the rank of $T$ using Lemma 4.1 we have to compute the rank of $T_1$ at each eigenspace of $\mathbf{A}^*$. Since $\widetilde{\mathbf{S}^*}$ maps eigenspaces into eigenspaces (see Lemma 3.3), it suffices to study $T_1$ restricted to some convenient eigenspaces:

**Lemma 4.2.** *Let $V \in \mathbf{V}_d$ and let $\mathbb{I} \in \mathbb{Z}^{n+1}_{\geq 0}$ such that $|\mathbb{I}| = d$. Then $T(\widetilde{\mathbf{S}^*}(V)) = T(V)$ and $T_1(x^{\mathbb{I}}\partial_k) = T_1(x^{\mathbb{S}^{-1}(\mathbb{I})}\partial_{k-1})$, for $k = 1, \dots, n+1$. Furthermore, $E_d \subset \ker(T)$.*

*Proof.* To obtain the first assertion of the lemma, let $Q \in \mathbb{C}[x_1, \dots, x_n]$ of degree less than or equal to $d$. For $t \in \mathbb{C}$, we have

$$x_n^{d-1}\mathbf{S}^*(X_{\mathcal{J}_d} + tQ\partial_1) = X_{\mathcal{J}_d} - tQ(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n})x_n^d\mathcal{R}.$$

Consider $\rho_j(t)$ the singular point of $X_{\mathcal{J}_d} + tQ\partial_1$ such that $\rho_j(0) = p_j$. Note that $\mathbf{S}^{-1}(\rho_j(t))$ goes to $p_j$ as $t$ goes to zero. Therefore

$$T(Q\partial_1) = T\big( - Q(\frac{1}{x_n}, \frac{x_1}{x_n}, \dots, \frac{x_{n-1}}{x_n})x_n^d\mathcal{R}\big).$$

The proof for the other cases is similar.

To prove $T_1(x^{\mathbb{I}}\partial_k) = T_1(x^{\mathbb{S}^{-1}(\mathbb{I})}\partial_{k-1})$, we apply the first part of this lemma and Lemma 3.6.

Finally, in order to get the last conclusion, we observe that

$$BB(\mathrm{X}_{\mathcal{J}_d} + t\mathrm{X}_{\mathcal{J}_d}, \rho_1(t)) = BB(\mathrm{X}_{\mathcal{J}_d}, p_1),$$

then we get $T(\mathrm{X}_{\mathcal{J}_d}) = 0$. From Lemma 3.3, we have

$$T(\mathrm{X}_{\mathcal{J}_d}) = T(\sum_{k=0}^{n}(\widetilde{\mathbf{S}^*})^k(\partial_n)) = (n+1)T(\partial_n).$$

Therefore $T(\partial_n) = 0$ and $E_d \subset \ker(T)$. □

Let $V \in \mathcal{V}_d$, we define $\mathcal{P}(V) = \{x^I V \in \mathbf{B}_d \mid I \in \mathbb{Z}^n_{\geq 0} \text{ and } |I| \leq d\}$. From the previous lemma it is enough to calculate the map $T_1$ on vectors in $\mathcal{P}(\partial_1)$. All we need is find the derivative of the elementary symmetric functions of the eigenvalues with respect to those vectors.

**Lemma 4.3.** *Let $Q$ be a polynomial in $n$ variables such that $Q(p_1) = 0$ and let $t \in \mathbb{C}$. Set $X(t) = \mathrm{X}_{\mathcal{J}_d} + tQ\partial_1$ and define $C_i(t) = C_i(DX(t)(p_1))$. Then we have*

$$\begin{aligned} C_1'(0) = {} & \partial_1 Q(p_1), \\ C_i'(0) = {} & (-1)^i \Big( -\tbinom{n-1}{i-1}\partial_1 Q(p_1) + \\ & \sum_{k=0}^{i-2} \tbinom{n-i+k}{k} (\partial_2 Q(p_1) + \ldots + \partial_{n-i+k+2}Q(p_1)) d^{i-k-1} \Big) \end{aligned}$$

*for $i = 2, \ldots, n$.*

*Proof.* Let $P(\lambda) = \det(\lambda \mathrm{Id} - DX(t)(p_1))$ be the characteristic polynomial of the matrix $DX(t)(p_1)$. For abbreviation, we write $\partial_i Q$ instead of $\partial_i Q(p_1)$ for $i = 1, \ldots, n$. Thus

$$P(\lambda) = \begin{vmatrix} (\lambda + (d+1) - t\partial_1 Q) & (-d - t\partial_2 Q) & -t\partial_3 Q & -t\partial_4 Q & \ldots & -t\partial_{n-1}Q & -t\partial_n Q \\ d & (\lambda+1) & -d & 0 & \ldots & 0 & 0 \\ d & 0 & (\lambda+1) & -d & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d & 0 & 0 & 0 & \ldots & -d & 0 \\ d & 0 & 0 & 0 & \ldots & (\lambda+1) & -d \\ d & 0 & 0 & 0 & \ldots & 0 & (\lambda+1) \end{vmatrix}.$$

Set $W_1 = \lambda + (d+1) - t\partial_1 Q$, $W_2 = -d - t\partial_2 Q$, and $W_i = -t\partial_i Q$, for $i = 3, \ldots, n$. To estimate the determinant, we divide the last row of the above matrix by $\lambda + 1$. Hence we have

$$P(\lambda) = (\lambda+1) \begin{vmatrix} W_1 & W_2 & W_3 & W_4 & \ldots & W_{n-1} & W_n \\ d & (\lambda+1) & -d & 0 & \ldots & 0 & 0 \\ d & 0 & (\lambda+1) & -d & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d & 0 & 0 & 0 & \ldots & -d & 0 \\ d & 0 & 0 & 0 & \ldots & (\lambda+1) & -d \\ \left(\dfrac{d}{\lambda+1}\right)^1 & 0 & 0 & 0 & \ldots & 0 & 1 \end{vmatrix}.$$

We take the $n$-th row multiplied by $-W_n$ and add it to the first row. We now consider the $n$-th row multiplied by $d$ and we add it to the $(n-1)$-th row. Dividing the $(n-1)$-th row by $\lambda + 1$

gives

$$P(\lambda) = (\lambda+1)^2 \begin{vmatrix} \left(W_1 - \left(\frac{d}{\lambda+1}\right) W_n\right) & W_2 & W_3 & W_4 & \dots & W_{n-2} & W_{n-1} \\ d & (\lambda+1) & -d & 0 & \dots & 0 & 0 \\ d & 0 & (\lambda+1) & -d & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ d & 0 & 0 & 0 & \dots & (\lambda+1) & -d \\ \left(\frac{d}{\lambda+1} + \left(\frac{d}{\lambda+1}\right)^2\right) & 0 & 0 & 0 & \dots & 0 & 1 \end{vmatrix}.$$

We apply similar arguments to the new matrix. Iterating this process leads to

$$\begin{aligned} P(\lambda) =& (\lambda+1)^{n-1} \begin{vmatrix} \left(W_1 - \left(\frac{d}{\lambda+1}\right) W_n - \dots - \left(\frac{d}{\lambda+1}\right)^{n-2} W_3\right) & W_2 \\ \left(\frac{d}{\lambda+1} + \dots + \left(\frac{d}{\lambda+1}\right)^{n-1}\right) & 1 \end{vmatrix} \\ =& (\lambda+1)^{n-1} W_1 - d(\lambda+1)^{n-2} W_n + \dots + \\ & - \left(d(\lambda+1)^{n-2} + d^2(\lambda+1)^{n-3} + \dots + d^{n-2}(\lambda+1)\right) W_3 + \\ & - \left(d(\lambda+1)^{n-2} + d^2(\lambda+1)^{n-3} + \dots + d^{n-2}(\lambda+1) + d^{n-1}\right) W_2. \end{aligned}$$

This yields

$$\begin{aligned} P(\lambda) =& (\lambda+1)^n + (\lambda+1)^{n-1}(d - t\partial_1 Q) + \\ & d(\lambda+1)^{n-2}(d + t(\partial_2 Q \dots + \partial_n Q)) + \\ & d^2(\lambda+1)^{n-3}(d + t(\partial_2 Q \dots + \partial_{n-1} Q)) + \dots + \\ & d^{n-2}(\lambda+1)(d + t(\partial_2 Q + \partial_3 Q)) + d^{n-1}(d + t\partial_2 Q). \end{aligned}$$

Since $P(\lambda) = \sum_{i=0}^{n} (-1)^i \lambda^{n-i} C_i(t)$, we get $\partial_t P(\lambda)|_{t=0} = \sum_{i=1}^{n} (-1)^i \lambda^{n-i} C_i'(0)$. From this the formulas of the lemma follow. □

We can identify which vectors are not in the kernel of the linear map $T_1$.

**Lemma 4.4.** *Let $d, n \in \mathbb{Z}$ such that $d \geq 2$ and $n \geq 3$. Let $Q \neq x_2^d$ be the monomial $Q = x_1^{i_1} \dots x_n^{i_n}$ of degree less than or equal to $d$. Assume that $Q \neq x_1^{r+1} x_2^r \dots x_n^r$, where $r \in \mathbb{Z}_{>0}$ such that $d = (n+1)r + 1$, whenever $r$ exists. Then $Q\partial_1 \notin \ker(T_1)$.*

*Proof.* Let

$$C_i(t) = C_i(D(X_{\mathcal{J}_d} + t(Q - x_2^d)\partial_1)(p_1)).$$

Observe that $p_1$ is a singularity of $X_{\mathcal{J}_d} + t(Q - x_2^d)\partial_1$. Suppose the assertion of the lemma is false. Then $(Q - x_2^d)\partial_1 \in \ker(T_1)$ and we have

$$i\frac{C_1'(0)}{C_1(0)} - \frac{C_i'(0)}{C_i(0)} = 0 \text{ for } i = 2, \dots, n.$$

Set $i_{n+1} = d - (i_1 + \ldots + i_n)$. Writing $i_2 = d - i_1 - (i_3 + \ldots + i_{n+1})$ and applying Lemma 4.3 and equation (2.1) to the above equation yields

$$(i_{n+1} - i_1)d + n(i_1 + i_{n+1}) - i_1(n-1) = 0, \tag{4.2}$$

$$\begin{aligned}(i_1(1-i) + i_{n-i+3} + \ldots + i_{n+1})d^{i-1} + \\ \sum_{k=1}^{i-2}\Bigg(\binom{n-i+k}{k}((1-i)i_1 + i_{n-i+3+k} + \ldots + i_{n+1}) + \\ n\binom{n-i+k-1}{k-1}(i_1 + i_{n-i+2+k} + \ldots + i_{n+1})\Bigg)d^{i-k-1} + \\ \binom{n-2}{i-2}(ni_{n+1} + i_1) = 0\end{aligned} \tag{4.3}$$

for $i = 3, \ldots, n$. From (4.2) it follows that

$$i_1 + ni_{n+1} = (i_1 - i_{n+1})d \geq 0 \text{ and} \tag{4.4}$$

$$i_1(d-1) = i_{n+1}(d+n) \tag{4.5}$$

Replacing (4.4) in (4.3) for $i = 3$ we get

$$2i_1(d-1) = (i_n + i_{n+1})(d+n).$$

From this and (4.5), we deduce that $i_n = i_{n+1}$. We replace $i_n$ by $i_{n+1}$ in (4.3) for $i = 4$ and use (4.4) and (4.5) to conclude that $i_{n-1} = i_{n+1}$. We deduce that $i_3 = \ldots = i_{n+1}$. Substituting this into (4.4) we obtain

$$(i_1 - i_{n+1})(d-1) = (n+1)i_{n+1}.$$

Since $d = i_1 + i_2 + \ldots + i_{n+1}$, we have

$$d = (i_1 - i_{n+1})d + i_2 - i_{n+1}. \tag{4.6}$$

Observe that if $i_1 = i_{n+1}$, then $i_2 = d$, which is a contradiction. Combining (4.4) and (4.6) we obtain $i_1 = i_{n+1} + 1$ and $i_2 = i_{n+1}$. Thus $d = (n+1)i_{n+1} + 1$, a contradiction. This finishes the proof. □

The rank of the linear map $T_1$ restricted to an eigenspace of dimension $n$ depends on the exponents of the monomial that define the eigenspace.

**Lemma 4.5.** *Let $n, d \in \mathbb{Z}$ such that $d \geq 2$ and $n \geq 3$. Let $x^I = x_1^{i_1} \ldots x_n^{i_n}$ be a monomial of degree less than or equar to $d-1$. Then we have*

$$\operatorname{rank} T_1|_{\{x_j x^I \partial_j | j=1,\ldots,n\}} = \operatorname{rank} M_{d,I}.$$

*Proof.* Recall that $i_{n+1} = d - 1 - (i_1 + \ldots + i_n)$. By Lemma 4.2 we have

$$\begin{aligned}T_1(x_2 x^I \partial_2) &= T_1(x_1^{i_2+1} x_2^{i_3} \ldots x_n^{i_{n+1}} x_{n+1}^{i_1} \partial_1),\\ T_1(x_3 x^I \partial_3) &= T_1(x_1^{i_3+1} x_2^{i_4} \ldots x_n^{i_1} x_{n+1}^{i_2} \partial_1), \ldots,\\ T_1(x_n x^I \partial_n) &= T_1(x_1^{i_n+1} x_2^{i_{n+1}} \ldots x_{n+1}^{i_{n-1}} \partial_1).\end{aligned}$$

It will cause no confusion if we denote $i_j$ for $i_{j \bmod (n+1)}$. Set

$$Q_j(x_1, \ldots, x_{n+1}) = x_1^{i_j+1} x_2^{i_{j+1}} \ldots x_n^{i_{j+n-1}} x_{n+1}^{i_{j+n}} - x_2^d.$$

Let

$$C_{i,Q_j}(t) = C_i(D(X_{\mathcal{J}_d} + tQ_j\partial_1)(p_1)),$$

where $t \in \mathbb{C}$. For simplicity of notation, set $C_i(0) = C_{i,Q_j}(0)$, for $i = 1, \ldots, n-1$, $j = 1, \ldots, n$. Note that

$$p_1 \in \operatorname{Sing}(\mathrm{X}_{\mathcal{J}_d} + tQ_j\partial_1).$$

Observe that the linear transformation $T_1$ restricted to the space generated by

$$\{x_j x^I \partial_j \mid j = 1, \ldots, n\}$$

has the same rank as

$$\begin{bmatrix} 2\dfrac{C'_{1,Q_j}(0)}{C_1(0)} - \dfrac{C'_{2,Q_j}(0)}{C_2(0)} \\ 3\dfrac{C'_{1,Q_j}(0)}{C_1(0)} - \dfrac{C'_{3,Q_j}(0)}{C_3(0)} \\ \vdots \\ n\dfrac{C'_{1,Q_j}(0)}{C_1(0)} - \dfrac{C'_{n,Q_j}(0)}{C_n(0)} \end{bmatrix}_{1 \leq j \leq n} . \tag{4.7}$$

Since $Q_j(p_1) = 0$, we can use Lemma 4.3 to get

$$\begin{aligned} \partial_1 Q_j &= i_j + 1, \\ \partial_2 Q_j &= i_{j+1} - d, \\ \partial_k Q_j &= i_{j+k-1} \text{ for } k = 3, \ldots, n+1 \text{ and } j = 1, \ldots, n. \end{aligned}$$

Therefore for $i = 2, \ldots, n$ and $j = 1, \ldots, n$, we get

$$\begin{aligned} iC_i(0)C'_{1,Q_j}(0) - C_1(0)C'_{i,Q_j}(0) =& i(i_j+1)\sum_{k=0}^{i}\binom{n-i+k}{k}d^{i-k} + \\ & (d+n)\left(\sum_{k=0}^{i-2}\binom{n-i+k}{k}(i_{j+1} - d + i_{j+2} + \right. \\ & \left. \ldots + i_{j+n+k-i+1})d^{i-k-1} - \binom{n-1}{i-1}(i_j+1)\right). \end{aligned}$$

Replacing $i_{j+1} - d + i_{j+2} + \ldots + i_{j+n+k-i+1}$ by $-(i_j + 1 + i_{j+n+k-i+2} + \ldots + i_{j+n})$ into the above expression yields

$$\begin{aligned} 2C_2(0)C'_{1,Q_j}(0) - C_1(0)C'_{2,Q_j}(0) &= d((i_j + 1 - i_{j+n})d - (i_j + 1 + ni_{j+n})), \\ iC_i(0)C'_{1,Q_j}(0) - C_1(0)C'_{i,Q_j}(0) &= \\ & d(((i-1)(i_j+1) - (i_{j+n-i+2} + \ldots + i_{j+n}))d^{i-1} + \\ & \sum_{k=1}^{i-2}\left(\binom{n-i+k}{k}((i-1)(i_j+1) - (i_{j+n+k-i+2} + \ldots + i_{j+n})) + \right. \\ & \left. -n\binom{n-i+k-1}{k-1}(i_j + 1 + i_{j+n+k-i+1} + \ldots + i_{j+n})\right) d^{i-k-1} + \\ & -(i_j + 1 + ni_{j+n})\binom{n-2}{i-2}), \end{aligned}$$

for $i = 3, \ldots, n$ and $j = 1, \ldots, n$. This implies that we have to calculate the rank of the following matrix

$$\begin{bmatrix} (i_j+1-i_{j+n})d-(i_j+1+ni_{j+n}) \\ [(2(i_j+1)-(i_{j+n-1}+i_{j+n}))d^2+ \\ ((n-2)(2(i_j+1)-i_{j+n})-n(i_j+1+i_{j+n-1}+i_{j+n}))d-(i_j+1+ni_{j+n})(n-2)] \\ \frac{1}{d}(4C_4(0)C'_{1,Q_j}(0)-C_1(0)C'_{4,Q_j}(0)) \\ \vdots \\ \frac{1}{d}(nC_n(0)C'_{1,Q_j}(0)-C_1(0)C'_{n,Q_j}(0)) \end{bmatrix}_{1\le j\le n}.$$

We first take the first row multiplied by $-(n-2)$ and add it to the second row. We divide the second row by $d$. Thus the above matrix is similar to

$$\begin{bmatrix} d(i_j+1-i_{j+n})-(i_j+1+ni_{j+n}) \\ (2(i_j+1)-(i_{j+n-1}+i_{j+n}))d-(2(i_j+1)+n(i_{j+n-1}+i_{j+n})) \\ \frac{1}{d}(4C_4(0)C'_{1,Q_j}(0)-C_1(0)C'_{4,Q_j}(0)) \\ \vdots \\ \frac{1}{d}(nC_n(0)C'_{1,Q_j}(0)-C_1(0)C'_{n,Q_j}(0)) \end{bmatrix}_{1\le j\le n}.$$

Take the first row multiplied by $-2$ and add it to the second row. We divide the second row by $(d+n)$. Then the above matrix is similar to

$$\begin{bmatrix} (i_j+1-i_{j+n})d-(i_j+1+ni_{j+n}) \\ i_{j+n}-i_{j+n-1} \\ \frac{1}{d}(4C_4(0)C'_{1,Q_j}(0)-C_1(0)C'_{4,Q_j}(0)) \\ \vdots \\ \frac{1}{d}(nC_n(0)C'_{1,Q_j}(0)-C_1(0)C'_{n,Q_j}(0)) \end{bmatrix}_{1\le j\le n}.$$

Considering the last matrix, we see that the $j$-th entry of its third row is

$$\begin{aligned}
&3(i_j+1)-(i_{j+n-2}+i_{j+n-1}+i_{j+n}))d^3+\\
&((n-3)(3(i_j+1)-(i_{j+n-1}+i_{j+n-1}+i_{j+n}))+\\
&-n(i_j+1+i_{j+n-2}+i_{j+n-1}+i_{j+n}))d^2+\\
&\left(\binom{n-2}{2}(3(i_j+1)-i_{j+n})-n(n-3)(i_j+1+i_{j+n-1}+i_{j+n})\right)d+\\
&-(i_j+1+ni_{j+n})\binom{n-2}{2}.
\end{aligned}$$

We take the second row multiplied by $-(d^3+(2n-3)d^2+n(n-3)d)$ and add it to the third one. Next, we add $-\binom{n-2}{2}$ times the first row to the third one. Dividing by $d$ the third row, its $j$-th entry is

$$\begin{aligned}
&(3(i_j+1)-(i_{j+n-2}+2i_{j+n}))d^2+\\
&((n-3)(3(i_j+1)-2i_{j+n})-n(i_j+1+i_{j+n-2}+2i_{j+n}))d+\\
&-2(n-3)(i_j+1+ni_{j+n}).
\end{aligned}$$

We add $-2(n-3)$ times the first row to the third one and divide it by $d$. We then add $-3$ times the first row to the third one and divide it by $(d+n)$. The resulting third row has $j$-th entry

$$i_{j+n} - i_{j+n-2}.$$

In this way the matrix (4.7) is similar to

$$\begin{bmatrix} (i_j + 1 - i_{j+n})d - (i_j + 1 + n i_{j+n}) \\ i_{j+n} - i_{j+n-1} \\ i_{j+n} - i_{j+n-2} \\ \frac{1}{d}(5C_5(0)C'_{1,Q_j}(0) - C_1(0)C'_{5,Q_j}(0)) \\ \vdots \\ \frac{1}{d}(nC_n(0)C'_{1,Q_j}(0) - C_1(0)C'_{n,Q_j}(0)) \end{bmatrix}_{1\le j\le n}.$$

We proceed analogously with the following rows and notice that the matrix (4.7) has the same rank as

$$\begin{bmatrix} (i_j - i_{j+n})(d-1) + d - 1 - (n+1)i_{j+n} \\ i_{j+n} - i_{j+n-1} \\ \cdots \\ i_{j+n} - i_{j+2} \end{bmatrix}_{1\le j\le n}.$$

Since $d - 1 = i_j - i_{j+n} + i_{j+1} - i_{j+n} + \ldots + i_{j+n-1} - i_{j+n} + (n+1)i_{j+n}$, the above matrix is similar to $M_{d,I}$. $\square$

We can now estimate the rank of the local Baum-Bott map at the Jouanolou foliation.

**Proof of Theorem 2.14.** By Lemma 4.1 we need to find $\operatorname{rank} T_1|_{E_j}$ for each eigenspace $E_j$ of $\mathbf{A}^*$. From Theorem 3.8 we see that there are only eigenspaces of dimension 1, 2, $n$ and $n+1$. We will examine each case.

$(a)$ We will see that

$$\sum_{\dim E_j = 1} \operatorname{rank}(T_1|_{E_j}) = (n+1)\left[\binom{n+d-1}{n-1} - 3(n-1)\right].$$

From item 4 of Theorem 3.8, eigenspaces of dimension 1 are generated by $(\mathbb{S}^*)^k(x^{\mathbb{I}}\partial_1)$ for $k = 0, \ldots, n$ such that $x^{\mathbb{I}} \notin \mathsf{D}$. Using Lemmas 3.6 and 4.2 yields

$$\sum_{\dim E_i = 1} \operatorname{rank}(T_1|_{E_i}) = (n+1)\sum_{x^{\mathbb{I}}\notin \mathsf{D}} \operatorname{rank}(T_1(x^{\mathbb{I}}\partial_1)).$$

Lemma 4.4 implies that $x^{\mathbb{I}}\partial_1 \notin \ker(T_1)$, for every $x^{\mathbb{I}} \notin \mathsf{D}$. Then the assertion follows.

$(b)$ We will show that the total rank at the eigenspaces of dimension 2 of $\mathbf{A}^*$ is

$$\sum_{\substack{j=1 \\ \dim E_j = 2}}^{N} \operatorname{rank} T_1|_{E_j} = \begin{cases} (n+1)(2n-3) & \text{, if } n \text{ is even.} \\ (n+1)(2n-4) + \dfrac{n+1}{2} & \text{, if } n \text{ is odd.} \end{cases}$$

Remember that $\mathcal{E}_r^2 =< x_2^{d-1}x_r\partial_1, x_{1+r}^d\partial_2 >$ and $\mathcal{E}^r =< x_r^{d-1}x_{n+1}\partial_1, x_1^{d-1}x_{r-1}\partial_r >$. From item 3 of Theorem 3.8 and Lemma 4.2 we have:

If $n$ is even, then

$$\sum_{\substack{j=1 \\ \dim E_j = 2}}^{N} \operatorname{rank} T_1|_{E_j} = (n+1)\left\{\sum_{r=3}^{\frac{n+2}{2}} \operatorname{rank} T_1|_{\mathcal{E}^r} + \sum_{r=3}^{n+1} \operatorname{rank} T_1|_{\mathcal{E}_r^2}\right\}.$$

If $n$ is odd, then

$$\sum_{\substack{j=1\\ \dim E_j=2}}^{N} \operatorname{rank} T_1|_{E_j} = (n+1)\left\{\sum_{r=3}^{\frac{n+1}{2}} \operatorname{rank} T_1|_{\mathcal{E}^r} + \sum_{r=3}^{n+1} \operatorname{rank} T_1|_{\mathcal{E}^2_r}\right\} + \frac{n+1}{2}\operatorname{rank} T_1|_{\mathcal{E}^{\frac{n+3}{2}}}.$$

We have three cases:

$(b.1)$ We will see that $\operatorname{rank} T_1|_{\mathcal{E}^2_r} = 1$, for $r = 3,\dots,n+1$. Indeed, by Lemma 4.2 we have

$$\operatorname{rank} T_1|_{\mathcal{E}^2_r} = \operatorname{rank} T_1|_{\langle x_r x_2^{d-1}\partial_1, x_r^d\partial_1\rangle}.$$

Let us define

$$Q(x_1,\dots,x_n) = \begin{cases} dx_r x_2^{d-1} - x_r^d - (d-1)x_2^d & \text{, if } r = 3,\dots,n.\\ dx_2^{d-1} - 1 - (d-1)x_2^d & \text{, if } r = n+1.\end{cases}$$

Observe that $p_1 \in \operatorname{Sing}(\mathrm{X}_{\mathcal{J}_d} + tQ\partial_1)$, for all $t \in \mathbb{C}$. Since

$$\det(\lambda \mathrm{Id} + D(\mathrm{X}_{\mathcal{J}_d} + tQ\partial_1)(p_1)) = \det(\lambda \mathrm{Id} + D\mathrm{X}_{\mathcal{J}_d}(p_1)),$$

we see that $BB(\mathrm{X}_{\mathcal{J}_d} + tQ\partial_1, p_1) = BB(\mathrm{X}_{\mathcal{J}_d}, p_1)$, for all $t \in \mathbb{C}$. Hence $Q\partial_1 \in \ker(T_1)$. From Lemma 4.4 we conclude that $\operatorname{rank} T_1|_{\mathcal{E}^2_r} = 1$, for $r = 3,\dots,n+1$.

$(b.2)$ We assert that if $n = 2K+1$ or $n = 2K$ for some positive integer $K$, then

$$\operatorname{rank} T_1|_{\mathcal{E}^r} = 2, \text{ for } r = 3,\dots,K+1.$$

By Lemma 4.2, $T_1(x_{r-1}x_1^{d-1}\partial_r) = T_1(x_{n+1}x_{n-r+3}^{d-1}\partial_1)$. Since $3 \le r \le K+1$, the polynomials $x_{n-r+3}^{d-1}x_{n+1}$ and $x_r^{d-1}x_{n+1}$ are different. Suppose there exist some $\alpha \in \mathbb{C}$ such that $T_1(x_r^{d-1}x_{n+1}\partial_1) = \alpha T_1(x_{n-r+3}^{d-1}x_{n+1}\partial_1)$. Let $Q_1 = x_r^{d-1}x_{n+1} - x_2^d$, and $Q_2 = x_{n-r+3}^{d-1}x_{n+1} - x_2^d$. Then

$$\partial_2 Q_1 = -d,\ \partial_r Q_1 = d-1,\ \partial_2 Q_2 = -d \text{ and } \partial_{n-r+3}Q_2 = d-1.$$

We see that $T_1(Q_1\partial_1) = \alpha T_1(Q_2\partial_1)$. Observe that $p_1 \in \operatorname{Sing}(\mathrm{X}_{\mathcal{J}_d} + tQ_j\partial_1)$, for $j = 1,2$ and $t \in \mathbb{C}$. Let us denote

$$C_{i,Q_j}(t) = C_i(D(X_{\mathcal{J}_d} + tQ_j\partial_1)(p_1)).$$

Using Lemma 4.3 we obtain $C'_{i,Q_1}(0) = \alpha C'_{i,Q_2}(0)$, for $i = 2,\dots,n$. Therefore

$$C'_{2,Q_1}(0) = C'_{2,Q_2}(0) = -d$$

and we conclude that $\alpha = 1$. Hence $0 = C'_{r,Q_1}(0) - C'_{r,Q_2}(0) = (-1)^r(d-1)d^{r-1}$, a contradiction.

$(b.3)$ Lemmas 4.2 and 4.4 imply that

$$\operatorname{rank} T_1|_{\mathcal{E}^{\frac{n+3}{2}}} = \dim\left\langle T_1\left(x_{\frac{n+3}{2}}^{d-1}x_{n+1}\partial_1\right)\right\rangle = 1.$$

$(c)$ From Theorem 3.8 each eigenspace of dimension $n$ is generated by

$$\{x_j x^I \partial_j \mid j = 1,\dots,n\}, \text{ where } I \in \mathbb{Z}^n_{\ge 0} \text{ and } |I| \le d-1.$$

We use Lemma 4.5 to calculate the rank of $T_1$ restricted to those eigenspaces.

$(d)$ Theorem 3.8 shows that there is only one eigenspace of dimension $n+1$, which is $E_d$. By Lemma 4.2, it is contained in the kernel of $T_1$. $\square$

For degree-2 foliations on even-dimensional projective spaces, we estimate the generic rank of the Baum-Bott map using the Jouanolou foliation.

**Proof of Theorem 2.15.** Let $n \geq 4$ be an even number. It suffices to show that the rank of the local Baum-Bott map $BB : \mathcal{F}\mathrm{ol}_{\mathrm{red}}(n, 2) \to (\mathbb{C}^{n-1})^N$ at the degree-2 Jouanolou foliation is equal to the upper bound given in Proposition 2.7. By Theorem 2.14, if we prove that $\operatorname{rank} M_{2,I} = n - 1$, the assertion follows. Denote $O_{k\times j}$ the zero matrix $k \times j$. We observe that

$$M_{d,I} = \begin{bmatrix} d+1 & d & 0_{1\times(n-2)} \\ 0_{(n-2)\times 1} & 0_{(n-2)\times 1} & \mathrm{Id}_{n-2} \end{bmatrix} \begin{bmatrix} i_{j+1} - i_j \\ i_{j+2} - i_{j+1} \\ \vdots \\ i_{j+n} - i_{j+n-1} \end{bmatrix}_{1\leq j\leq n},$$

where $i_1 + \ldots + i_{n+1} = d - 1$. We only need to show that

$$\det[(i_{j+i} - i_{j+i-1})]_{1\leq i,j\leq n} \neq 0.$$

Set

$$M_k = \begin{bmatrix} -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & -1 & 1 \\ 0 & 0 & 0 & \ldots & 0 & -1 \end{bmatrix}_{k\times k},$$

and let us denote by $M_{n-k}^T$ the transpose of the matrix $M_{n-k}$. Since $d = 2$ and $i_1 + \ldots + i_{n+1} = 1$, it follows that

$$\det[(i_{j+i} - i_{j+i-1})]_{1\leq i,j\leq n} = \begin{cases} (-1)^{\frac{(n-1)n}{2}} \begin{vmatrix} 0_{k\times n-k} & M_k \\ -M_{n-k}^T & 0_{(n-k)\times k} \end{vmatrix} & \text{, if } i_k = 1 \text{ and } 1 \leq k \leq n. \\ -(-1)^{\frac{(n-1)n}{2}} M_n^T & \text{, if } i_{n+1} = 1. \end{cases}$$

In any case, we conclude that $\det[(i_{j+i} - i_{j+i-1})]_{1\leq i,j\leq n} \neq 0$. ☐

**Proof of Theorem 2.18.** Let $I = (i_1, i_2, i_3) \in \mathbb{Z}^3_{\geq 0}$ such that $|I| \leq d - 1$. By Theorem 2.14, we have to calculate the rank of

$$M_{d,I} = \begin{bmatrix} d+1 & d & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i_{j+1} - i_j \\ i_{j+2} - i_{j+1} \\ i_{j+3} - i_{j+2} \end{bmatrix}_{1\leq j\leq 3}.$$

Let us analize the determinant of the matrix

$$\begin{bmatrix} i_2 - i_1 & i_3 - i_2 & i_4 - i_3 \\ i_3 - i_2 & i_4 - i_3 & i_1 - i_4 \\ i_4 - i_3 & i_1 - i_4 & i_2 - i_1 \end{bmatrix}$$

If $a = i_3 - i_1$, $b = i_4 - i_2$ and $c = i_3 - i_2$, then the determinant of the above matrix is $-(a + b - 2c)(a^2 + b^2)$. If the determinant is equal to zero, we have two cases.

First case: $a = b = 0$. This implies $i_3 = i_1$, $i_4 = i_2$. If $\operatorname{rank} M_{d,I} = 1$, then $d = 2r + 1$ and $i_1 \neq i_2$, where $r = i_1 + i_2$. If $\operatorname{rank} M_{d,I} = 0$, then $r = i_1 = i_2 = i_3 = i_4$ and $d = 4r + 1$.

Second case: $a + b - 2c = 0$. This means that $i_4 = i_3 - i_2 + i_1$. If $\operatorname{rank} M_{d,I} = 0$, then $i_1 = i_2 = i_3 = i_4 = r$ and $d = 4r + 1$. If $\operatorname{rank} M_{d,I} > 0$, then $\operatorname{rank} M_{d,I} = 2$.

In view of the results, if $d = 4r + 1$, there are $2r$ eigenspaces of dimension 3 such that the rank of $T_1$ restricted to each of those eigenspaces is one, and there is one eigenspace of dimension 3 that is in the kernel of the linear map $DBB(\mathrm{X}_{\mathcal{J}_d}, p_1)$. If $d = 4r + 3$, there are $2r + 2$ eigenspaces of dimension 3 such that the rank of the linear map $T_1$ restricted to each of those eigenspaces is one. We obtain the result by replacing this information in Theorem 2.14. ☐

In some cases it appears another eigenspace that is in the kernel of $T_1$.

**Corollary 4.6.** *Let $n, d \geq 3$. If there exists $r \in \mathbb{Z}_{>0}$ such that $d = (n+1)r + 1$, then $x_j x_1^r x_2^r \ldots x_n^r \partial_j$ is in the kernel of $T_1 : \mathbf{V}_d \to \mathbb{C}^{n-1}$, for $j = 1, \ldots, n$.*

**Remark 4.7.** The Baum-Bott Residues Theorem 2.1 holds for higher dimensional foliations on complex manifolds. More precisely, let $\mathcal{F}$ be a singular holomorphic foliation of dimension $p$ on a compact complex manifold $M$ of dimension $n$ and $\phi$ be a homogeneous symmetric polynomial of degree $n-p+1$. Suppose that the singular set $\mathrm{Sing}(\mathcal{F})$ of $\mathcal{F}$ has dimension at most $p-1$. Let $Z$ be an irreducible component of dimension $p-1$ of $\mathrm{Sing}(\mathcal{F})$ satisfying Baum-Bott's generic condition. Let $B_q$ be a ball centered at $q$ of dimension $n-p+1$, sufficiently small and transversal to $Z$ in $q$. Then, the Baum-Bott residue is given by

$$\mathrm{Res}_\phi(\mathcal{F}, Z) = \mathrm{Res}_\phi(\mathcal{F}|_{B_q}; q)[Z],$$

where $\mathrm{Res}_\phi(\mathcal{F}|_{B_q}; q) \in \mathbb{C}$ represents the Grothendieck residue at $q$ of the one dimensional foliation $\mathcal{F}|_{B_q}$ on $B_q$, see [2, Theorem 3]. In a recent work, Corrêa and Lourenço [5] showed that the identity above holds regardless Baum-Bott's generic condition. Now, consider the component $\mathcal{H}$ of the space of foliation of dimension $p$ and degree $d$ on $\mathbb{P}^n$ whose generic element is the linear pullback of a one-dimensional foliation of degree $d$ on $\mathbb{P}^{n-p+1}$, see [7]. Therefore, since $\mathrm{Res}_\phi(\mathcal{F}|_{B_q}; q)$ does not depend on the choices, we have a well defined Baum-Bott map

$$BB(\mathcal{H}) : \mathcal{H} \dashrightarrow (\mathbb{P}^{n-p})^N / S_N$$

defined in the natural way, where $N$ denotes the number of singularities of a foliation of degree $d$ on $\mathbb{P}^{n-p+1}$. The rank of $BB(\mathcal{H})$ is clearly the same of

$$BB : \mathcal{F}\mathrm{ol}(n-p+1, d) \dashrightarrow (\mathbb{P}^{n-p})^N / S_N.$$

**Remark 4.8.** It will be interesting to consider the problem of determining the rank of the orbital Baum-Bott map on weighted projective spaces $\mathbb{P}(a_0, \ldots, a_n)$, by using the Baum-Bott residues theorem in the complex orbifold context, which follows from [6].

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