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COMBINATORIALLY DETERMINED ZEROES OF BERNSTEIN–SATO IDEALS FOR TAME AND FREE ARRANGEMENTS

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Abstract. For a central, not necessarily reduced, hyperplane arrangement f equipped with any factorization $f = f_1 \cdots f_r$ and for f' dividing f, we consider a more general type of Bernstein–Sato ideal consisting of the polynomials $B(S) \in \mathbb{C}[s_1, \ldots, s_r]$ satisfying the functional equation $B(S)f' f_1^{s_1} \cdots f_r^{s_r} \in \mathrm{A}_n(\mathbb{C})[s_1, \ldots, s_r] f_1^{s_1+1} \cdots f_r^{s_r+1}$.

Generalizing techniques due to Maisonobe, we compute the zero locus of the standard Bernstein–Sato ideal in the sense of Budur (i.e. $f' = 1$) for any factorization of a free and reduced f and for certain factorizations of a non-reduced f. We also compute the roots of the Bernstein–Sato polynomial for any power of a free and reduced arrangement. If f is tame, we give a combinatorial formula for the roots lying in $[-1, 0)$.

For $f' \neq 1$ and any factorization of a line arrangement, we compute the zero locus of this ideal. For free and reduced arrangements of larger rank, we compute the zero locus provided $\deg(f') \leq 4$ and give good estimates otherwise. Along the way we generalize a duality formula for $\mathscr{D}_{X,\mathfrak{x}}[S]f' f_1^{s_1} \cdots f_r^{s_r}$ that was first proved by Narváez-Macarro for f reduced, $f' = 1$, and $r = 1$.

As an application, we investigate the minimum number of hyperplanes one must add to a tame f so that the resulting arrangement is free. This notion of freeing a divisor has been explicitly studied by Mond and Schulze, albeit not for hyperplane arrangements. We show that small roots of the Bernstein–Sato polynomial of f can force lower bounds for this number.

1. Introduction

Consider a central, not necessarily reduced, hyperplane arrangement cut out by

$$f \in \mathbb{C}[X] = \mathbb{C}[x_1, \ldots, x_n].$$

Given a factorization $f = f_1 \cdots f_r$, not necessarily into linear terms, and letting $F = (f_1, \ldots, f_r)$, there is a free $\mathbb{C}[X][\frac{1}{f}][s_1, \ldots, s_r]$-module generated by the symbol $F^S = f_1^{s_1} \cdots f_r^{s_r}$. This module has an $\mathrm{A}_n(\mathbb{C})[S] = \mathrm{A}_n(\mathbb{C})[s_1, \ldots, s_r]$-module structure, where $\mathrm{A}_n(\mathbb{C})[S]$ is a polynomial ring extension over the Weyl algebra, given by the formal rules of calculus. We will denote the $\mathrm{A}_n(\mathbb{C})[S]$-module generated by F^S as $\mathrm{A}_n(\mathbb{C})[S]F^S$. For f' and $g \in \mathbb{C}[X]$ dividing f we study the polynomials $B(S) \in \mathbb{C}[S] = \mathbb{C}[s_1, \ldots, s_r]$ satisfying the functional equation

$$B(S)f'F^S \in \mathrm{A}_n(\mathbb{C})[S]gf'F^S. \tag{1.1}$$

The ideal populated by said polynomials is the *Bernstein–Sato ideal* $B_{f'F}^g$. When $f' = 1$ and $g = f$ this defines the multivariate Bernstein–Sato ideal in the sense of Budur [7] and we simply write B_F; if we further restrict to the trivial factorization $F = (f)$ then we obtain the classical functional equation whose corresponding ideal, which we denote by B_f, has as its monic generator the *Bernstein–Sato polynomial*.

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The roots of the Bernstein–Sato polynomial encode various data about the singular locus of f. Malgrange and Kashiwara, cf. [19], [15], famously proved that exponentiating the local version of the Bernstein–Sato polynomial's roots recovers the eigenvalues of the algebraic monodromy action on nearby Milnor fibers. In [7], Budur conjectured the analogous claim for the multivariate Bernstein–Sato ideal B_F associated to a factorization of f into irreducibles: exponentiating the ideal's zero locus recovers the cohomology support locus of the complement of $\mathrm{Var}(f)$. A proof of this (for germs f that need not be arrangements) has recently been announced by Budur, Veer, Wu, and Zhou, cf. [8]. Beyond these monodromy results, zeroes of Bernstein–Sato polynomials are related to many other invariants: multiplier ideals, log canonical thresholds, F-pure thresholds, etc.

However, even in the case of arrangements, formulae for Bernstein–Sato ideals, polynomials, or their zero loci are very rare. Walther has found a formula for the Bernstein–Sato polynomial for generic arrangements in [28], Maisonobe has shown the Bernstein–Sato ideal B_F for a generic arrangement factored into linear forms is principal and found the corresponding formula for a generator, cf. [17], and Saito has shown that the roots of the Bernstein–Sato polynomial of a reduced and central arrangement f lie in $(-2+\frac{1}{\deg(f)}, 0) \cap \mathbb{Q}$, cf. [23]. On the other hand, Walther has shown that, in general, the roots of the Bernstein–Sato polynomial are not combinatorially determined, that is, they cannot be computed from the arrangement's intersection lattice, cf. [29] and Example 4.22. The multivariate Bernstein–Sato ideal B_F is not even guaranteed to be principal, cf. [2] for a counter-example in the local case. To our knowledge, there are no systematic studies of the more general type of Bernstein–Sato ideal $B_{f'F}^{g}$ though it does play a role in [28].

Our starting point is the program of Maisonobe in [18] wherein he proves the Bernstein–Sato ideal of a central, reduced, and free (in the sense of Saito [22]) arrangement equipped with its factorization into linear forms is principal and gives a combinatorial formula for its generator. While the approach is similar, we encounter many technical difficulties because our results are significantly more general: we consider the more general functional equation (1.1) and we often relax the assumptions of f being factored into linear forms, being free, and being reduced.

In Section 2, we consider a larger class of analytic germs $f \in \mathscr{O}_X$ than just central, reduced, and free arrangements and we consider any factorization $f = f_1 \cdots f_r$. In [3], we proved that $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S$ is generated by derivations, that is, by differential operators of order at most one under a natural filtration, under the hypotheses of tameness (a sliding condition on projective dimension), strongly Euler-homogeneous (a hypothesis that a particular logarithmic derivation exists locally everywhere), and Saito-holonomicity (a finiteness condition on the logarithmic stratification). We use similar techniques to generalize these results from [3] in Theorem 2.21:

Theorem 1.1. *Suppose $f = f_1 \cdots f_r$ is tame, strongly Euler-homogeneous, and Saito-holonomic, $f' \in \mathscr{O}_{X,\mathfrak{x}}[\frac{1}{f}]$ is compatible with f, and $F = (f_1, \ldots, f_r)$. Then the $\mathscr{D}_{X,\mathfrak{x}}[S]$-annihilator of $f'F^S$ is generated by derivations.*

In Section 3, we replace the hypothesis of tame with free and prove a version of the symmetry of $B_{f'F}^{g}$ that was first identified by Narváez-Macarro in [21] in the case of Bernstein–Sato polynomials and generalized to B_F by Maisonobe in [18]. This follows from computing the $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual of $\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S$. Without freeness, computing these $\mathscr{D}_{X,\mathfrak{x}}[S]$-duals is currently intractible. While we are certain one could use Narváez-Macarro's Lie-Rinehart strategy, we instead opt for Maisonobe's approach, which itself relies on a computation of the trace of an adjoint action first proved by Castro–Jiménez and Ucha in Theorem 4.1.4 of [9]; we give a different proof of this in Appendix A. With $\mathbb{D}$ denoting the $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual $\mathrm{RHom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(-, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}}$, in Theorem 3.9 we prove:

Theorem 1.2. *Suppose $f = f_1 \cdots f_r \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, and Saito-holonomic and $f_{\text{red}} \in \mathscr{O}_{X,\mathfrak{x}}$ is a Euler-homogeneous reduced defining equation for f at $\mathfrak{x}$. Let $F = (f_1, \ldots, f_r)$, let $f' \in \mathscr{O}_{X,\mathfrak{x}}$ be compatible with f, and let $g \in \mathscr{O}_{X,\mathfrak{x}}$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g$. Then*

$$\mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]\cdot gf'F^S}\right) \simeq \frac{\mathscr{D}_{X,\mathfrak{x}}[S](gf'f_{\text{red}})^{-1}F^{-S}}{\mathscr{D}_{X,\mathfrak{x}}[S](f'f_{\text{red}})^{-1}F^{-S}}[n+1].$$

The main application is Theorem 3.16 which identifies technical conditions on f', g, and F such that $B^g_{f'F}$ is invariant under a non-trivial involution of $\mathbb{C}[S]$.

In Section 4 we return to hyperplane arrangements and first show that the nice structure of $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}S} f'F^S$ from Theorem 2.21 allows us to adapt Maisonobe's arguments to estimate $B^g_{f'F}$ for any factorization. In particular we complement Walther's result that the roots of Bernstein–Sato polynomial are not combinatorial for even tame arrangements, cf. [29]. Namely, we prove in Theorem 4.21 the roots lying in $[-1, 0)$ are combinatorial:

Theorem 1.3. *Let f be a central, not necessarily reduced, tame hyperplane arrangement. Suppose f' divides f; let $g = \frac{f}{f'}$. Then the roots $\mathrm{V}(B^g_{f'f})$ lying in $[-1, 0)$ are combinatorially determined:*

$$\mathrm{V}(B^g_{f'f}) \cap [-1, 0) = \bigcup_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \bigcup_{j_X = r(X) + d'_X}^{d_X} \frac{-j_X}{d_X}.$$

Setting $f' = 1$ gives all the roots of the Bernstein–Sato polynomial of f lying in $[-1, 0)$.

If we assume further that f is free, then we can use the symmetry property of Theorem 3.16 to more accurately estimate $\mathrm{V}(B^g_{f'F})$, where $\mathrm{V}(-)$ always refers to the zero locus of the ideal in question. In this setting there is a computation for the multivariate Bernstein–Sato ideal of a reduced, free f that has been factored into linear forms due to Maisonobe [18], but no results about other factorizations, non-reduced f, or even the Bernstein–Sato polynomial. We fill in much of this gap. With $P^g_{f'F,X} \in \mathbb{C}[S]$ the explicit linear polynomial from Definition 4.10, we obtain the following, which in particular shows that the roots of the Bernstein–Sato polynomial for any power of a reduced, central, and free arrangement are combinatorially determined:

Theorem 1.4. *Suppose $f = f_1 \cdots f_r$ is a central, not necessarily reduced, free hyperplane arrangement, $F = (f_1, \cdots, f_r)$, f' divides f, and $g = \frac{f}{f'}$. If (f', F) is an unmixed pair up to units and if $\deg(f') \leq 4$, then $\mathrm{V}(B^g_{f'F})$ is a hypersurface and*

$$\mathrm{V}(B^g_{f'F}) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X = 0}^{d_{X,\text{red}} + d_X - 2r(X) - d'_X} \left(P^g_{f'F,X} + j_X\right)\right). \tag{1.2}$$

If L is a factorization of $f = l_1 \cdots l_d$ into irreducibles and $\deg(f') \leq 4$, then

$$B^g_{f'L} = \prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X = 0}^{d_{X,\text{red}} + d_X - 2r(X) - d'_X} \left(P^g_{f'L,X} + j_X\right)$$

and so $B^g_{f'L}$ principal. If $f' = 1$ and f is reduced, then for any F

$$\mathrm{V}(B_F) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X = 0}^{d_{X,\text{red}} + d_X - 2r(X)} \left(P^g_{F,X} + j_X\right)\right). \tag{1.3}$$

In particular, if f is reduced or is a power of a central, reduced, and free hyperplane arrangement, then the roots of the Bernstein–Sato polynomial of f are given by (1.3)*.*

In Remark 4.28 we discuss how to use new results to get a combinatorial formula for the roots of the Bernstein–Sato polynomial corresponding to any central, free f, that is, to f that may not be a power of a reduced arrangement. In the case of line arrangements, we are also able to compute $\mathrm{V}(B_{f'F}^{g})$ for any suitable choice of f', g, and F without the technical condition of unmixed up to units, cf. Theorem 4.25 and Definition 3.14.

Unfortunately our methods are not appropriate for determining the multiplicity of roots of the Bernstein–Sato polynomial so we cannot conclude this polynomial is combinatorial for free arrangements. These multiplicities are mysterious, although in [23] Saito proves various results about them in the general (i.e. in the non-free) setting. Notably he shows that -1 has multiplicity equal to the arrangement's rank.

In Section 5 we make use of our results involving the more general functional equation (1.1) to study the smallest arrangement $\mathrm{V}(f')$ that when added to the arrangement $\mathrm{V}(g)$ makes $\mathrm{V}(f'g)$ free, i.e. the smallest arrangement f' that *frees* g. For arbitrary divisors g, it is unknown whether or not such a divisor f' exists. There are some positive results, but the methodologies are very particular to the type of divisors considered. For example, Mond and Schulze identified certain classes of germs that are freed by a adjoint divisors–these germs are related to discrimants of versal deformations, cf. [20]. Other cases of freeing divisors are considered in [25] and [6]. However, Yoshinaga [30] has communicated to us a way, based on the combinatorics of g, to find an arrangement f' that frees an arrangement g. In Theorem 5.4 we prove the degree of f' is related to roots of the Bernstein–Sato polynomial of g.

Theorem 1.5. *Suppose that g is a central, reduced, tame hyperplane arrangement of rank n, v an integer such that $1 < v \leq n-1$, and* $\deg(g)$ *is co-prime to v. If $\frac{-2\deg(g)+v}{\deg(g)}$ is a root of the Bernstein–Sato polynomial of g and if f' is a central arrangement that frees g, then* $\deg(f') \geq n - v$.

In Appendix B we prove a conjecture of Budur's in the case of central, reduced, and free hyperplane arrangments. The recently announced paper [8] gives a general proof using entirely different methods.

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2. Bernstein–Sato Ideals and the $\mathscr{D}_{X,\mathfrak{r}}[S]$-module $\mathscr{D}_{X,\mathfrak{r}}[S]f'F^S$

In this section we introduce some of our working hypotheses on $f \in \mathscr{O}_X$. These are needed to utilize results from [3] and [29] which will be needed throughout the paper. We generalize Theorem 2.29 of [3] and discuss how Bernstein–Sato varieties attached to different factorizations of f relate to each other.

2.1. Hypotheses on f.

Let X be a smooth analytic space or $\mathbb{C}$-scheme of dimension n and $\mathscr{O}_X$ be the analytic structure sheaf. Pick $f \in \mathscr{O}_X$ to be regular with divisor $Y = \mathrm{Div}(f)$ and ideal sheaf $\mathscr{I}_Y$. In general, we make no reducedness assumption on Y.

Definition 2.1. Let $\mathrm{Der}_X(-\log Y)$ be the $\mathscr{O}_X$-sheaf of *logarithmic derivations on Y*, that is, the sheaf generated locally by the vector fields δ such that $\delta \bullet \mathscr{I}_Y \subseteq \mathscr{I}_Y$. If $Y = \mathrm{Div}(f)$ then we also label $\mathrm{Der}_X(-\log f) = \mathrm{Der}_X(-\log Y)$. Define the *derivations that kill f* to be

$$\mathrm{Der}_X(-\log_0 f) = \{\delta \in \mathrm{Der}_X(-\log f) \mid \delta \bullet f = 0\}.$$

Remark 2.2. (a) It is easily checked that $\mathrm{Der}_X(-\log Y)$ depends on $\mathscr{I}_Y$ and not the choice of generators of $\mathscr{I}_Y$.
(b) By Lemma 3.4 of [13], $\mathrm{Der}_{X,\mathfrak{x}}(-\log fg) = \mathrm{Der}_{X,\mathfrak{x}}(-\log f) \cap \mathrm{Der}_{X,\mathfrak{x}}(-\log g)$. This is not always true when restricting to derivations that kill f.
(c) $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ is closed under taking commutators.

At points we will be interested in when $\mathrm{Der}_X(-\log Y)$ has a particularly nice structure.

Definition 2.3. The divisor $Y = \mathrm{Div}(f)$ is *free* when $\mathrm{Der}_X(-\log Y)$ is locally everywhere a free $\mathscr{O}_X$-module. Similarly $f \in \mathscr{O}_{X,\mathfrak{x}}$ is free when $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ is a free $\mathscr{O}_{X,\mathfrak{x}}$-module.

In [22], Saito introduced the logarithmic differential forms which are, in some sense, a dual notion to logarithmic derivations.

Definition 2.4. Let Ω_X^k be the sheaf of differential k-forms on X and $d : \Omega_X^k \to \Omega_X^{k+1}$ the standard differential. Define the sheaf of *logarithmic k-forms along f* by

$$\Omega_X^k(\log f) = \{w \in \frac{1}{f}\Omega_X^k \mid df \wedge w \in \Omega_X^{k+1}\}.$$

An element $f \in \mathscr{O}_X$ is *tame* if the projective dimension of the logarithmic k-forms along f is at most k in each stalk. A divisor Y is tame if it locally everywhere admits tame defining equations.

Remark 2.5. (a) The logarithmic 1-forms are dual to the logarithmic differentials:

$$\mathrm{Hom}_{\mathscr{O}_{X,\mathfrak{x}}}(\mathrm{Der}_{X,\mathfrak{x}}(-\log f), \mathscr{O}_{X,\mathfrak{x}}) \simeq \Omega_X^1(\log f).$$

When f is free, $\Omega_X^k(\log f) \simeq \bigwedge^k \Omega_X^1(\log f)$, cf. 1.6 and page 270 of [22].
(b) If $\dim(X) = n \leq 3$ then any divisor Y is automatically tame. This follows from the reflexivity of logarithmic k-forms, cf. [22].

The logarithmic derivations can also be used to stratify X:

Definition 2.6. (Compare to 3.3 and 3.8 of [22]) There is a relation on X induced by the logarithmic derivations along Y. Two points $\mathfrak{x}$ and $\mathfrak{y}$ are equivalent if there exists an open U containing them and a $\delta \in \mathrm{Der}_U(-\log Y \cap U)$ such that: (i) δ vanishes nowhere on U; (ii) an integral curve of δ passes through $\mathfrak{x}$ and $\mathfrak{y}$. The transitive closure of this relation stratifies X into equivalence classes whose irreducible components are the *logarithmic strata*. These strata constitute the *logarithmic stratification*.

We say Y is *Saito-holonomic* when the logarithmic stratification is locally finite.

Example 2.7. By 3.14 of [22] hyperplane arrangements are Saito-holonomic.

Finally, we define some homogeneity conditions on $f \in \mathscr{O}_X$.

Definition 2.8. We say $f \in \mathscr{O}_{X,\mathfrak{x}}$ is *Euler-homogeneous* when there exists $\delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ such that $\delta \bullet f = f$. If δ may be picked to vanish at $\mathfrak{x}$, then f is *strongly Euler-homogeneous*.

The element $f \in \mathscr{O}_X$ is (strongly) Euler-homogeneous if it is so at each point. The divisor Y is (strongly) Euler-homogeneous if it locally everywhere admits a defining equation that is (strongly) Euler-homogeneous.

Remark 2.9. If $f \in \mathscr{O}_{X,\mathfrak{x}}$ and $u \in \mathscr{O}_{X,\mathfrak{x}}$ is a unit, then f is strongly Euler-homogeneous if and only if uf is, cf. Remark 2.8 of [29].

Example 2.10. Hyperplane arrangements are strongly Euler-homogeneous.

Our working hypotheses on f will often be "tame, strongly Euler-homogeneous, and Saito-holonomic" or "free, strongly Euler-homogeneous, and Saito-holonomic." In light of Examples 2.7 and 2.10, if f cuts out a hyperplane arrangement only tameness or freeness need be assumed.

2.2. **The $\mathscr{D}_{X,\mathfrak{x}}[S]$-Annihilator of $f'F^S$.**

Let $\mathscr{D}_X$ be the sheaf of $\mathbb{C}$-linear differential operators with coefficients in $\mathscr{O}_X$ and $\mathscr{D}_X[S]$ be the polynomial ring extension induced by adding r central variables $S = s_1, \dots, s_r$.

Definition 2.11. Consider the free $\mathscr{O}_X[S][\frac{1}{f}]$-module generated by the symbol $F^S = f_1^{s_1} \cdots f_r^{s_r}$. This is endowed with a $\mathscr{D}_X[S]$-action by specifying the action of a $\mathbb{C}$-linear derivation δ on $\mathscr{O}_X$. For any $g \in \mathscr{O}_X[\frac{1}{f}]$, declare

$$\delta \bullet (s_i g F^S) = s_i(\delta \bullet g)F^S + s_i g(\sum_k \frac{\delta \bullet f_k}{f_k} s_k)F^S.$$

Let $\mathscr{D}_X[S]F^S$ be the $\mathscr{D}_X[S]$-module generated by F^S. For $g \in \mathscr{O}_X[\frac{1}{f}]$, let $\mathscr{D}_X[S]gF^S$ be the $\mathscr{D}_X[S]$-module generated by gF^S.

Remark 2.12. When executing the above construction with only one s, we use the notation $\mathscr{D}_X[s]f^s$. This is the classical, univariate situation.

In Proposition 2.7 of [3] we showed both that there is a canonical way to associate elements of $\operatorname{Der}_X(-\log f)$ to elements of $\operatorname{ann}_{\mathscr{D}_X[S]} F^S$ and that when f is tame, strongly Euler-homogeneous, and Saito-holonomic, $\operatorname{ann}_{\mathscr{D}_{X,x}[S]} F^S$ is generated by said elements. In this subsection we prove the analogous claims for $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$, provided f' is chosen such that $f^N f' \in \mathscr{O}_{X,\mathfrak{x}}$ and $f^M \in \mathscr{O}_{X,\mathfrak{x}} \cdot f^N f'$ for suitable choices of $N, M \geq 0$. First, we show how to associate elements of $\operatorname{Der}_{X,x}(-\log f)$ to $\operatorname{ann}_{\mathscr{D}_{X,x}[S]} f'F^S$ in an entirely similar way as in the prequel; second, we show that these elements generate $\operatorname{ann}_{\mathscr{D}_{X,x}[S]} f'F^S$ when f is tame, strongly Euler-homogeneous, and Saito-holonomic.

Definition 2.13. The *total order filtration* $F_{(0,1,1)}$ on $\mathscr{D}_{X,\mathfrak{x}}[S]$ assigns, in local coordinates, every ∂_{x_k} weight one, every s_k weight one, and every element of $\mathscr{O}_X$ weight zero. We will denote the elements of weight at most l by $F^l_{(0,1,1)}$ or $F^l_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$.

Definition 2.14. Write $f \in \mathscr{O}_{X,\mathfrak{x}}$ as $f = u l_1^{p_1} \cdots l_q^{p_q}$ where the l_t are pairwise distinct irreducibles, $p_t \in \mathbb{Z}_+$, and u is a unit in $\mathscr{O}_{X,\mathfrak{x}}$. We say $f' \in \mathscr{O}_{X,\mathfrak{x}}[\frac{1}{f}]$ is *compatible* with f if there exists a unit $u' \in \mathscr{O}_{X,\mathfrak{x}}$ and integers $v_t \in \mathbb{Z}$ such that

$$f' = u l_1^{v_1} \cdots l_q^{v_q}.$$

In this case, v_t is the *multiplicity of* l_t.

By Remark 2.2, if $f = u l_1^{p_1} \cdots l_q^{v_q}$ a factorization of f into irreducibles at $\mathfrak{x}$, u a unit, then if $\delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log f)$, $\frac{\delta \bullet l_t}{l_t} \in \mathscr{O}_{X,\mathfrak{x}}$. So for f' compatible with f,

$$\delta \bullet f'F^S = (\delta \bullet f')F^S + f'(\sum_k \frac{\delta \bullet f_k}{f_k} s_k)F^S = (\frac{\delta \bullet f'}{f'} + \sum_k \frac{\delta \bullet f_k}{f_k} s_k) f'F^S,$$

where $(\frac{\delta \bullet f'}{f'} + \sum_k \frac{\delta \bullet f_k}{f_k} s_k) \in \mathscr{O}_{X,\mathfrak{x}}[S]$. Indeed, $\frac{\delta \bullet f'}{f'} = \sum v_t \frac{\delta \bullet l_t}{l_t} \in \mathscr{O}_{X,\mathfrak{x}}$ and similarly $\frac{\delta \bullet f_k}{f_k} \in \mathscr{O}_X$.

Definition 2.15. Suppose f' is compatible with f. If $f = f_1 \cdots f_r$ and $F = (f_1, \dots, f_r)$, then there is a map of $\mathscr{O}_{X,x}$-modules

$$\psi_{f'F,\mathfrak{x}} : \operatorname{Der}_{X,x}(-\log f) \to \operatorname{ann}_{\mathscr{D}_{X,x}[S]} f'F^S \cap F^1_{(0,1,1)}$$

given by

$$\psi_{f'F,\mathfrak{x}}(\delta) = \delta - \sum_k \frac{\delta \bullet f_k}{f_k} s_k - \frac{\delta \bullet f'}{f'}.$$

The $\mathscr{O}_{X,\mathfrak{x}}$-module of *annihilating derivations* along $f'F$ is defined as

$$\theta_{f'F,\mathfrak{x}} = \psi_{f'F,\mathfrak{x}}(\mathrm{Der}_{X,\mathfrak{x}}(-\log f))$$

and $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$ is *generated by derivations* when

$$\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S = \mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}}.$$

When $f' = 1$ we write $\psi_{F,\mathfrak{x}}$ and $\theta_{F,\mathfrak{x}}$.

Arguing as in Proposition 2.7 of [3] we see that:

Proposition 2.16. (Compare to Proposition 2.7 of [3]) *Suppose f' is compatible with f. If $f = f_1 \cdots f_r$ and $F = (f_1, \ldots, f_r)$, then $\psi_{f'F,\mathfrak{x}}$ is an isomorphism.*

Proof. Suppose $\delta - \sum_k b_k s_k - b \in \mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F \cap F^1_{(0,1,1)}$ where $b_k, b \in \mathscr{O}_{X,\mathfrak{x}}$. Since $f'F^S$ generates a free $\mathscr{O}_{X,\mathfrak{x}}[S][\frac{1}{f}]$-module we deduce

$$(\sum_k \frac{\delta \bullet f_k}{f_k} s_k - b_k s_k) + (\frac{\delta \bullet f'}{f'} - b) = 0$$

and hence

$$\delta \in \bigcap_k \mathrm{Der}_{X,\mathfrak{x}}(-\log f_k) = \mathrm{Der}_{X,\mathfrak{x}}(-\log f).$$

So the map $\delta - \sum_k b_k s_k - b \mapsto \delta$ sends $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F \cap F^1_{(0,1,1)}$ to $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$. Its inverse is $\psi_{f'F,\mathfrak{x}}$. ☐

Remark 2.17. By definition, $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$ is closed under taking commutators; hence $\theta_{f'F,\mathfrak{x}}$ is as well. As $\psi_{f'F,\mathfrak{x}}$ is an isomorphism, a basic computation shows $\psi_{f'F,\mathfrak{x}}$ respects taking commutators.

In [3] we generalized an approach of Walther's in [29]: we looked at the associated graded object of $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S$ under the total order filtration $F_{(0,1,1)}$. As

$$\psi_{F,\mathfrak{x}}(\mathrm{Der}_{X,\mathfrak{x}}(-\log f)) \subseteq \mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S,$$

the following definition is natural:

Definition 2.18. Suppose f is strongly Euler-homogeneous. The *generalized Liouville ideal* $\widetilde{L_{F,\mathfrak{x}}} \subseteq \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ is generated by the symbols of elements in $\psi_F(\mathrm{Der}_{X,\mathfrak{x}}(-\log f))$ under the total order filtration. That is,

$$\widetilde{L_{F,\mathfrak{x}}} = \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \mathrm{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\mathrm{Der}_{X,\mathfrak{x}}(-\log f))).$$

Remark 2.19. (a) The strongly Euler-homogeneous assumption in the above definition ensures that algebraic properties of $\widetilde{L_{F,x}}$ do not depend on choice of defining equations for each f_k at x. See Remark 2.15 of [3] for details.

(b) By Corollary 2.28 of [3], if $f \in \mathscr{O}_X$ is tame, strongly Euler-homogeneous, and Saito-holonomic then $\widetilde{L_{F,\mathfrak{x}}} = \mathrm{gr}_{(0,1,1)}(\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} F^S)$.

(c) For $\delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f)$, note that

$$\begin{aligned}\mathrm{gr}_{(0,1,1)}(\psi_{f'F,\mathfrak{x}}(\delta)) &= \mathrm{gr}_{(0,1,1)}(\delta - \sum_k \frac{\delta \bullet f_k}{f_k} s_k - \frac{\delta \bullet f'}{f'}) \\ &= \mathrm{gr}_{(0,1,1)}(\delta - \sum_k \frac{\delta \bullet f_k}{f_k} s_k) \\ &= \mathrm{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta)).\end{aligned}$$

Since $\widetilde{L_{F,\mathfrak{x}}} \subseteq \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ has, by definition, generators

$$\{\mathrm{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta)) \mid \delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f)\},$$

we deduce

$$\begin{aligned}\widetilde{L_{F,\mathfrak{x}}} &= \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \{\mathrm{gr}_{(0,1,1)}(\psi_{f'F,\mathfrak{x}}(\delta)) \mid \delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f)\} \\ &= \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \mathrm{gr}_{(0,1,1)}(\theta_{f'F,\mathfrak{x}}) \\ &\subseteq \mathrm{gr}_{(0,1,1)}(\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]}\, f'F^S).\end{aligned}$$

By the preceding remark, $\widetilde{L_{F,\mathfrak{x}}}$ approximates $\mathrm{gr}_{(0,1,1)}(\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]}\, f'F^S)$. Arguing as in Corollary 2.28 of [3] we prove the approximation is in fact an equality:

Theorem 2.20. *Suppose $f = f_1 \cdots f_r$ is tame, strongly Euler-homogeneous and Saito-holonomic. Let $F = (f_1, \ldots, f_r)$ and suppose $f' \in \mathscr{O}_{X,\mathfrak{x}}[\frac{1}{f}]$ is compatible with f. Then*

$$\mathrm{gr}_{(0,1,1)}(\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]}\, f'F^S) = \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \mathrm{gr}_{(0,1,1)}(\theta_{f'F,\mathfrak{x}}).$$

Proof. For the first part of this proof we mimic Proposition 2.25 of [3]. In Definition 2.24 of loc. cit., we introduced a $\mathscr{O}_{X,\mathfrak{x}}$-linear ring homomorphism

$$\phi_{F,\mathfrak{x}} : \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \to R(\mathrm{Jac}(f_1), \ldots, \mathrm{Jac}(f_r)),$$

where $R(\mathrm{Jac}(f_1), \ldots, \mathrm{Jac}(f_r))$ is the multi-Rees algebra associated to the r Jacobian ideals $\mathrm{Jac}(f_1), \ldots, \mathrm{Jac}(f_r)$. Using local coordinates ∂_{x_i} and identifying $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ with $\mathscr{O}_{X,\mathfrak{x}}[Y][S]$ via $\mathrm{gr}_{(0,1,1)}(\partial_{x_i}) = y_i$, the map $\phi_{F,\mathfrak{x}}$ is given by

$$y_i \mapsto \sum_k \frac{f}{f_k}(\partial_{x_i} \bullet f_k) s_k \text{ and } s_k \mapsto f s_k.$$

Proposition 2.26 of loc. cit. shows $\ker(\phi_{F,\mathfrak{x}})$ is a prime ideal of dimension $n + r$.

Select $P \in \mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]}\, f'F$ of weight l under the total order filtration $F_{(0,1,1)}$. For any Q of weight l, $f^l Q \bullet f'F^S \in \mathscr{O}_{X,\mathfrak{x}}[S]F^S$. Now, for $g \in \mathscr{O}_{X,\mathfrak{x}}[S][\frac{1}{f}]$, write

$$\partial_{x_i} \bullet g f' F^S = (\partial_{x_i} \bullet g + g\frac{\partial_{x_i} \bullet f'}{f'} + g \sum_k \frac{\partial_{x_i} \bullet f_k}{f_k} s_k) f'F^S.$$

Thus, if applying a partial derivative to $gf'F^S$ causes the s-degree (under the natural filtration) of the $\mathscr{O}_{X,\mathfrak{x}}[S]$-coefficient of $f'F^S$ to increase, the terms of higher s-degree are precisely $g \sum_k \frac{\partial_{x_i} \bullet f_k}{f_k}$. A straightforward computation then shows that the S-lead term of $f^l Q \bullet f'F^S$ is exactly $\phi_{F,\mathfrak{x}}(\mathrm{gr}_{(0,1,1)}(Q)) f'F^S \in \mathscr{O}_{X,\mathfrak{x}}[S] f'F^S$. Since $f'F^S$ generates a free $\mathscr{O}_{X,\mathfrak{x}}[S][\frac{1}{f}]$-module and since $P \bullet f'F^S = 0$, we conclude $\mathrm{gr}_{(0,1,1)}(P) \in \ker(\phi_{F,\mathfrak{x}})$.

By Remark 2.19 we deduce:

$$\begin{aligned}\widetilde{L_{F,\mathfrak{x}}} \subseteq \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot \mathrm{gr}_{(0,1,1)}(\theta_{f'F,\mathfrak{x}}) &\subseteq \mathrm{gr}_{(0,1,1)}(\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]}\, f'F^S) \\ &\subseteq \ker(\phi_{F,\mathfrak{x}}).\end{aligned} \tag{2.1}$$

Since f is tame, strongly Euler-homogeneous, and Saito-holonomic, by Theorem 2.23 of loc. cit., $\widetilde{L_{F,\mathfrak{x}}}$ is a prime ideal of dimension $n+r$. So the outer ideals of (2.1) are prime ideals of dimension $n+r$ and the containments are equalities. ☐

Theorem 2.21. *Suppose $f = f_1 \cdots f_r$ is tame, strongly Euler-homogeneous, and Saito-holonomic, $f' \in \mathscr{O}_{X,\mathfrak{x}}[\frac{1}{f}]$ is compatible with f, and $F = (f_1, \dots, f_r)$. Then the $\mathscr{D}_{X,\mathfrak{x}}[S]$-annihilator of $f'F^S$ is generated by derivations.*

Proof. By Theorem 2.20, for $P \in \operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$, we can find $L \in \mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}}$ such that P and L have the same initial term with respect to the total order filtration. Since $P - L$ annihilates $f'F^S$ and, by construction, has a smaller weight than P, we can argue inductively as in Theorem 2.29 of [3] now using Theorem 2.20 instead of Corollary 2.28 of [3]. The induction argument therein will also terminate in this setting since $\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S \cap \mathscr{O}_{X,\mathfrak{x}} = 0$. ☐

The following corollary will let us study the Weyl algebra version of the annihilator of $f'F^S$ when f' and f are global algebraic.

Corollary 2.22. *If X is the analytic space of a smooth $\mathbb{C}$-scheme, then the statement of Theorem 2.21 holds in the algebraic category.*

Proof. See Corollary 2.30 of [3]. ☐

We will also be interested in the $\mathscr{D}_{X,\mathfrak{x}}[S]$-module generated by the symbol $F^{-S} = f_1^{-s_1} \cdots f_r^{-s_r}$ which is defined in the same way as $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$. Most of our previous definitions apply to F^{-S} as well, in particular, if f' is compatible with f let $\psi_{f'F,\mathfrak{x}}^{-S}$ and $\theta_{F,\mathfrak{x}}^{-S}$ be as before, except with the signs of the s_k switched.

Theorem 2.23. *Suppose $f = f_1 \cdots f_r$ is tame, strongly Euler-homogeneous, and Saito-holonomic, f' is compatible with f, and $F = (f_1, \cdots, f_r)$. Then the $\mathscr{D}_{X,\mathfrak{x}}[S]$-annihilator of $f'F^{-S}$ is generated by derivations in that*

$$\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^{-S} = \mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}}^{-S}.$$

If X is the analytic space of a smooth $\mathbb{C}$-scheme, then this holds in the algebraic category as well.

Proof. It is sufficient to prove the generated by derivations statement. For this argue as in Theorem 2.21 except replace $\widetilde{L_{F,\mathfrak{x}}}$ and $\phi_{F,\mathfrak{x}}$ with their images under the $\operatorname{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ automorphism induced by $s_k \mapsto -s_k$. ☐

2.3. Bernstein–Sato Ideals.

Recall the univariate *functional equation*, with $b(s) \in \mathbb{C}[s]$, $P(s) \in \mathscr{D}_{X,\mathfrak{x}}[s]$:

$$b(s)f^s = P(s)f^{s+1}.$$

The polynomials $b(s)$ generate the *Bernstein–Sato ideal* $B_{f,\mathfrak{x}}$ of f. The monic generator of this ideal is the *Bernstein–Sato polynomial*; the reduced locus of its variety is $\operatorname{V}(B_{f,\mathfrak{x}})$. We will be interested in multivariate generalizations of this functional equation.

Definition 2.24. Let $f', g_1, \dots, g_u \in \mathscr{O}_{X,\mathfrak{x}}$ and I the ideal generated by the $g_1, \dots g_u$. Consider the functional equation

$$B(S)f'F^S = \sum_t P_t g_t f'F^S \in \mathscr{D}_{X,\mathfrak{x}}[S] \cdot If'F^S$$

where $f = f_1 \cdots f_r$, $F = (f_1, \dots, f_r)$, $P_t \in \mathscr{D}_{X,\mathfrak{x}}[S]$, and $B(S) \in \mathbb{C}[S]$. The polynomials $B(S)$ satisfying this functional equation constitute the *Bernstein–Sato ideal* $B_{f'F,\mathfrak{x}}^I$. Note that

$$B_{f'F,\mathfrak{x}}^I = \mathbb{C}[S] \cap (\operatorname{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I).$$

When $I = (f)$ we will write $B^I_{f'F,\mathfrak{x}} = B_{f'F,\mathfrak{x}}$ and when $I = (g)$ we will write $B^g_{f'F,\mathfrak{x}}$. When in the univariate case, i.e. $r = 1$, we will write $B_{f'F,\mathfrak{x}} = B_{f'f,x}$ and $B^g_{f'F,\mathfrak{x}} = B^g_{f'f,\mathfrak{x}}$. When in the global algebraic case we define similar objects using $\mathrm{A}_n(\mathbb{C})[S]$ instead of $\mathscr{D}_{X,\mathfrak{x}}[S]$–in this case we drop the $(-)_\mathfrak{x}$ subscript. Finally by $\mathrm{V}(-)$ we always mean the reduced locus of the appropriate variety.

We will want to compare the Bernstein–Sato ideals corresponding to different factorizations.

Definition 2.25. Let $f = f_1 \cdots f_r$ and $F = (f_1, \dots, f_r)$. Write $[r]$ as the disjoint union of the intervals I_t where $1 \leq t \leq m$ and consider the *coarser* factorization $H = (h_1, \dots, h_m)$ where $f = h_1 \cdots h_m$ and $h_t = \prod_{i \in I_t} f_i$. Define S_H to be the ideal of $\mathbb{C}[S]$ generated by $s_i - s_j$ for all $i, j \in I_t$ and for all t.

Proposition 2.26. *Let $f = f_1 \cdots f_r$ be tame, strongly Euler-homogeneous, and Saito-holonomic. Let $F = (f_1, \dots, f_r)$, let $I \subseteq \mathscr{O}_{X,\mathfrak{x}}$, and let H be a coarser factorization. If $f' \in \mathscr{O}_{X,\mathfrak{x}}$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot f'$, then the image of $B^I_{f'F,\mathfrak{x}}$ modulo S_H lies in $B^I_{f'H,\mathfrak{x}}$.*

Proof. As f' is compatible with f, $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'F^S$ and $\mathrm{ann}_{\mathscr{D}_{X,\mathfrak{x}}[S]} f'H^S$ are both generated by derivations. Since $\mathrm{Der}_{X,\mathfrak{x}}(-\log f) \subseteq \mathrm{Der}_{X,\mathfrak{x}}(-\log f')$, we can easily get a result similar to Proposition 2.33 of [3] and, from that, a result similar to Proposition 2.32 of loc. cit. The argument is essentially the same as the proof of Proposition 5.3 of this paper. □

Example 2.27. For $f = xy^2(x+y)^2$ and $F = (xy, y(x+y), x+y)$,

$$B_F = (s_1 + 1) \prod_{j=0}^{1} (s_1 + s_2 + 1 + j)(s_2 + s_3 + 1 + j)\Big(\prod_{m=0}^{4} (2s_1 + 2s_2 + s_3 + 2 + m).$$

While Proposition 2.26 can estimate B_f, it estimates multiplicities poorly. Indeed, going modulo $(s_1 - s_2, s_1 - s_3, s_2 - s_3)$ we find

$$(s+1)^3(2s+1)^2 \prod_{m=0}^{4} (5s + 2 + m) \in B_f = \mathbb{C}[s] \cdot (s+1)(2s+1) \prod_{m=0}^{4} (5s + 2 + m).$$

3. $\mathscr{D}_{X,\mathfrak{x}}[S]$-Dual of $\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S$

In [21], Narváez-Macarro computed the $\mathscr{D}_{X,\mathfrak{x}}[s]$-dual of $\mathscr{D}_{X,\mathfrak{x}}[s]f^s$ when f is reduced, free, and quasi-homogeneous; in [18] Maisonbe generalized this approach to compute the $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual of $\mathscr{D}_{X,\mathfrak{x}}[S]F^S$ where f is as in [21], $f = f_1 \cdots f_r$, and $F = (f_1, \dots, f_r)$. In this section we will use Maisonobe's approach to compute the $\mathscr{D}_{X,x}[S]$-dual of $\mathscr{D}_{X,x}[S]f'F^S$ where $f \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, Saito-holonomic, not necessarily reduced but admitting a reduced Euler-homogeneous defining equation f_{red} at $\mathfrak{x}$, $f' \in \mathscr{O}_{X,\mathfrak{x}}$ is compatible with f, and

$$F = (f_1, \dots, f_r)$$

corresponds to any factorization, not necessarily into irreducibles, of $f = f_1 \cdots f_r$. The strategy hinges on a formula for the trace of the adjoint first proved by Castro–Jiménez and Ucha in Theorem 4.1.4 of [9]. We supply a different proof in Proposition A.12.

In the second subsection, we note that this duality computation lets us argue as in Maisonobe's Proposition 20 of [16] and prove that the radical of $B_{f'F,\mathfrak{x}}$ is principal. In the third subsection, we show that $B^g_{f'F,\mathfrak{x}}$ is fixed under a non-trivial involution when f', F, and g satisfy a technical condition, cf. Definition 3.14.

Convention 3.1. A resolution is a (co)-complex with a unique (co)-homology module at its end. An acyclic (co)-complex has no non-trivial (co)-homology. Given a (co)-complex ($C^\bullet$) $C_\bullet$ resolving A, the augmented (co)-complex ($C^\bullet \to A$) $C_\bullet \to A$ is acyclic.

3.1. **Computing the Dual.**

Our argument begins at essentially the same place as Narváez-Macarro's and Maisonobe's: the Spencer co-complex.

Definition 3.2. Let $f = f_1 \cdots f_r \in \mathscr{O}_{X,\mathfrak{x}}$ be free, let $F = (f_1, \ldots, f_r)$, and let $f' \in \mathscr{O}_{X,\mathfrak{x}}$ be compatible with f. Consider $g_1, \ldots, g_u \in \mathscr{O}_{X,\mathfrak{x}}$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g_j$ for all $1 \leq j \leq u$, and let $I \subseteq \mathscr{O}_{X,\mathfrak{x}}$ be the ideal generated by $g_1, \ldots, g_u$. We will define $\mathrm{Sp}^I_{\theta_{f'F,\mathfrak{x}}}$, the *extended Spencer co-complex* associated to f' and I. When $I = (g)$, write $\mathrm{Sp}^g_{f'F}$. This will be a mild generalization of the normal Spencer complex, cf. A.18 of [21].

Let E be the free submodule of $\mathscr{O}^u_{X,\mathfrak{x}}$ prescribed by the basis $e_1, \ldots, e_u$ where

$$e_j = (0, \ldots, g_j, \ldots, 0).$$

We define an anti-commutative map

$$\sigma : (\theta_{f'F,\mathfrak{x}} \oplus E) \times (\theta_{f'F,\mathfrak{x}} \oplus E) \to \theta_{f'F,\mathfrak{x}} \oplus E$$

that is essentially the commutator on $F^1_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$. The map is determined by its anti-commutativity and the following assignments:

$$\sigma(\lambda_i, \lambda_j) = \begin{cases} [\lambda_i, \lambda_j], & \lambda_i, \lambda_j \in \theta_{f'F,\mathfrak{x}}, \\ 0, & \lambda_i, \lambda_j \in E, \\ \frac{\delta \bullet (b g_j)}{g_j} e_j, & \lambda_i = \psi_{f'F,\mathfrak{x}}(\delta_i) \text{ for } \delta_i \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f),\ \lambda_j = b e_j. \end{cases}$$

Abbreviate $\mathrm{Sp}^I_{\theta_{f'F,\mathfrak{x}}}$ as $\mathrm{Sp}^\bullet$. Then the objects of our complex are

$$\mathrm{Sp}^{-m} = \mathscr{D}_{X,\mathfrak{x}}[S] \otimes_{\mathscr{O}_{X,\mathfrak{x}}} \bigwedge^m (\theta_{f'F,\mathfrak{x}} \oplus E)$$

and the differentials $d^{-m} : \mathrm{Sp}^{-m} \mapsto \mathrm{Sp}^{-m+1}$ are given by

$$\begin{aligned} d^{-m}(P \otimes \lambda_1 \wedge \cdots \wedge \lambda_m) = &\sum_{i=1}^{r} (-1)^{i-1} P \lambda_i \otimes \widehat{\lambda_i} \\ &+ \sum_{1 \leq i < j \leq m} (-1)^{i+j} P \otimes \sigma(\lambda_i, \lambda_j) \wedge \widehat{\lambda_{i,j}}. \end{aligned}$$

Here $\widehat{\lambda_i}$ is the wedge, in increasing order, of all the $\lambda_1, \ldots, \lambda_r$ except for λ_i; $\widehat{\lambda_{i,j}}$ is the same except now excluding both λ_i and λ_j. To be clear, we interpret Pe_j as $Pg_j \in \mathscr{D}_{X,\mathfrak{x}}[S]$; in particular, $d^{-1}(P \otimes e_j) = Pg_j$. There is a natural augmentation map

$$\mathrm{Sp}^0 = \mathscr{D}_{X,\mathfrak{x}}[S] \mapsto \frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I}.$$

Remark 3.3. (a) Since $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ is closed under taking commutators, so is $\theta_{f'F,\mathfrak{x}}$, see also Example 4.7 of [3]. And as g_j divides f for all $1 \leq j \leq u$, we know

$$\mathrm{Der}_{X,\mathfrak{x}}(-\log f) \subseteq \mathrm{Der}_{X,\mathfrak{x}}(-\log g_j)$$

for all j. Thus σ, and consequently the differentials, are well-defined.

(b) That the extended Spencer co-complex is in fact a co-complex is a straightforward computation mirroring the case of the standard Spencer co-complex.

(c) We have assumed f is free so that $\mathrm{Sp}^I_{\theta_{f'F,\mathfrak{x}}}$ will be a finite, free co-complex of $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules. We may fix a basis of $\theta_{f'F,\mathfrak{x}}$, extend it to a basis of $\theta_{f'F,\mathfrak{x}} \oplus E$ using the prescribed basis of E, and then compute differentials. Label this basis $\lambda_1, \ldots, \lambda_{n+u}$. Let $\sigma(\lambda_i, \lambda_j) = \sum_{k=1}^{n+u} c_k^{i,j} \lambda_k$ be the unique expression of $\sigma(\lambda_i, \lambda_j)$. Then

$$\begin{aligned} d^{-m}(\lambda_1 \wedge \cdots \wedge \lambda_m) &= \sum_{i=1}^{m} (-1)^{i-1} \lambda_i \otimes \widehat{\lambda_i} \\ &+ \sum_{1 \leq i < j \leq m} (-1)^{i+j} c_i^{i,j} \otimes (-1)^{i-1} \widehat{\lambda_j} + (-1)^{i+j} c_j^{i,j} \otimes (-1)^j \widehat{\lambda_i} \\ &= \sum_{i=1}^{m} \left((-1)^{i-1} \lambda_i + \sum_{j<i} (-1)^{i-1} c_j^{j,i} + \sum_{i<j} (-1)^i c_j^{i,j} \right) \otimes \widehat{\lambda_i}. \end{aligned}$$

We can naturally encode this as matrix multiplication on the right.

The following calculation relies on Castro–Jiménez and Ucha's formula for adjoints appearing in Theorem 4.1.4 of [9]; cf. Proposition A.12 for our proof. See also Lemma 1 and Proposition 6 of [18]. Before stating the Proposition, let us recall the side-changing functor for $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules. We use the notation of Appendix A of [21].

Definition 3.4. (Compare to Appendix A of [21]) We will define the equivalence of categories between right $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules and left $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules. First, regard $\mathrm{Der}_{X,\mathfrak{x}}[S]$ as a free $\mathscr{O}_{X,\mathfrak{x}}[S]$-module of rank n. Then the dualizing module $\omega_{\mathrm{Der}_{X,\mathfrak{x}}[S]}$ of $\mathrm{Der}_{X,\mathfrak{x}}[S]$ is defined as

$$\omega_{\mathrm{Der}_{X,\mathfrak{x}}[S]} = \mathrm{Hom}_{\mathscr{O}_{X,\mathfrak{x}}[S]} \left(\bigwedge^n \mathrm{Der}_{X,\mathfrak{x}}[S], \mathscr{O}_{X,\mathfrak{x}}[S] \right).$$

This naturally carries a right $\mathscr{D}_{X,\mathfrak{x}}[S]$-module structure by A.20 of [21]. The aforementioned equivalence of categories is given by associated to every right $\mathscr{D}_{X,\mathfrak{x}}[S]$-module Q the left $\mathscr{D}_{X,\mathfrak{x}}[S]$-module Q^{left} defined by

$$Q^{\mathrm{left}} = \mathrm{Hom}_{\mathscr{O}_{X,\mathfrak{x}}[S]} \left(\omega_{\mathrm{Der}_{X,\mathfrak{x}}[S]}, Q \right).$$

That Q^{left} is a left $\mathscr{D}_{X,\mathfrak{x}}[S]$-module follows from A.2 of [21]; that this gives an equivalence of categories follows from the discussion before A.25 of loc. cit.

Remark 3.5. Despite the s-terms, this side-changing functor is defined entirely similarly to the side-changing functor for $\mathscr{D}_{X,\mathfrak{x}}$-modules. So just as in the $\mathscr{D}_{X,\mathfrak{x}}[S]$-module case, if we fix coordinates (x, ∂_x) we can describe the transition from right to left $\mathscr{D}_{X,\mathfrak{x}}[S]$-modules in elementary terms. Define $\tau : \mathscr{D}_{X,\mathfrak{x}}[S] \to \mathscr{D}_{X,\mathfrak{x}}[S]$ by $\tau(x^\alpha \partial_x^\beta s^\gamma) = (-\partial_x^\beta) x^\alpha s^\gamma$ where α, β, and γ are multi-indices. Then $(-)^{\mathrm{left}}$ sends the cyclic right $\mathscr{D}_{X,\mathfrak{x}}[S]$-module $\mathscr{D}_{X,\mathfrak{x}}[S]/J$ is to the left $\mathscr{D}_{X,\mathfrak{x}}[S]$-module $\mathscr{D}_{X,\mathfrak{x}}[S]/\tau(J)$. See 1.2 of [26] for details in a similar case.

Proposition 3.6. *Let $f = f_1 \cdots f_r \in \mathscr{O}_{X,\mathfrak{x}}$ be free, $F = (f_1, \dots, f_r)$, $f_{\mathrm{red}} \in \mathscr{O}_{X,\mathfrak{x}}$ a Euler-homogeneous reduced defining equation for f at $\mathfrak{x}$, and $I \subseteq \mathscr{O}_{X,\mathfrak{x}}$ the ideal generated by $g_1, \dots, g_u$ with $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g_v$ for each g_v. Write $g = g_1 \cdots g_u$. Then we can compute the terminal homology module of $\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(Sp^I_{\theta_{f'F,x}}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}}$:*

$$H_{-n-u} \left(\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(Sp^I_{\theta_{f'F,x}}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}} \right) \simeq \frac{\mathscr{D}_{X,\mathfrak{x}}[S]}{\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f'gf_{\mathrm{red}})^{-1}F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I}.$$

Proof. We will show that the image of $\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(d^{-n-u}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}}$ is

$$\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f'gf_{\mathrm{red}})^{-1}F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I.$$

It suffices to do this in local coordinates $x_1, \dots, x_n$. Select a basis $\delta_1, \dots, \delta_n$ of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$, label $\lambda_i = \psi_{f'F,\mathfrak{x}}(\delta_i)$ and label $\lambda_{n+j} = e_j = (0, \dots, g_j, \dots, 0)$ for $1 \leq j \leq u$, cf. Definition 3.2.

Then $\lambda_1, \ldots, \lambda_{n+u}$ is a basis of $\psi_{f'F,\mathfrak{x}} \oplus E$. Consequently, we may uniquely write

$$\sigma(\lambda_i, \lambda_j) = \sum_{k=1}^{n+u} c_k^{i,j} \lambda_k$$

with $c_k^{i,j} \in \mathscr{O}_{X,\mathfrak{x}}$.

Let us compute the $c_k^{i,j}$ terms in cases. First assume $i, j \leq n$. Then

$$\sigma(\lambda_i, \lambda_j) = [\psi_{f'F,\mathfrak{x}}(\delta_i), \psi_{f'F,\mathfrak{x}}(\delta_j)] = [\delta_i, \delta_j],$$

where the last equality follows since $\psi_{f'F,\mathfrak{x}}$ respects taking commutators, cf. Remark 2.17. Thus $c_1^{i,j}, \ldots, c_n^{i,j}$ satisfy $[\delta_i, \delta_j] = \sum_{k=1}^{n} c_k^{i,j} \delta_k$; moreover, if $k \geq n+1$, then $c_k^{i,j} = 0$. Second, assume $i \leq n$ and $j \leq u$. By definition $\sigma(\lambda_i, \lambda_{n+j}) = \frac{\delta \bullet g_j}{g_j} \lambda_{n+j}$ and so $c_{n+j}^{i,n+j} = \frac{\delta_i \bullet g_j}{g_j}$ and $c_k^{i,n+j} = 0$ for $k \neq n+j$. Similarly for $j \leq n$ and $i \leq u$, $c_{n+j}^{n+j,i} = -\frac{\partial_i \bullet g_j}{g_j}$ and $c_k^{n+j,i} = 0$ for all $k \neq n+j$. Finally, assume $i, j \leq u$. Then $\sigma(\lambda_{n+i}, \lambda_{n+j}) = 0$ and $c_k^{n+i,n+j} = 0$ for all k.

Using Remark 3.3, d^{-n-u} is given, where $i \leq n$ and $v \leq u$, by multiplying on the right by the matrix

$$\begin{bmatrix} \cdots & (-1)^{i-1}(\psi_{f'F,\mathfrak{x}}(\delta_i) - \sum\limits_{j=1}^{n} c_j^{i,j} - \sum\limits_{v=1}^{u} \frac{\delta \bullet g_v}{g_v}) & \cdots & (-1)^{n+v-1} g_v & \cdots \end{bmatrix}. \tag{3.1}$$

The dual map is given by transposing (3.1) and applying τ, the standard right-to-left map (cf. Remark 3.5), to each each entry where τ is inert on $\mathscr{O}_{X,\mathfrak{x}}[S]$ and sends $h\partial_{x_i}$ to $-\partial_{x_i} h$, $h \in \mathscr{O}_{X,\mathfrak{x}}[S]$. Write $\delta_i = \sum_e h_{e,i} \partial_{x_e}$ and observe that $\tau(\delta_i) = -\delta_i - \sum_e \partial_{x_e} \bullet h_{e,i}$. Therefore $\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(d^{-n-u}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}}$ is given by right multiplication by

$$\begin{bmatrix} \vdots \\ (-1)^{i-1}(-\delta_i - \sum\limits_{k=1}^{r} \frac{\delta_i \bullet f_k}{f_k} s_k - \frac{\delta_i \bullet f'}{f'} - \sum\limits_{e=1}^{n} \partial_{x_e} \bullet h_{e,i} - \sum\limits_{j=1}^{n} c_j^{i,j} - \sum\limits_{v=1}^{u} \frac{\delta \bullet g_v}{g_v}) \\ \vdots \\ (-1)^{n+v-1} g_v \\ \vdots \end{bmatrix} \tag{3.2}$$

Assume $n \geq 2$. We could have chosen $\delta_1, \ldots, \delta_n$ to be a preferred basis of

$$\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}}) = \mathrm{Der}_{X,\mathfrak{x}}(-\log f),$$

cf. Definition A.11, making $\delta_1, \ldots, \delta_{n-1} \in \mathrm{Der}_{X,\mathfrak{x}}(-\log_0 f)$ and δ_n a Euler-homogeneity for f_{red}. By the trace-adjoint formula of Proposition A.12:

$$\sum_j c_j^{i,j} = -\sum_e \partial_{x_e} \bullet h_{e,i} \text{ for } i \neq n; \ \sum_j c_j^{n,j} = -\sum_e \partial_{x_e} \bullet h_{e,n} + 1 \text{ for } i = n.$$

Recall $g = g_1 \cdots g_u$. Since $\delta_i \bullet f_{\text{red}} = 0$ for $i \leq n-1$ and since δ_n is Euler-homogeneous on f_{red}, (3.2) simplifies to

$$\begin{bmatrix} \vdots \\ (-1)^i(\psi^{-S}_{(f'gf_{\text{red}})^{-1}F,\mathfrak{x}})(\delta_i) \\ \vdots \\ (-1)^n(\psi^{-S}_{(f'gf_{\text{red}})^{-1}F,\mathfrak{x}}(\delta_n) \\ \vdots \\ (-1)^{n+v-1}g_v \\ \vdots \end{bmatrix}.$$

Thus the image of $\text{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(d^{-n-u}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\text{left}}$ is $\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f'gf_{\text{red}})^{-1}F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot I$, proving the proposition for $n \geq 2$.

As for $n = 1$, we can assume $f_{\text{red}} = x$ and $\text{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ is freely generated by its Euler-homogeneity. Simplifying (3.2) is then an easy calculation. □

We endow $\text{Sp}^I_{f'F,\mathfrak{x}}$ with a chain co-complex filtration that is based on a construction of Gros and Narváez-Macarro, cf. page 85 of [14].

Proposition 3.7. *Let $f = f_1 \cdots f_r$ be free, $F = (f_1, \ldots, f_r)$, and let f' and I be as in Definition 3.2. Abbreviate $Sp^I_{\theta_{f'F,\mathfrak{x}}}$ to $Sp^\bullet$. Define a filtration $G^\bullet$ on $Sp^\bullet$ by*

$$G^p\, Sp^{-m} = \bigoplus_j \left(F^{p-m+j}_{(0,1,1)} \mathscr{D}_{X,\mathfrak{x}}[S] \otimes_{\mathscr{O}_{X,\mathfrak{x}}} \bigwedge^{m-j} \theta_{f'F,\mathfrak{x}} \wedge \bigwedge^{j} E \right).$$

If $\delta_1, \ldots, \delta_n$ is a basis of $\text{Der}_{X,\mathfrak{x}}(-\log f)$, then $\text{gr}_G(Sp^\bullet)$ is isomorphic to the following Koszul co-complex on $\text{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$:

$$K^\bullet(\text{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta_1)), \ldots, \text{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta_n)), g_1, \ldots, g_u; \text{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])). \tag{3.3}$$

Moreover, $G^\bullet$ naturally gives a filtration on $Hom_{\mathscr{D}_{X,\mathfrak{x}}[S]}(Sp^\bullet, \mathscr{D}_{X,\mathfrak{x}}[S])^{left}$ whose associated graded complex is isomorphic to

$$K^\bullet(\text{gr}_{(0,1,1)}(-\psi_{F,\mathfrak{x}}(\delta_1)), \ldots, \text{gr}_{(0,1,1)}(-\psi_{F,\mathfrak{x}}(\delta_n)), g_1, \ldots, g_u; \text{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])). \tag{3.4}$$

Proof. That $G^\bullet$ is a chain filtration and that the associated graded co-complex is isomorphic to the Koszul complex (3.3) follows from the definitions. As for the dual statement, it is enough to note that τ, the standard right-to-left map (cf. Lemma 4.13 of [3]), preserves weight 0 entries (under the total order filtration) and sends weight 1 entries $\delta + p(S)$ to $-\delta + p(S)+$ error terms, where δ is a derivation and both $p(S)$ and the error terms lie in $\mathscr{O}_{X,\mathfrak{x}}[S]$. □

We now add hypotheses to the settings of Propositions 3.6 and 3.7. First, we assume $I = \mathscr{O}_{X,x} \cdot g$ is principal; second, we assume f is not only free but also strongly Euler-homogeneous and Saito-holonomic. This will let us use results from [3]. The filtration $G^\bullet$ will demonstrate that $\text{Sp}^g_{f'F}$ and its dual are resolutions.

Definition 3.8. For M a left $\mathscr{D}_{X,\mathfrak{x}}[S]$-module, denote the $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual of M by

$$\mathbb{D}(M) = \text{RHom}_{\mathscr{D}_{X,x}[S]}(M, \mathscr{D}_{X,\mathfrak{x}}[S])^{\text{left}}.$$

Theorem 3.9. *Suppose $f = f_1 \cdots f_r \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, and Saito-holonomic and $f_{\text{red}} \in \mathscr{O}_{X,\mathfrak{x}}$ is a Euler-homogeneous reduced defining equation for f at $\mathfrak{x}$. Let*

$F = (f_1, \ldots, f_r)$, *let* $f' \in \mathscr{O}_{X,\mathfrak{x}}$ *be compatible with* f, *and let* $g \in \mathscr{O}_{X,\mathfrak{x}}$ *such that* $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g$. *Then*

$$\mathbb{D}\left(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]\cdot gf'F^S}\right) \simeq \frac{\mathscr{D}_{X,\mathfrak{x}}[S](gf'f_{\mathrm{red}})^{-1}F^{-S}}{\mathscr{D}_{X,\mathfrak{x}}[S](f'f_{\mathrm{red}})^{-1}F^{-S}}[n+1].$$

Proof. We first show that (3.3) and (3.4) are both resolutions; in fact, showing (3.3) is a resolution proves (3.4) is as well. Let $\delta_1, \ldots, \delta_n$ be a basis of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$. Since $\mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S])$ is graded local and $\mathrm{gr}_{(0,1,1)}(\psi_{F,\mathfrak{x}}(\delta_i))$ and f all live in the graded maximal ideal, it is sufficient to prove that the Koszul co-complex (3.3) is a resolution after localization at the graded maximal ideal. By Theorem 2.23 of [3], $\widetilde{L_{F,\mathfrak{x}}}$ is Cohen–Macaulay and prime of dimension $n+r$. Therefore $\widetilde{L_{F,\mathfrak{x}}} + \mathrm{gr}_{(0,1,1)}(\mathscr{D}_{X,\mathfrak{x}}[S]) \cdot f$ has dimension $n + r - 1$. Moreover, this ideal's dimension does not change after localization at the graded maximal ideal. Theorem 2.1.2 of [5] then implies (3.3) is a resolution after said localization, finishing this part of the proof.

Since (3.3) is a resolution, a standard spectral sequence argument associated to the filtered co-complex of $\mathrm{Sp}^g_{f'F,\mathfrak{x}}$ implies $\mathrm{Sp}^g_{f'F,\mathfrak{x}}$ is a resolution. By Theorem 2.21 and the definition of the augmentation map it resolves $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$. Similar reasoning verifies that

$$\mathrm{Hom}_{\mathscr{D}_{X,\mathfrak{x}}[S]}(\mathrm{Sp}^g_{f'F,\mathfrak{x}}, \mathscr{D}_{X,\mathfrak{x}}[S])^{\mathrm{left}}$$

is a resolution. Because f_{red} is Euler homogeneous, the claim follows by Proposition 3.6 and Theorem 2.23. ☐

Remark 3.10. We are skeptical that (3.3) is a resolution for any non-principal, non-pathological I. Possible candidates are linear free divisors f with many factors, even though the non-pathological examples in $n \leq 4$ fail, cf. [12].

3.2. Principality of $\sqrt{B^g_{f'F,\mathfrak{x}}}$.

Here we discuss the principality of the radical of $B^g_{f'F,\mathfrak{x}}$. The argument is essentially the same as Proposition 20 of [16], but we do not have to appeal to tame pure extensions because of our hypotheses on f.

We will need some homological definitions for modules over non-commutative rings, cf. Appendix IV of [4] for a detailed treatment. We say a $\mathscr{D}_{X,\mathfrak{x}}[S]$-module M has *grade* j if

$$\mathrm{Ext}^k_{\mathscr{D}_{X,\mathfrak{x}}[S]}(M, \mathscr{D}_{X,\mathfrak{x}}[S])$$

vanishes for all $k < j$ and is nonzero for $k = j$. We say M is *pure* of grade j if every nonzero submodule of M has grade j. We also need the following filtration on $\mathscr{D}_{X,\mathfrak{x}}[S]$:

Definition 3.11. Define the *order filtration* $F_{(0,1,0)}$ on $\mathscr{D}_X[S]$ by designating, in local coordinates, every ∂_{x_k} weight one and every element of $\mathscr{O}_X[S]$ weight zero. Let $\mathrm{gr}_{(0,1,0)}(\mathscr{D}_X[S])$ denote the associated graded object and note that locally $\mathrm{gr}_{(0,1,0)}(\mathscr{D}_X[S]) \simeq \mathscr{O}_X[Y][S]$, with $\mathrm{gr}_{(0,1,0)}(\partial_{x_k}) = y_k$. For a coherent $\mathscr{D}_X[S]$-module M and any good filtration Γ on M relative to $F_{(0,1,0)}$, the *characteristic ideal* $J^{\mathrm{rel}}(M) \subseteq \mathrm{gr}_{(0,1,1)}(\mathscr{D}_X[S])$ is defined as

$$J^{\mathrm{rel}}(M) = \sqrt{\mathrm{ann}_{\mathrm{gr}_{(0,1,0)}(\mathscr{D}_X[S])}\,\mathrm{gr}_\Gamma(M)}$$

and is independent of the choice of good filtration.

Proposition 3.12. (Compare to Proposition 20 of [16]) *Suppose* $f = f_1 \ldots f_r \in \mathscr{O}_X$ *is free, strongly Euler-homogeneous, and Saito-holonomic such that the reduced divisor of* f *is Euler-homogeneous. Let* $F = (f_1, \ldots, f_r)$ *and select* $f' \in \mathscr{O}_X$ *and* $g \in \mathscr{O}_X$ *such that* f *lies in both* $\mathscr{O}_X \cdot f'$ *and* $\mathscr{O}_X \cdot g$. *Then for all* $\mathfrak{x}$, $\sqrt{B^g_{f'F,\mathfrak{x}}}$ *is principal.*

Proof. Since f' is a section generating a holonomic $\mathscr{D}_X$-module, by Proposition 13 of [16] there is a conical Lagrangian variety $\Lambda \subseteq T^\star X$ so that $\mathrm{V}(J^{\mathrm{rel}}(\mathscr{D}_X[S]f'F^S)) = \Lambda \times C^r$. So $\mathrm{V}(J^{\mathrm{rel}}(\frac{\mathscr{D}_X[S]f'F^S}{\mathscr{D}_X[S]gf'F^S})) \subseteq \Lambda \times \mathbb{C}^r$, that is, in the language of Maisonobe, $\frac{\mathscr{D}_X[S]f'F^S}{\mathscr{D}_X[S]gf'F^S}$ is *majoré par une Lagrangian.* By Proposition 8 of [16], there exist conical Lagrangians $T^\star_{X_\alpha}X$ and algebraic varieties $S_\alpha \subseteq \mathbb{C}^r$ such that

$$\mathrm{V}\left(J^{\mathrm{rel}}(\frac{\mathscr{D}_X[S]f'F^S}{\mathscr{D}_X[S]gf'F^S})\right) = \cup_\alpha T^\star_{X_\alpha}X \times S_\alpha. \tag{3.5}$$

By Proposition 9 of [16], $\mathrm{V}(B^g_{f'F,\mathfrak{x}}) = \cup_{\mathfrak{x}\in X_\alpha} S_\alpha$.

Now to show the radical of $B^g_{f'F,\mathfrak{x}}$ is principal, it suffices to show S_α is of dimension $r-1$ for each α such that $\mathfrak{x} \in X_\alpha$; that is, by the description of $T^\star_{X_\alpha}X$, it suffices to show $J^{\mathrm{rel}}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S})$ is equidimensional of dimension $n+r-1$. By Theorem 3.9, $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ has grade $n+1$. Using Theorem 3.9 again and the characterization of pure modules in terms of double Ext modules, cf. Proposition IV.2.6 of [4], we deduce $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ is a pure $\mathscr{D}_{X,\mathfrak{x}}[S]$-module of grade $n+1$. By Theorem IV.5.2 of [4], $J^{\mathrm{rel}}(\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S})$ is equidimensional and every minimal prime of the characteristic ideal has codimension $n+1$, completing the proof. ☐

The next proposition lays out a criterion for $B^g_{f'F,\mathfrak{x}}$ to be principal. The argument is that of the last paragraph of Theorem 2 of [18].

Proposition 3.13. (Compare with Theorem 2 of [18]) *Let f, F, f', and g be as in Proposition 3.12 and suppose that $\sqrt{B^g_{f'F,\mathfrak{x}}} = \mathbb{C}[S]\cdot b(S)$, i.e. it is principal. Suppose that $(B_{f'F,\mathfrak{x}} : \sqrt{B_{f'F,\mathfrak{x}}})$ contains a polynomial $a(S)$ such that $\mathrm{V}(\mathbb{C}[S]\cdot b(S)) \cap \mathrm{V}(\mathbb{C}[S]\cdot a(S))$ has irreducible components of dimension at most $r-2$. Then $B^g_{f'F,\mathfrak{x}}$ equals its radical and is principal.*

Proof. It suffices to show $b(S)\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ is zero. If it is nonzero, it is a submodule of the pure module $\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ of grade $n+1$ and so is itself pure of the same grade. Reasoning as in Proposition 3.12, cf. Proposition 9 of [16] in particular, all the minimal primes of $\mathbb{C}[S]$-annihilator of $b(S)\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ have dimension $r-1$. But the variety of this annihilator is contained inside $\mathrm{V}(\mathbb{C}[S]\cdot b(S)) \cap \mathrm{V}(\mathbb{C}[S]\cdot a(S))$ which is of dimension $r-2$ by hypothesis. As this is impossible, $b(S)\frac{\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S}{\mathscr{D}_{X,\mathfrak{x}}[S]gf'F^S}$ must be zero. ☐

3.3. **Symmetry of Some Bernstein–Sato Varieties.**

As Theorem 3.9 generalizes Corollary 3.6 of [21] and Proposition 6 of [18], one would hope $B^g_{f'F,\mathfrak{x}}$ has a symmetry generalizing Theorem 4.1 of [21] and Proposition 8 of [18]. However, without reducedness and with the addition of f', symmetry seems to depend on the factorization of f.

Definition 3.14. Suppose f has a factorization into irreducibles $l_1^{v_1}\cdots l_q^{v_q}$ at $\mathfrak{x}$ where the l_t are distinct and $v_t \in \mathbb{Z}_+$. Let $f = f_1 \cdots f_r$ be some other factorization of f and let $F = (f_1, \dots, f_r)$. We say the factorization $f = f_1 \cdots f_r$ is *unmixed* if the following hold:

(i) for each k, there exists $d_k \in \mathbb{Z}_+$ and $J_k \subseteq [q]$ such that $f_k = \prod_{j \in J_k} l_j^{d_k}$;
(ii) if $i, j \in J_k$, then $v_i = v_j$.

F is *unmixed* when it corresponds to an unmixed factorization; F is *unmixed up to units* if there exists units $u_1, \dots, u_r$ such that $uF = (u_1 f_1, \dots, u_r f_r)$ is unmixed. Given an unmixed factorization, let the *repeated multiplicity* of F be $\{m_k\}_k$ where, for any $j \in J_k$ (and thus all), m_k is the multiplicity of l_j with respect to f.

For $f' \in \mathscr{O}_{X,\mathfrak{x}}$ compatible with f, we say (f', F) is an *unmixed pair* if:

(i)′ F is unmixed;

(ii)′ $f' = \prod_k \prod_{j \in J_k} l_j^{d'_k}$ for $d'_k \in \mathbb{Z}$.

The pair (f', F) is an *unmixed pair up to units* if F is unmixed up to units and f' satisfies (ii′) after possibly multiplying by a unit. For (f', F) an unmixed pair up to units, the *pairs of repeated powers* of (f', F) are $\{(d'_k, d_k)\}_k$.

Lemma 3.15. *Write $f = l_1^{v_1} \cdots l_q^{v_q}$ where the l_i are distinct and irreducible; $f_k = \prod_{j \in J_k} l_j^{d_k}$; $f_{\text{red}} = l_1 \cdots l_q$. Assume that f' and g are compatible with f, $F = (f_1, \ldots, f_r)$ a factorization of f, (f', F) and (g, F) are unmixed pairs with pairs of repeated powers $\{(d'_k, d_k)\}_k$ and $\{(d''_k, d_k)\}_k$, and $\{m_k\}_k$ the repeated multiplicities of F. If $\varphi : \mathbb{C}[S] \to \mathbb{C}[S]$ is the automorphism of $\mathbb{C}$-algebras induced by*

$$\varphi(s_k) = -s_k - \frac{1}{m_k} - \frac{2d'_k}{d_k} - \frac{d''_k}{d_k},$$

then for $\delta \in \operatorname{Der}_{X,\mathfrak{x}}(-\log f)$, and after extending φ to $\mathscr{D}_{X,\mathfrak{x}}[S]$,

$$\varphi(\psi^{-S}_{(f'gf_{\text{red}})^{-1}F,\mathfrak{x}}(\delta)) = \psi^{S}_{f'F,\mathfrak{x}}(\delta).$$

Proof. This is a straightforward computation once we observe that v_j is the sum of all the d_k such that l_j divides f_k. □

Theorem 3.16. *Suppose $f = f_1 \cdots f_r \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, and Saito-holonomic, and while f is not necessarily reduced, suppose that it admits a strongly Euler-homogeneous reduced defining equation at $\mathfrak{x}$. Let $F = (f_1, \cdots, f_r)$ and select $g \in \mathscr{O}_{X,\mathfrak{x}}$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g$. Assume that f' and g are compatible with f, (f', F) and (g, F) are unmixed pairs up to units with pairs of repeated powers $\{(d'_k, d_k)\}_k$ and $\{(d''_k, d_k)\}_k$, and $\{m_k\}_k$ are the repeated multiplicities of F. If $\varphi : \mathbb{C}[S] \to \mathbb{C}[S]$ is the automorphism of $\mathbb{C}$-algebras induced by*

$$\varphi(s_k) = -s_k - \frac{1}{m_k} - \frac{2d'_k}{d_k} - \frac{d''_k}{d_k},$$

then

$$B(S) \in B^g_{f'F,\mathfrak{x}} \iff \varphi(B(S)) \in B^g_{f'F,\mathfrak{x}}.$$

Proof. We first reduce to the case that (f', F) and (g, F) are unmixed pairs. It follows from the functional equation that if u is a unit in $\mathscr{O}_{X,\mathfrak{x}}$, then $B^g_{f'F,\mathfrak{x}} = B^g_{uf'F,\mathfrak{x}}$ and $B^g_{f'F,\mathfrak{x}} = B^{ug}_{f'F,\mathfrak{x}}$. To finish the reduction, we must also verify that if $F' = (u_1 f_1, \ldots, u_r f_r)$ for units $u_1, \ldots, u_r$ in $\mathscr{O}_{X,\mathfrak{x}}$, then $B^g_{f'F,\mathfrak{x}} = B^g_{f'F',\mathfrak{x}}$. This follows by arguing as in Lemma 10 (i) of [2] wherein the claim is proved for $f' = 1$ and $g = f$.

By the $\mathbb{C}[S]$-linearity of $\mathbb{D}$, cf. Remark 3.2 of [21], and by Theorem 3.9,

$$B(S) \in \operatorname{ann}_{\mathbb{C}[S]} \frac{\mathscr{D}_{X,\mathfrak{x}}[S] f' F^S}{\mathscr{D}_{X,\mathfrak{x}}[S] \cdot g f' F^S} \implies B(S) \in \operatorname{ann}_{\mathbb{C}[S]} \frac{\mathscr{D}_{X,\mathfrak{x}}[S](gf'f_{\text{red}})^{-1} F^{-S}}{\mathscr{D}_{X,\mathfrak{x}}[S] \cdot (f' f_{\text{red}}) F^{-S}}$$

where we may assume f_{red} is as in Lemma 3.15, cf. Remark 2.9. In other words,

$$\begin{aligned} B(S) &\in \mathbb{C}[S] \cap (\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot g) \\ &\implies B(S) \in \mathbb{C}[S] \cap (\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f'gf_{\text{red}})^{-1}F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot g). \end{aligned}$$

By Lemma 3.15, φ induces a $\mathscr{D}_{X,\mathfrak{x}}$-automorphism that sends $\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta^{-S}_{(f' g f_{red})^{-1} F, \mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot g$ to $\mathscr{D}_{X,\mathfrak{x}}[S] \cdot \theta_{f'F,\mathfrak{x}} + \mathscr{D}_{X,\mathfrak{x}}[S] \cdot g$. Therefore $\varphi(B^I_{f'F,\mathfrak{x}}) \subseteq B^I_{f'F,\mathfrak{x}}$. The reverse containment follows from the fact φ is an involution. □

Remark 3.17. Suppose f, f', and F are as in Theorem 3.16, and I is the ideal generated by $g_1, \dots, g_u$ such that $f \in \mathscr{O}_{X,\mathfrak{x}} \cdot g_j$. If $\mathrm{Sp}^g_{f'F,\mathfrak{x}}$ and its $\mathscr{D}_{X,\mathfrak{x}}[S]$-dual are both resolutions, then φ fixes $B^I_{f'F,\mathfrak{x}}$. Note that φ depends only on the product of the g_j.

Let us catalogue some of the most useful versions of the theorem:

Corollary 3.18. *Suppose $f = f_1 \cdots f_r \in \mathscr{O}_X$ is free, strongly Euler-homogeneous, and Saito-holonomic, and while f is not necessarily reduced, suppose that it admits a strongly Euler-homogeneous reduced defining equation at $\mathfrak{x}$. Let $F = (f_1, \dots, f_r)$ and φ be as in Theorem 3.16.*

(a) Suppose that $F = (l_1, \dots, l_1, \dots, l_q)$ with each l_t appearing v_t times, and f' and g any elements of $\mathscr{O}_{X,\mathfrak{x}}$ dividing f. Then $\varphi(B^g_{f'F,\mathfrak{x}}) = B^g_{f'F,\mathfrak{x}}$.

(b) Suppose f is reduced, F corresponds to any factorization, $f' = \prod_{k' \in K'} f'_k$, $g = \prod_{k \in K} f_k$, for $K', K \subseteq [r]$. Then $\varphi(B^g_{f'F,\mathfrak{x}}) = B^g_{f'F,\mathfrak{x}}$.

(c) Suppose f' divides $f = f_1 \cdots f_r$, $F = (f_1, \dots, f_r)$ and $g = \frac{f}{f'}$. If (f', F) is an unmixed pair up to units, then $\varphi(B^g_{f'F,\mathfrak{x}}) = B^g_{f'F,\mathfrak{x}}$

(d) Suppose $f = f^k_{\mathrm{red}}$ and $F = (f^k_{\mathrm{red}})$. Then $\varphi(s) = -s - 1 - \frac{1}{k}$ and $\varphi(B_{f^k,\mathfrak{x}}) = B_{f^k,\mathfrak{x}}$.

Proof. All that must be checked is that the appropriate things are unmixed pairs up to units. For example, in (a) and (b), F is unmixed up to units because it is a factorization into irreducibles, possibly with repetition, and because f is reduced, respectively. In both cases, d_k, d'_k, and d''_k are all 1. □

The symmetry property for the Bernstein–Sato polynomial of a reduced divisor forces all its roots to lie inside $(-2, 0)$, cf. [21]. We have the following generalization for powers of reduced divisor:

Corollary 3.19. *Suppose f is reduced, free, strongly Euler-homogeneous, and Saito-holonomic. Then $\mathrm{V}(B_{f^k}) \subseteq (-1 - \frac{1}{k}, 0)$. If $b_{f^k,\min}$ is the smallest root of the Bernstein–Sato polynomial of f^k, then $b_{f^k,\min} \to -1$ as $k \to \infty$.*

Proof. Since freeness, strongly Euler-homogeneous, and Saito-holonomicity pass from f_{red} to f^k we may use Corollary 3.18 to improve the well known containment $\mathrm{V}(B_{f^k,\mathfrak{x}}) \subseteq (-\infty, 0)$ to $\mathrm{V}(B_{f^k,\mathfrak{x}}) \subseteq (-1 - \frac{1}{k}, 0)$. The rest follows since $-1 \in \mathrm{V}(B_{f^k,\mathfrak{x}})$. □

4. Bernstein–Sato Varieties for Tame and Free Arrangements

In this section we study the global Bernstein–Sato ideals $B^g_{f'F}$ where f is a central, not necessarily reduced, tame hyperplane arrangement, f' divides f, $g = \frac{f}{f'}$, and F corresponds to the factorization $f = f_1 \cdots f_r$, which need not be into linear forms. We always assume $\mathscr{O}_{X,\mathfrak{x}} \cdot f' \neq \mathscr{O}_{X,\mathfrak{x}} \cdot f$. We revisit the arguments of Maisonobe in [18] giving full details for our versions of Lemma 2 and Proposition 9 in the first subsection and Proposition 10 in the second. We generalize the strategy of Lemma 2 and Proposition 9 to compute a principal ideal containing $B^g_{f'F}$ for tame hyperplane arrangements and any F; we generalize Proposition 10 to find an element of $B^g_{f'F}$ when f is not necessarily reduced, not necessarily tame, and F is the total factorization of f into linear forms. As Maisonobe does in Theorem 2 of loc. cit., in the third subsection we use the symmetry of $B^g_{f'F}$ when f is free and (f', F) is an unmixed pair up to units to provide rather precise estimates of $\mathrm{V}(B^g_{f'F})$. In certain situations, these estimates compute $\mathrm{V}(B^g_{f'F})$.

Definition 4.1. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ be a central, not necessarily reduced, hyperplane arrangement of degree d whose factorization into homogeneous linear forms is $f = l_1 \cdots l_d$. Associated to f is the intersection lattice $L(A)$, partially ordered by reverse inclusion and with smallest element $\mathbb{C}^n$. We call any $X \in L(A)$ an *edge* of $L(A)$. The *rank* of X is the length of a maximal chain in $L(A)$ with smallest element $\mathbb{C}^n$ and largest element X. We denote the rank of X by $r(X)$; for example, $r(\mathrm{V}(l_i)) = 1$. Given an edge $X \in L(A)$ we define $J(X)$ to be the subset of $[d]$ identifying the hyperplanes that contain X, that is:

$$X = \bigcap_{j \in J(X)} \mathrm{V}(l_j).$$

Note that because f is not necessarily reduced $J(X)$ may contain indices i and j such that $\mathrm{V}(l_i) = \mathrm{V}(l_j)$. Given an edge X, there is the subarrangement A_X which has the defining equation

$$f_X = \prod_{j \in J(X)} l_j.$$

The degree of f_X is denoted d_X. So $d_X = |J(X)|$. The edge X is *decomposable* if there is a change of coordinates $y_1 \sqcup y_2$, y_1 and y_2 disjoint, such that $f_X = pq$ where p and q are hyperplane arrangements using variables only from y_1 and y_2 respectively. Otherwise X is *indecomposable*.

Consider a potentially different factorization $f = f_1 \cdots f_r$ where each f_k is of degree d_k. Since each f_k is a product of some of the l_m, let $S_k \subseteq [d]$ identify the linear forms comprising f_k, that is,

$$f_k = \prod_{m \in S_k} l_m.$$

The factorization $f = f_1 \cdots f_r$ induces a factorization of f_X. Define $S_{X,k} \subseteq [d]$ by

$$S_{X,k} = J_X \cap S_k.$$

Then f_X inherits the factorization $f_X = f_{X,1} \cdots f_{X,r}$ where

$$f_{X,k} = \prod_{j \in S_{X,k}} l_j.$$

We say $f_{X,k}$ has degree $d_{X,k}$. We also write $F_X = (f_{X,1}, \ldots, f_{X,r})$.

Any hyperplane arrangement has a reduced equation f_{red} of degree d_{red}. We define $f_{X,\mathrm{red}}$, $d_{X,\mathrm{red}}$, $f_{X,k,\mathrm{red}}$, and $d_{X,k,\mathrm{red}}$ similarly.

If f' of degree d' divides f, then all the previous constructions apply to f'. Define $f'_{\mathrm{red}}, d'_{\mathrm{red}}$, f'_X, d'_X, $f'_{X,\mathrm{red}}, d'_{X,\mathrm{red}}, f'_{X,k}$, $d'_{X,k}$, $f'_{X,k,\mathrm{red}}$, $d'_{X,k,\mathrm{red}}$ in the natural ways.

We will be working with the Weyl algebra $\mathrm{A}_n(\mathbb{C}) = \mathbb{C}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]$ where the *global Bernstein–Sato ideal* $B^g_{f'F}$ is defined similarly to $B^g_{f'F,\mathfrak{x}}$ except using $\mathrm{A}_n(\mathbb{C})[S]$ operators. Write $B^g_{f'f}$ when $F = (f)$ corresponds to the trivial factorization $f = f$. We use the notation $\theta_{f'F}$ and $\psi_{f'F}$ for the algebraic, global versions of $\theta_{f'F,\mathfrak{x}}$ and $\psi_{f'F,\mathfrak{x}}$.

By Corollary 2.22 and Examples 2.7 and 2.10, if f is tame and f' divides f, then $\mathrm{ann}_{\mathrm{A}_n(\mathbb{C})[S]} f'F^S$ is generated by derivations. Moreover, f_{red} is strongly Euler-homogeneous itself. Finally, since f is central, the $\mathbb{C}^\star$-action on $\mathrm{V}(f)$ can be used to show $B^g_{f'F} = B^g_{f'F,0}$. Therefore we can apply the results of the previous sections.

Finally, recall that for any central hyperplane arrangement $f \in \mathbb{C}[x_1, \ldots, x_n]$ of degree d, the *Euler derivation* $E = x_1\partial_1 + \cdots x_n\partial_n$ satisfies $E \bullet f = df$. Thus $\frac{1}{d}E$ is a strong Euler-homogeneity for f at the origin.

4.1. **An Ideal Containing $B^g_{f'F}$.**

We compute a principal ideal containing $B^g_{f'F}$ where f is a central, indecomposable, and tame hyperplane arrangement, f' divides f, $g = \frac{f}{f'}$, and F corresponds to any factorization. The argument tracks Lemma 2 and Proposition 9 of [18] but we have replaced freeness with tameness, reduced with non-reduced, added f', and we will use any factorization F instead of the factorization into linear forms. Though the approach is similar to Maisonobe's, we provide detail for the sake of the reader.

Definition 4.2. The *right normal form* of $P \in \mathrm{A}_n(\mathbb{C})[S]$ is the unique expression

$$P = \sum_{\mathbf{u}} \partial^{\mathbf{u}} P_{\mathbf{u}}$$

where $P_{\mathbf{u}} \in \mathbb{C}[X][S]$. The *right constant term* of P is $P_{\mathbf{0}}$. Note that for $P, Q \in \mathrm{A}_n(\mathbb{C})[S]$, the right constant term of $P + Q$ is $P_{\mathbf{0}} + Q_{\mathbf{0}}$.

Convention 4.3. Let $\mathbb{C}[X]_t$ be the subspace of homogeneous polynomials in $\mathbb{C}[X]$ of degree t and let $\mathbb{C}[X]_{\geq t}$ be the ideal of $\mathbb{C}[X]$ generated by the homogeneous polynomials of degree at least t. Denote by $\mathbb{C}[X]_t[S]$ and $\mathbb{C}[X]_{\geq t}[S]$ the $\mathbb{C}[S]$-modules generated by $\mathbb{C}[X]_t$ and $\mathbb{C}[X]_{\geq t}$ respectively.

Lemma 4.4. *Consider a derivation $\delta = \sum_i a_i \partial_{x_i}$ and a polynomial $c \in \mathbb{C}[X][S]$. If $P \in A_n(\mathbb{C})[S]$ has right constant term $P_{\mathbf{0}}$, then $P \cdot (\delta - c)$ has right constant term*

$$-(\sum_i \partial_{x_i} \bullet a_i) P_{\mathbf{0}} - \delta \bullet (P_{\mathbf{0}}) - c P_{\mathbf{0}}.$$

Proof. Consider the right normal form $\sum \partial^{\mathbf{u}} P_{\mathbf{u}}$ of P. Then

$$\begin{aligned} P \cdot (\delta - c) &= \sum_{\mathbf{u}} \partial^{\mathbf{u}} (\delta P_{\mathbf{u}} - \delta \bullet P_{\mathbf{u}} - P_{\mathbf{u}} c) \\ &= \sum_{\mathbf{u}} \partial^{\mathbf{u}} ((\sum_i \partial_i a_i - \sum_i \partial_i \bullet a_i) P_{\mathbf{u}} - \delta \bullet P_{\mathbf{u}} - P_{\mathbf{u}} c) \\ &= \sum_{\mathbf{u}} \partial^{\mathbf{u}} \sum_i \partial_i a_i P_{\mathbf{u}} + \sum_{\mathbf{u}} \partial^{\mathbf{u}} ((- \sum_i \partial_i \bullet a_i) P_{\mathbf{u}} - \delta \bullet (P_{\mathbf{u}}) - c P_{\mathbf{u}}). \end{aligned}$$

Because $\sum_{\mathbf{u}} \partial^{\mathbf{u}} \sum_i \partial_i a_i P_{\mathbf{u}}$ has constant term 0, the lemma follows. □

Lemma 4.5. *Suppose $\delta \in \mathrm{Der}_X(-\log f)$ can be written as $\sum_{i=1}^n a_i \partial_i$ where each a_i is a homogeneous polynomial of degree t in $\mathbb{C}[X]$. Let $f = f_1 \cdots f_r$ where each f_k is homogeneous, $F = (f_1, \ldots, f_r)$, and f' is a homogeneous polynomial dividing f. If $P \in A_n(\mathbb{C})[S]$, then the right constant term of $P \cdot \psi_{f'F}(\delta)$ lies in $\mathbb{C}[X]_{\geq t-1}[S]$.*

Proof. Recall $\psi_{f'F}(\delta) = \delta - \sum \frac{\delta \bullet f_k}{f_k} s_k - \frac{\delta \bullet f'}{f'}$. By the choice of δ,

$$- \sum_{k=1}^{r} \frac{\delta \bullet f_k}{f_k} s_k - \frac{\delta \bullet f'}{f'} \in \mathbb{C}[X]_{t-1}[S].$$

By Lemma 4.4, the right constant term of $P \cdot \psi_F(\delta)$ is

$$(- \sum_i \partial_i \bullet a_i) P_0 - \delta \bullet P_0 - (\sum_k \frac{\delta \bullet f_k}{f_k} s_k) P_0 - \frac{\delta \bullet f'}{f'} P_{\mathbf{0}}.$$

Let m be the smallest nonnegative integer such that $P_0 \in \mathbb{C}[X]_{\geq m}[S]$. Because $\partial_i \bullet a_i \in \mathbb{C}[X]_{t-1}$ and $\delta \bullet P_0 \in \mathbb{C}[X]_{\geq t+m-1}[S]$ the claim follows. □

There is a natural $\mathbb{C}[X]$-isomorphism between $\operatorname{Der}_X(-\log_0 f)$ and the first syzygies of the Jacobian ideal $J(f)$, i.e. the ideal of $\mathbb{C}[X]$ generated by the partials of f. If f is homogeneous, so is $J(f)$ and so is its first syzygy module.

Definition 4.6. For f homogeneous, define $\operatorname{mdr}(f)$ to be

$$\operatorname{mdr}(f) = \min\{t \mid \text{ there exists a homogeneous syzygy of } J(f) \text{ of degree } t\}.$$

Remark 4.7. (a) It known that a central hyperplane arrangement of f of rank ≥ 2 is indecomposable if and only if $\operatorname{mdr}(f) \geq 2$. For one direction use the first part of Theorem 5.13 of [29]; for the other, use the two disjoint Euler derivations induced by the coordinate change.

(b) Identify $\operatorname{Der}_X(-\log_0 f)$ and first syzygies of $J(f)$ to conclude that we may pick a generating set $\delta_1, \dots, \delta_m$ of $\operatorname{Der}_X(-\log_0 f)$ such that $\delta_j = \sum_{i=1}^r a_{j,i}\partial_i$ and each $a_{j,i} \in \mathbb{C}[X]$ is homogeneous of degree at least $\operatorname{mdr}(f)$.

We can now prove our version of Lemma 2 from [18]. The argument is similar but we defer applying any symmetry of $B^g_{f'F}$ until later.

Theorem 4.8. (Compare to Lemma 2 in [18]) *Let f be a central, not necessarily reduced, indecomposable and tame hyperplane arrangement of rank $n \geq 2$ and let $F = (f_1, \dots, f_r)$ correspond to any factorization $f = f_1 \cdots f_r$. If f' divides f and $g = \frac{f}{f'}$, then*

$$B^g_{f'F} \subseteq \mathbb{C}[S] \cdot \prod_{j=0}^{\operatorname{mdr}(f)+d-d'-3} \left(\sum_k d_k s_k + n + d' + j \right).$$

Proof. To begin, we choose two polynomials. First fix $0 \neq B(S) \in B^g_{f'F}$. By definition of $B^g_{f'F,\mathfrak{x}}$, the polynomial $B(S)$ lies in $\operatorname{ann}_{\mathrm{A}_n(\mathbb{C})[S]} f'F + \mathrm{A}_n(\mathbb{C})[S] \cdot g$. Second, pick a nonzero homogeneous polynomial $v \in \mathbb{C}[X]$ such that (i) $\deg(v) \leq \operatorname{mdr}(f) - 2$ and (ii) there exists a point $\alpha \in \mathrm{V}(g) \setminus \mathrm{V}(v)$. By Remark 4.7 such a choice of v is possible. Note that

$$vB(S) \in \operatorname{ann}_{\mathrm{A}_n(\mathbb{C})[S]} f'F + \mathrm{A}_n(\mathbb{C})[S] \cdot g.$$

Let $\delta_1, \dots, \delta_m$ generate $\operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f)$ where $\delta_j = \sum_j a_{j,i}\partial_i$; let E by the Euler derivation. By Remark 4.7, we may assume $\{a_{j,i}\}_i$ are all homogeneous polynomials of the same degree where that degree is at least $\operatorname{mdr}(f)$. Corollary 2.22 implies there exist $L, P, Q_2, \dots, Q_m \in \mathrm{A}_n(\mathbb{C})[S]$ such that

$$vB(S) = Lg + P\psi_{f'F}(E) + \sum_{j=2}^m Q_j \psi_{f'F}(\delta_j). \tag{4.1}$$

Express both sides of (4.1) in their right normal form. First consider the right hand side of (4.1). By Lemma 4.5, the right constant term of $Q_j\psi_{f'F}(\delta_j)$ is in $\mathbb{C}[X]_{\geq \operatorname{mdr}(f)-1}[S]$. Write the right constant term $L_{\mathbf{0}}$ of L as $L_{\mathbf{0}} = \sum_t L^t_{\mathbf{0}}$ where $L^t_{\mathbf{0}} \in \mathbb{C}[X]_t[S]$; similarly, write the right constant term $P_{\mathbf{0}}$ of P as $P_{\mathbf{0}} = \sum_t P^t_{\mathbf{0}}$ where $P^t_{\mathbf{0}} \in \mathbb{C}[X]_t[S]$. The right constant term of Lg is $L_{\mathbf{0}}g$. By Lemma 4.4, the right constant term of $P\psi_{f'F}(E)$ is

$$\sum_t -nP^t_{\mathbf{0}} - E \bullet P^t_{\mathbf{0}} - \left(\sum_k \frac{E \bullet f_k}{f_k} s_k \right) P^t_{\mathbf{0}} - \frac{E \bullet f'}{f'} P^t_{\mathbf{0}}$$
$$= \sum_t \left(-n - t - \sum_k d_k s_k - d' \right) P^t_{\mathbf{0}}.$$

On the other hand, the right constant term of $vB(S)$ is $vB(S)$ itself. Note that

$$vB(S) \in \mathbb{C}[X]_{\deg(v)}[S]$$

and, by the choice of v, $\deg(v) < \mathrm{mdr}(f) - 1$. So when we write the right constant term of both sides of (4.1), the left hand side is $vB(S)$ and the right hand side can be written using only terms in $\mathbb{C}[X]_{\deg(v)}[S]$. We deduce

$$vB(S) = L_{\mathbf{0}}^{\deg(v)} g + \left(-n - \deg(v) - d' - \sum_k d_k s_k \right) P_{\mathbf{0}}^{\deg(v)}. \tag{4.2}$$

The equation (4.2) occurs in $\mathbb{C}[X]_{\deg(v)}[S]$ and so the equality is still true when regarding all the elements as belonging to $\mathbb{C}[X][S]$. By the choice of v, there exists $\alpha \in \mathrm{V}(g) \setminus \mathrm{V}(v)$. The polynomial $P_{\mathbf{0}}^{\deg(v)}$ cannot vanish at α, lest $B(S) = 0$. By evaluating (4.2) at α we see

$$B(S) \in \mathbb{C}[S] \cdot \left(-n - \deg(v) - d' - \sum_k d_k s_k \right). \tag{4.3}$$

As $\deg(v)$ is flexible,

$$B_{f'F,\mathfrak{x}}^{g} \subseteq \mathbb{C}[S] \cdot \prod_{j=0}^{\mathrm{mdr}(f)-2} \left(\sum_k d_k s_k + n + d' + j \right). \tag{4.4}$$

Now suppose $(f) \subseteq (f'') \subseteq (f')$ and let $g'' = \frac{f}{f''}$. Since f is a hyperplane arrangement we can choose f'' to be of any degree between d' and $d-1$. Because $B_{f'F}^{g} \subseteq B_{f'',F}^{g''}$, the containment (4.4) can be improved to

$$B_{f'F}^{g} \subseteq \mathbb{C}[S] \cdot \prod_{j=0}^{\mathrm{mdr}(f)+d-d'-3} \left(\sum_k d_k s_k + n + d' + j \right).$$

□

Remark 4.9. (a) It is easy to see, see Corollary 6 in [2] for the B_F statement, that

$$B_{f'F}^{g} = \bigcap_{\mathfrak{x} \in \mathbb{C}^n} B_{f'F,\mathfrak{x}}^{g}.$$

(b) Recall the notation of Definition 4.1. Given an edge $X \in L(A)$, there exists a $\mathfrak{x} \in X$ such that $\mathfrak{x} \notin \mathrm{V}(l_m)$ for all $m \notin J(X)$. By definition,

$$F_X = (f_{X,1}, \ldots, f_{X,r}) = (\prod_{j \in S_{x,1}} l_j, \ldots, \prod_{j \in S_{X,r}} l_j).$$

We may write F as

$$F = (\prod_{m \in S_1 \setminus S_{X,1}} l_m \prod_{j \in S_{X,1}} l_j, \ldots, \prod_{m \in S_r \setminus S_{X,r}} l_m \prod_{j \in S_{X,r}} l_j).$$

So at $\mathfrak{x}$, the decompositions F and F_X differ by multiplying each component by a unit at $\mathfrak{x}$. Arguing as in Lemma 10 of [2] (see also the first paragraph of the proof of Theorem 3.16), we deduce

$$B_{f'F,\mathfrak{x}}^{g} = B_{f_X' F_X,\mathfrak{x}}^{g_X}.$$

Since $\mathfrak{x}$ and 0 both lie in the maximal edge of f_X, $B_{f_X' F_X,0}^{g_X} = B_{f_X' F_X,\mathfrak{x}}^{g_X}$. The centrality of f_X, and the consequent $\mathbb{C}^{\star}$-action on $\mathrm{V}(f_X)$, implies

$$B_{f_X' F_X,0}^{g_X} = B_{f_X' F_X}^{g_X}.$$

(c) Putting (a) and (b) together yields

$$B^{g}_{f'F} = \bigcap_{X \in L(A)} B^{g_X}_{f'_X F_X}.$$

The following definition will help simplify notation.

Definition 4.10. Let $f = f_1 \cdots f_r$ be any factorization of a central hyperplane arrangement and $F = (f_1, \ldots, f_r)$. Suppose f' divides f; $g = \frac{f}{f'}$. For any indecomposable edge X define the polynomial

$$P^{g}_{f'F,X} = \sum_{k} d_{X,k} s_k + r(X) + d'_X \in \mathbb{C}[S].$$

Remark 4.9 and Theorem 4.8 prove our version of Proposition 9 in [18]:

Theorem 4.11. (Compare to Proposition 9 of [18]) *Suppose f is a central, tame, not necessarily reduced, hyperplane arrangement of rank n and let $F = (f_1, \ldots, f_r)$ correspond to any factorization $f = f_1 \cdots f_r$. Let f' divide f and $g = \frac{f}{f'}$. For indecomposable edges X of rank ≥ 2 define*

$$p_{f'F,X}(S) = \prod_{j_X=0}^{mdr(f_X)+d_X-d'_X-3} \left(P^{g}_{f'F,X} + j_X\right).$$

For indecomposable edges X of rank one define

$$p_{f'F,X}(S) = \prod_{j_X=0}^{d_X-d'_X-1} \left(P^{g}_{f'F,X} + j_X\right).$$

Then

$$B^{g}_{f'F} \subseteq \mathbb{C}[S] \cdot \operatorname{lcm}\left\{p_{f'F,X}(S) \mid X \in L(A),\ X \text{ indecomposable}\right\}.$$

Proof. By Remark 4.9,

$$B^{g}_{f'F} = \left(\bigcap_{\substack{X \in L(A) \\ r(X) \geq 2}} B^{g_X}_{f'_X F_X} \right) \bigcap \left(\bigcap_{\substack{X \in L(A) \\ r(X) = 1}} B^{g_X}_{f'_X F_X} \right)$$

If X is an edge of rank ≥ 2, then Theorem 4.8 combined with Definition 4.10 says

$$B^{g_X}_{f'_X F_X} \subseteq \mathbb{C}[S] \cdot \prod_{j_X=0}^{\mathrm{mdr}(f_X)+d_X-d'_x-3} (P^{g}_{f'F,X} + j_X).$$

Therefore, once we prove that for rank one edges X

$$B^{g_X}_{f'_X F_X} \subseteq \mathbb{C}[S] \cdot \prod_{j_X=0}^{d_X-d'_X-1} \left(P^{g}_{f'F,X} + j_X\right),$$

then the claim will follow.

For the rank one edges, argue as in Theorem 4.8. Since the rank is one, we can get an equation resembling (4.1) without any $\psi_{f'F}(\delta)$ terms and with $v = 1$. Now looking at the right constant terms, since $B(S) \in \mathbb{C}[S]$ and $L_{\mathbf{0}}g$ is not, we deduce (4.3) holds with $\deg(v) = 0$. The other factors of $p_{f'F}$ are found using the containment $B^{g}_{f'F} \subseteq B^{g''}_{f''F}$, as in the final paragraph of Theorem 4.8. ☐

4.2. **An Element of $B^g_{f'F}$.**

Here we drop the assumption of tameness and compute an element of $B^g_{f'F}$ for $f = f_1 \cdots f_r$ any factorization of a central, not necessarily reduced, hyperplane arrangement f and where f' and g are as before. The bulk of the argument tracks Proposition 10 of [18], however we have removed the reducedness hypothesis. Again, we provide detail for the reader's sake.

We begin with some basic facts about differential operators. First, consider a product of functions fg with factorizations $f = f_1 \dots f_r$ and $g = g_1 \dots, g_u$. Let $F = (f_1, \dots, f_r)$ and $G = (g_1, \dots, g_u)$ and $FG = (f_1, \dots, f_r, g_1, \dots, g_u)$.

Definition 4.12. Let $P \in \mathrm{A}_n(\mathbb{C})[S]$ and consider $\mathrm{A}_n(\mathbb{C})[S](FG)^S$. Relabel the s_k so that we may write $\mathrm{A}_n(\mathbb{C})[S,T]F^S G^T = \mathrm{A}_n(\mathbb{C})[S] f_1^{s_1} \cdots f_r^{s_r} g_1^{t_1} \cdots g_u^{t_u}$ and consider P as in $\mathrm{A}_n(\mathbb{C})[S,T]$. As there is an $\mathrm{A}_n(\mathbb{C})[S]$-action on F^S there is a naturally defined $\mathrm{A}_n(\mathbb{C})[S,T]$ action. Denote by $P \bullet F^S$ the result of letting P act on F^S.

Lemma 4.13. *Let $P \in A_n(\mathbb{C})[S]$ of total order k, i.e. $P \in F^k_{(0,1,1)} A_n(\mathbb{C})[S]$. Then*

$$PF^S G^T - (P \bullet F^S) G^T \in A_n(\mathbb{C})[S,T] F^S G^{T-k}.$$

Proof. It is sufficient to prove the following:

Claim: If $h \in \mathbb{C}[X][S][T]$, there exists $Q_{\mathbf{u}}$ of total order at most $|\mathbf{u}|$ such that

$$\partial^{\mathbf{u}} h F^S G^T - h(\partial^{\mathbf{u}} \bullet F^S) G^T = Q_{\mathbf{u}} F^S G^{T-|\mathbf{u}|}.$$

We prove this by induction on $|\mathbf{u}|$. The base case is straightforward. For the inductive step, observe:

$$\begin{aligned} \partial_1 \partial^{\mathbf{u}} h F^S G^T &= \partial_1 [h(\partial^{\mathbf{u}} \bullet F) G^T + Q_{\mathbf{u}} F^S G^{T-|\mathbf{u}|}] \\ &= (\partial_1 \bullet h)(\partial^{\mathbf{u}} \bullet F^S) G^T + h(\partial_1 \partial^{\mathbf{u}} \bullet F^S) G^T \\ &+ h(\partial^{\mathbf{u}} \bullet F)(g \sum_k t_k \frac{\partial_1 \bullet g_k}{g_k}) G^{T-1} + \partial_1 Q_{\mathbf{u}} F^S G^T. \end{aligned} \tag{4.5}$$

Since $\partial_1 \bullet h \in \mathbb{C}[X][S][T]$ the induction hypothesis implies

$$(\partial_1 \bullet h)(\partial^{\mathbf{u}} \bullet F^S) G^T \in F^{|\mathbf{u}|}_{(0,1,1)} \mathrm{A}_n(\mathbb{C})[S][T] F^S G^{T-|\mathbf{u}|}.$$

Similarly, since $h(g \sum_k t_k \frac{\partial_1 \bullet g_k}{g_k}) \in \mathbb{C}[S][T]$, by induction

$$h(\partial^{\mathbf{u}} \bullet F^S)(g \sum_k t_k \frac{\partial_1 \bullet g_k}{g_k}) G^{T-1} \in F^{|\mathbf{u}|}_{(0,1,1)} \mathrm{A}_n(\mathbb{C})[S][T] F^S G^{T-|\mathbf{u}|-1}.$$

Rearranging (4.5) proves the claim and hence the lemma. □

We also need the following elementary lemma.

Lemma 4.14. *Let $E = x_1 \partial_1 + \cdots + x_n \partial_n$ be the Euler derivation. Then*

$$\prod_{j=0}^{t} (E + n + j) = \sum_{\substack{u_1, \dots, u_n \\ u_1 + \cdots + u_n = t+1}} \binom{t+1}{u_1, \cdots, u_n} \partial^{\boldsymbol{u}} x^{\boldsymbol{u}}.$$

Proof. This also succumbs to induction on t after utilizing Pascal's formula for multinomial coefficients. □

Definition 4.15. Consider a central, essential, not necessarily reduced, hyperplane arrangement of rank n defined by $f = l_1 \cdots l_d$, where the l_k are homogeneous linear forms. Write

$$L = (l_1, \ldots, l_d).$$

For an edge $X \in L(A)$ and with $J(X)$ as in Definition 4.1, define the ideal $\Gamma_L \subseteq \mathbb{C}[x_1, \ldots, x_n]$ by

$$\Gamma_L = \sum_{\substack{X \in L(A) \\ r(X)=n-1}} \mathbb{C}[x_1, \ldots, x_n] \cdot \prod_{k \notin J(X)} l_k.$$

Lemma 4.16. *Consider a central, essential, not necessarily reduced, hyperplane arrangement of rank n defined by $f = l_1 \cdots l_d$, where the l_k are homogeneous linear forms. Let $L = (l_1, \ldots, l_d)$ and denote the ideal of $\mathbb{C}[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$ by $\mathfrak{m}$. Then there exists an integer k such that $\mathfrak{m}^k \subseteq \Gamma_L$.*

Proof. It suffices to show Γ_L is $\mathfrak{m}$-primary since $\mathfrak{m}$ is maximal and $\mathbb{C}[x_1, \ldots, x_n]$ is Noetherian. So we need only show $\mathrm{V}(\Gamma_L) = \{0\}$. Suppose $0 \neq p \in \mathrm{V}(\Gamma_L)$. Since $\mathrm{V}(\Gamma_L)$ is the intersection of unions of central hyperplanes, we deduce $\mathrm{V}(\Gamma_L)$ contains a codimension $n-1$ line. We may find a largest edge X containing said line; if X is not of codimension $n-1$ enlarge X further to a codimension $n-1$ edge. So for all $k \notin J(X)$, $\mathrm{V}(l_k)$ will not contain this line and hence will not contain p. But $p \in \mathrm{V}(\Gamma_F) \subseteq \mathrm{V}(\prod_{k \notin J(X)} l_k) = \cup_{k \notin J(X)} \mathrm{V}(l_k)$, contradicting $p \in \mathrm{V}(\Gamma_L)$. □

Remark 4.17. We need essentiality in the above lemma lest the maximal edge of $L(A)$ have rank $n-1$ forcing $\Gamma_F = 1$. Without this condition, the X selected in the above proof could be the maximal edge of $L(A)$.

Recall the notation of Definition 4.1. We proceed to the subsection's main idea, which is a generalization of Proposition 10 of [18] and is proved similarly.

Theorem 4.18. (Compare to Proposition 10 of [18]) *Consider a central, not necessarily reduced, hyperplane arrangement $f = l_1 \cdots l_d$ where the l_k are linear terms and let $L = (l_1, \ldots, l_d)$. Suppose that f' divides f; let $g = \frac{f}{f'}$. Then there is a positive integer N such that*

$$\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j=0}^{N} \left(P^g_{f'L,X} + j \right) \in B^g_{f'F}.$$

Proof. We prove this by induction on the rank of $L(A)$ and first deal with the inductive step. So we may assume the rank is n and f is essential. If f is decomposable into $f_1 f_2$, then f' (resp. g) inherts a decomposition $f'_1 f'_2$ (resp. $g_1 g_2$). If F_1 (resp. F_2) is the associated factorization of f_1 (resp. f_2) into linear forms and if $b_1 \in B^{g_1}_{f'_1 F_1}$ and $b_2 \in B^{g_2}_{f'_2 F_2}$, then $b_1 b_2 \in B^g_{f'F}$. In this case the induction hypothesis applies to $B^{g_1}_{f'_1 F_1}$ and $B^{g_2}_{f'_2 F_2}$. So we may assume f is indecomposable.

Let $\mathfrak{m}$ be the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ generated by $x_1, \ldots, x_n$. On the one hand, Lemma 4.14 implies that for all positive integers t

$$\prod_{j=0}^{t} (s_1 + \cdots + s_d + n + d' + j) f' L^S = \prod_{j=0}^{t} (E + n + j) f' L^S \in \mathrm{A}_n(\mathbb{C}) \cdot \mathfrak{m}^{t+1} f' L^S.$$

By Lemma 4.16, for any positive integer m there exists an integer N large enough so that

$$\prod_{j=0}^{N} (s_1 + \cdots + s_d + n + d' + j) f' L^S \in \sum_{\substack{X \in L(A) \\ r(X)=n-1}} \mathrm{A}_n(\mathbb{C})[S] (\prod_{k \notin J(X)} l_k)^m f'_X L^S. \tag{4.6}$$

Note we have folded some of the factors of f' into $(\prod_{k \notin J(X)} l_k)^m$.

By induction, for each such edge X of rank less than n, there exists a differential operator P_X of total order k_X and a polynomial $b_X \in \mathbb{C}[S]$ such that $P_X \prod_{i \in J(X)} l_i^{s_i+1} = b_X f'_X \prod_{i\in J(X)} l_i^{s_i}$. Fix m large enough so that $m > \max\{k_X \mid X \in L(A),\ X \text{ codimension } n-1\}$. Consequently, choose N large enough so that (4.6) holds for this fixed m. Lemma 4.13 implies

$$\begin{aligned} b_X(\prod_{k\notin J(X)} l_k)^m f'_X L^S &= (b_X f'_X \prod_{i\in J(X)} l_i^{s_i})(\prod_{k\notin J(X)} l_k^{s_k+m}) \\ &\in \mathrm{A}_n(\mathbb{C})[S](\prod_{i\in J(X)} l_i^{s_i+1})(\prod_{k\notin J(X)} l_k^{s_k+m-k_X}) \\ &\subseteq \mathrm{A}_n(\mathbb{C})[S]L^{S+1}. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) we deduce

$$\prod_{j=0}^{N}(s_1+\cdots+s_d+n+d'+j)(\prod_{\substack{X\in L(A)\\ r(X)=n-1}} b_X) f' L^S \in \mathrm{A}_n(\mathbb{C})[S]L^{S+1}. \tag{4.8}$$

The result follows by the inductive description of each b_X and the definition of $P^g_{f'L,X}$. Note we may have to replace either the N chosen in (4.8) or the N coming from the inductive hypothesis with a larger integer so that the final polynomial is in the promised form. There is no harm in this as it can only only add linear factors to the polynomial appearing in (4.8) and does not change the containment.

All that remains is the base case, but this is obvious by a direct computation using Lemma 4.14. ∎

This theorem only gives an element of $B^g_{f'L}$ when L is a factorization into linear forms. If f is tame we can find an element no matter the factorization.

Corollary 4.19. *Let $f = f_1 \cdots f_r$ be a central, not necessarily reduced, tame hyperplane arrangement where the f_k are not necessarily linear forms. Let $F = (f_1, \ldots, f_r)$. Suppose f' divides f; let $g = \frac{f}{f'}$. If L corresponds to the factorization of f into linear terms, then there exists a positive integer N such that*

$$\prod_{\substack{X\in L(A)\\ X \text{ indecomposable}}} \prod_{j=0}^{N}\left(P^g_{f'L,X}+j\right) \text{ modulo } S_F \in B^g_{f'F},$$

where S_F is as in Definition 2.25.

Proof. Use Proposition 2.26. ∎

Just as in the last part of Theorem 2 of [18], 4.18 also implies $B^g_{f'L}$ is principal. (Here we very much need L to correspond to a factorization into linear forms.)

Corollary 4.20. *Consider the central, not necessarily reduced, free hyperplane arrangement $f = l_1 \cdots l_d$, where the l_k are linear forms, and let $L = (l_1, \cdots, l_d)$. Suppose f' divides f; let $0 \neq g$ divide $\frac{f}{f'}$. Then $B^g_{f'L}$ equals its radical and is principal.*

Proof. Let $P(S)$ be the polynomial of Theorem 4.18. If g divides $\frac{f}{f'}$, then by said theorem $P(S) \in B^g_{f'L}$. The claim then follows by Proposition 3.12 and Proposition 3.13 since $P(S)$ cuts out a reduced hyperplane arrangement. ∎

4.3. **Computations and Estimates.**

We now have combinatorial determined ideal subsets and supsets of $B^g_{f'F}$. In general, $\mathrm{V}(B_f)$ is not combinatorially determined. However, if f is tame, then $\mathrm{V}(B_f) \cap [-1, 0]$ is combinatorial.

Theorem 4.21. *Let f be a central, not necessarily reduced, tame hyperplane arrangement. Suppose f' divides f; let $g = \frac{f}{f'}$. Then the roots $\mathrm{V}(B^g_{f'f})$ lying in $[-1, 0)$ are combinatorially determined:*

$$\mathrm{V}(B^g_{f'f}) \cap [-1, 0) = \bigcup_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \bigcup_{j_X = r(X) + d'_X}^{d_X} \frac{-j_X}{d_X}.$$

Setting $f' = 1$ gives the roots of the Bernstein–Sato polynomial of f lying in $[-1, 0)$.

Proof. We find a subset and supset of $B^g_{f'F}$ using Corollary 4.19 and Theorem 4.11 respectively. Their varieties will be equal after intersecting with $[-1, 0)$ once we verify the following inequalities for indecomposable edges X: $r(X) + \mathrm{mdr}(f) + d_X - 3 \geq d_X$ if $r(X) \geq 2$; $1 + d_X - 1 \geq d_X$ if $r(X) = 1$. The second is trivial. The first is as well: since X is indecomposable $\mathrm{mdr}(f) \geq 2$. □

Example 4.22. In [29], Walther showed the Bernstein–Sato polynomial of an arrangement is not combinatorially determined. He gives the following two arrangements that have the same intersection lattice, but the former has $\frac{-18+2}{9}$ as a root and the latter does not:

$$f = xyz(x + 3z)(x + y + z)(x + 2y + 3z)(2x + y + z)(2x + 3y + z)(2x + 3y + 4z);$$
$$g = xyz(x + 5z)(x + y + z)(x + 3y + 5z)(2x + y + z)(2x + 3y + z)(2x + 3y + 4z).$$

Because these arrangements are rank 3 they are automatically tame, cf. Remark 2.5. The above theorem says the roots of the b-polynomials agree inside $[-1, 0)$. In Remark 4.14.(iv) of [23], Saito shows that their roots agree except for $\frac{-18+2}{9}$.

For the rest of the subsection we restrict to free hyperplane arrangments. In [18], Maisonobe used the symmetry of B_L, when L corresponded to a factorization of a reduced f into linear terms, to make his estimates of B_L so precise they actually computed B_L, cf. Theorem 2 in loc. cit. We use the symmetry of $B^g_{f'F}$ given by φ of Theorem 3.16 similarly, but our situation is more technical because of the addition of f', the lack of reducedness, and our focus on different factorizations F.

Lemma 4.23. *Let $f = f_1 \cdots f_r$ be an unmixed factorization of a central hyperplane arrangement and let $F = (f_1, \ldots, f_r)$. Suppose f' divides f; $g = \frac{f}{f'}$. If (f', F) is an unmixed pair and φ the $\mathbb{C}[S]$-automorphism prescribed in Theorem 3.16, then*

$$\varphi(P^g_{f'F,X}) = -(P^g_{f'F,X} + d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X).$$

Proof. First notation. Factor $f = l_1^{v_1} \cdots l_q^{v_q}$, where the l_t pairwise distinct irreducibles. Let $\{m_k\}$ be the repeated multiplicities of F; $\{d'_k, d_k\}_k$ and $\{d''_k, d_k\}_k$ the repeated powers of the unmixed pairs (f', F) and (g, F). Because $f'g = f$, the formulation of φ in Theorem 3.16 can be simplified:

$$\begin{aligned}
\varphi(\sum_k d_{X,k} s_k) &= -\sum_k d_{X,k}(s_k + \frac{1}{m_k} + \frac{2d'_k}{d_k} + \frac{d''_k}{d_k}) \\
&= -\sum_k d_{X,k}(s_k + \frac{1}{m_k} + \frac{d'_k}{d_k} + 1) \\
&= -\sum_k d_{X,k}(s_k + \frac{1}{m_k}) - \sum_k d_{X,k,\mathrm{red}} d'_k - d_X
\end{aligned}$$

$$= -\sum_k d_{X,k}(s_k + \frac{1}{m_k}) - d'_X - d_X.$$

After rearranging, we will be done once we show that $\sum_k \frac{d_{X,k}}{m_k} = d_{X,red}$.

Fix $k \in [r]$. Observe:

$$\prod_{\substack{t\in[q] \\ v_t = m_k}} l_t^{m_k} = \prod_{\substack{i\in[r] \\ m_i = m_k}} f_i = \prod_{\substack{i\in[r] \\ m_i = m_k}} \prod_{\substack{t\in[q] \\ f_i \in (l_t)}} l_t^{d_i}. \tag{4.9}$$

Equality will still hold in (4.9) if we further restrict t to the integers such that l_t divides f_X. The degrees of the resulting polynomials are equal:

$$\begin{aligned} m_k \ |\{l_t \mid v_t = m_k; f_X \in (l_t)\}| &= \sum_{\substack{i\in[r] \\ m_i = m_k}} d_i \ |\{l_t \mid f_i, f_X \in (l_t)\}| \\ &= \sum_{\substack{i\in[r] \\ m_i = m_k}} d_i \ d_{X,i,\text{red}} \\ &= \sum_{\substack{i\in[r] \\ m_i = m_k}} d_{X,i}. \end{aligned} \tag{4.10}$$

Therefore

$$\begin{aligned} \sum_k \frac{d_{X,k}}{m_k} = \sum_{p\in\{m_k\}} \sum_{\substack{i\in[r] \\ m_i = p}} \frac{d_{X,k}}{p} &= \sum_{p\in\{m_k\}} |\{l_t \mid v_t = p; f_X \in (l_t)\}| \\ &= \sum_{p\in\{v_t\}} |\{l_t \mid v_t = p; f_X \in (l_t)\}| \\ &= d_{X,\text{red}}. \end{aligned} \tag{4.11}$$

□

First we use Theorem 4.18 and the symmetry of $B^g_{f'L}$ to find an element of $B^g_{f'L}$ that more accurately approximates the Bernstein–Sato ideal.

Proposition 4.24. *Consider the central, not necessarily reduced, free hyperplane arrangement $f = l_1 \cdots l_d$, where the l_k are linear forms, and let $L = (l_1, \ldots, l_d)$. Suppose f' divides f; let $g = \frac{f}{f'}$. Then*

$$\prod_{\substack{X\in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{x,\text{red}}+d_X-2r(X)-d'_X} \left(P^g_{f'L,X} + j_X\right) \in B^g_{f'L}. \tag{4.12}$$

Proof. By Theorem 4.18 there exists a positive integer N such that

$$\prod_{\substack{X\in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{N} \left(P^g_{f'L,X} + j_X\right) \in B^g_{f'L}. \tag{4.13}$$

Since (f', L) are an unmixed pair up to units by virtue of L being a factorization into linear forms, by Theorem 3.16/Corollary 3.18 and Lemma 4.23

$$\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{N} \left(P^g_{f'L,X} + d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X - j_X \right) \in B^g_{f'L}. \tag{4.14}$$

By Corollary 4.20, $B^g_{f'L}$ is principal. Comparing the irreducible factors of the elements given in (4.13) and (4.14) proves the claim. □

When the rank of f is at most 2, and so f is automatically free, we can compute $\mathrm{V}(B^g_{f'F})$ for any factorization F of f and we can compute $B^g_{f'L}$ for L a factorization into linear terms.

Theorem 4.25. *Suppose that f is a central, not necessarily reduced, hyperplane arrangement of rank at most 2 and let $F = (f_1, \ldots, f_r)$ correspond to any factorization $f = f_1 \cdots f_r$. Let f' divide f and $g = \frac{f}{f'}$. Then*

$$\mathrm{V}(B^g_{f'F}) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P^g_{f'F,X} + j_X \right) \right). \tag{4.15}$$

If L is a factorization of $f = l_1 \cdots l_d$ into irreducibles, then

$$B^g_{f'L} = \prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P^g_{f'L,X} + j_X \right). \tag{4.16}$$

Proof. If f is indecomposable, then by Saito's criterion for freeness, cf. page 270 of [22],

$$\mathrm{mdr}(f) = d_{\mathrm{red}} - 1.$$

So in this case Theorem 4.11 implies

$$B^g_{f'F} \subseteq \sqrt{ \mathbb{C}[S] \cdot \prod_{j_0=0}^{d_{\mathrm{red}} + d - d' - 4} \left(P^g_{f'F,0} + j_0 \right) \prod_{\substack{X \in L(A) \\ r(X)=1}} \prod_{j_X=0}^{d_X - d'_X - 1} \left(P^g_{f'F,X} + j_X \right) }. \tag{4.17}$$

Proposition 4.24 and Proposition 2.26 together imply

$$\sqrt{ \mathbb{C}[S] \cdot \prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{x,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P^g_{f'F,X} + j_X \right) } \subseteq \sqrt{B^g_{f'F}}, \tag{4.18}$$

where we have included radicals because the image of a polynomial modulo S_F may have multiplicands with large multiplicities, cf. Example 2.27. Combining (4.17) and (4.18) and simplifying $d_{x,\mathrm{red}} + d_X - 2r(X) - d'_X$ for rank 2 and rank 1 edges proves (4.15).

Because L is a factorization into irreducibles, even if f is not reduced the polynomial on the right hand side of (4.16) is reduced. Therefore (4.15) and Corollary 4.20 implies (4.16). The case of f decomposable follows by similar reasoning. □

If f is of rank greater than 2, $\mathrm{mdr}(f)$ can be small and so the estimate in Theorem 4.11 will not be precise enough for our purposes. In this case, we impose symmetry on $B^g_{f'F}$ to obtain the following estimates:

Theorem 4.26. *Suppose that $f = f_1 \cdots f_r$ is a central, not necessarily reduced, free hyperplane arrangement, $F = (f_1, \cdots, f_r)$, f' divides f, and $g = \frac{f}{f'}$. Then*

$$\sqrt{\mathbb{C}[S] \cdot \prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X = 0}^{d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P^g_{f'F,X} + j_X \right)} \subseteq \sqrt{B^g_{f'F}}. \tag{4.19}$$

If we assume (f', F) is an unmixed pair up to units, then

$$B^g_{f'F} \subseteq \sqrt{\mathbb{C}[S] \cdot \prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X \in \Xi_X} \left(P^g_{f'F,X} + j_X \right)}, \tag{4.20}$$

where, for each indecomposable edge X, Ξ_X is the, possibly empty, set of nonnegative integers defined by

$$\begin{cases} [0, d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X] & r(X) \leq 2 \\ [0, d_X - d'_X - 1] \cup [d_{X,\mathrm{red}} - 2r(X) + 1, d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X] & r(X) \geq 3. \end{cases}$$

Proof. The inclusion (4.19) is proved in exactly the same way as (4.18), so we need to only prove (4.20). Arguing as in the beginning of Theorem 3.16, we may assume (f', F) is an unmixed pair. Theorem 4.11 implies

$$B^g_{f'F} \subseteq \sqrt{\mathbb{C}[S] \cdot \prod_{\substack{X \in L(A) \\ X \text{ indecomposable} \\ r(X) \geq 3}} \prod_{j_X = 0}^{d_X - d'_X - 1} \left(P^g_{f'F,X} + j_X \right)}. \tag{4.21}$$

The symmetry of $B^g_{f'F,X}$, cf. Theorem 3.16/Corollary 3.18, Lemma 4.23, and (4.21) imply

$$\begin{aligned} B^g_{f'F} &\subseteq \sqrt{\mathbb{C}[S] \cdot \prod_{\substack{X \in L(A) \\ X \text{ indecomposable} \\ r(X) \geq 3}} \prod_{j_X = 0}^{d_X - d'_X - 1} P^g_{f'F,X} + d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X - j_X} \\ &= \sqrt{\mathbb{C}[S] \cdot \prod_{\substack{X \in L(A) \\ X \text{ indecomposable} \\ r(X) \geq 3}} \prod_{j_X = d_{x,\mathrm{red}} - 2r(X) + 1}^{d_{x,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P^g_{f'F,X} + j_X \right)}. \end{aligned} \tag{4.22}$$

At the edges of rank two or one we have an ideal containment similar to (4.17). Combining this, (4.21), and (4.22) and using the fact that $\mathbb{C}[S]$ is a UFD proves (4.20). ☐

If d' is small enough, the previous result does not just estimate–it computes.

Corollary 4.27. (Compare to Theorem 2 of [18]) *Suppose $f = f_1 \cdots f_r$ is a central, not necessarily reduced, free hyperplane arrangement, $F = (f_1, \cdots, f_r)$, f' divides f, and $g = \frac{f}{f'}$. If (f', F) is an unmixed pair up to units and if $d' \leq 4$, then*

$$\mathrm{V}(B^g_{f'F}) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X = 0}^{d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X} \left(P^g_{f'F,X} + j_X \right) \right). \tag{4.23}$$

If L is a factorization of $f = l_1 \cdots l_d$ into irreducibles and $d' \leq 4$, then

$$B^g_{f'L} = \prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{X,\mathrm{red}}+d_X-2r(X)-d'_X} \left(P^g_{f'L,X} + j_X\right). \tag{4.24}$$

If $f' = 1$ and f is reduced, then for any F

$$\mathrm{V}(B_F) = \mathrm{V}\left(\prod_{\substack{X \in L(A) \\ X \text{ indecomposable}}} \prod_{j_X=0}^{d_{X,\mathrm{red}}+d_X-2r(X)} \left(P^g_{F,X} + j_X\right) \right). \tag{4.25}$$

In particular, if f is reduced or is a power of a central, reduced, and free hyperplane arrangement, then the roots of the Bernstein–Sato polynomial of f are given by (4.25).

Proof. Because of Theorem 4.26, proving (4.23) amounts to showing that

$$\Xi_X = [0, d_{X,\mathrm{red}} + d_X - 2r(X) - d'_X]$$

for each X of rank at least 3. This occurs if $d'_X \leq 2(r(X) - 1)$. So (4.23) is true. Since (f', L) is always an unmixed pair up to units, Corollary 4.20 proves (4.24). Equation (4.25) follows from (4.23) and the fact $(1, F)$ is always an unmixed pair up to units when f is reduced, cf. Corollary 3.18. For the final claim, it suffices to note that $(1, F)$ for $F = (f)$ is an unmixed pair up to units provided f is reduced or f is a power of a central, reduced hyperplane arrangement. □

Remark 4.28. (a) Let us outline how to strengthen the final claim of Corollary 4.27 to Bernstein–Sato polynomials for all non-reduced, free f. In the recently announced paper [8], Budur, Veer, Wu, and Zhou consider local, analytic f that satisfy a vanishing Ext criterion. Namely, that $\mathrm{Ext}^k_{\mathscr{D}_{X,\mathfrak{x}}[S]}(\mathscr{D}_{X,\mathfrak{x}}[S]F^S, \mathscr{D}_{X,\mathfrak{x}}[S])$ vanishes for all but one value of k. (We let F corresponds to any factorization of f.) In Proposition 3.4.3 they characterize elements of $\mathrm{V}(B_{F,\mathfrak{x}})$ in terms of the non-vanishing of a certain tensor product. It is easy to show that this is equivalent to the non-surjectivity of the $\mathscr{D}_{X,\mathfrak{x}}$-map ∇_A. This is the map

$$\mathscr{D}_{X,\mathfrak{x}}[S]F^S/(s_1 - a_1, \ldots, s_r - a_r) \cdot \mathscr{D}_{X,\mathfrak{x}}[S]F^S \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S/(s_1 - (a_1 - 1), \ldots, s_r - (a_r - 1)) \cdot \mathscr{D}_{X,\mathfrak{x}}[S]F^S$$

induced by sending each s_k to s_{k+1}. Here A corresponds to $(a_1, \ldots, a_r) \in \mathbb{C}^r$. See Section 3 of [3], Proposition 2 of [7], or Appendix B in this paper for more details on ∇_A. If f corresponds to a free, possibly non-reduced, arrangement, it follows from Theorem 3.9 that the vanishing Ext condition of [8] holds. Moreover, using the commutative diagram in Remark 3.3 of [3], the non-surjectivity of the map ∇_A is equivalent to the non-surjectivity of the classical map ∇_a. (This is the same as ∇_A for $r = 1$.) The non-surjectivity of ∇_a is known to characterize the roots of the Bernstein–Sato polynomial of an arbitrary f. So when L corresponds to a factorization of our possibly non-reduced arrangement f into irreducibles, we can use the above procedure to show that intersecting $\mathrm{V}(B_L)$ with the diagonal gives $\mathrm{V}(B_f)$, again, see Remark 3.3 of [3]. Using the formula for $\mathrm{V}(B_L)$ in (4.24), we then obtain the expected formula (4.25) for $\mathrm{V}(B_f)$ without requiring the reduced hypothesis.

(b) The above strategy for computing $\mathrm{V}(B_f)$ for f a central, reduced, free hyperplane arrangement can also be executed without appeal to [8] thanks to Proposition B.1.

(c) In light of Proposition 3.4.3 of [8], the assumption of "unmixed pair up to units" does not seem to be necessary. Rather, it seems there should be a version of this result for $f'F^S$ so that computing $B^g_{f'L}$ would be sufficient for computing $\mathrm{V}(B^g_{f'F})$.

5. Freeing Hyperplane Arrangements

In this short section we consider the problem of embedding a central hyperplane arrangement g inside a central, free hyperplane arrangement. Equivalently, given such a g we consider central hyperplane arrangements f such that fg is free. (Note that we have somewhat switched notation for reasons that will become clear in Proposition 5.3.)

Definition 5.1. We say the central arrangement f *frees* the central arrangement g if fg is free.

For g an arbitrary divisor, it is unknown if such an f exists. In [20], Mond and Schulze find some general instances of the freeing divisor f; see also [6], [10], [25]. Returning to arrangements g, both Abe and Wakefield identify some situations in [1] and [27] respectively where f is a hyperplane and fg is free. For g a central hyperplane arrangement, Masahiko Yoshinaga [30] has communicated to us an algorithm, depending only on the intersection lattice of g, that always produces such an f. Accordingly, we make the following definition, noting nothing is lost by assuming reducedness.

Definition 5.2. For g a central, reduced hyperplane arrangement, define

$$\mu_g = \min\{\deg(f) \mid f \text{ is a central arrangement that frees } g\}.$$

We will highlight a connection between small roots of the Bernstein–Sato polynomial of a tame g and lower bounds for μ_g. First some notation.

Consider a reduced hyperplane arrangement $l_1 \cdots l_d$ and write it as a product fg. Let $F = (f_1, \ldots, f_r)$ and $G = (g_1, \ldots, g_u)$ correspond to the factorizations $f = f_1 \cdots f_r$ and $g = g_1 \cdots g_u$ into linear terms and let FG correspond to the factorization

$$l_1 \cdots l_d = f_1 \cdots f_r \cdot g_1 \cdots g_u.$$

When considering the $\mathrm{A}_n(\mathbb{C})[S]$-module generated $(FG)^S$, we will re-label so this is an $\mathrm{A}_n(\mathbb{C})[S,T]$-module generated by $f_1^{s_1} \cdot f_r^{s_r} g_1^{t_1} \cdot g_u^{t_u}$. Finally, let $S+1$ denote the $\mathbb{C}[S]$ ideal generated by $s_1+1, \ldots, s_r+1$ and let $\Delta_{S+1} : \mathbb{C}^u \to \mathbb{C}^{r+u} = \mathbb{C}^d$ be the embedding given by

$$(a_1, \ldots, a_u) \mapsto (-1, \ldots, -1, a_1, \ldots, a_u).$$

We need the following result:

Proposition 5.3. *Let f, g, F, G be as in the preceding paragraph. Suppose fg is tame. Then*

$$\Delta_{S+1}(\mathrm{V}(B_G)) \subseteq \mathrm{V}(B_{fFG}^g) \cap \{s_1 = -1, \ldots, s_r = -1\} \subseteq \mathbb{C}^{u+r}.$$

Proof. Define $I = \mathrm{A}_n(\mathbb{C})[S,T] \cdot \mathrm{ann}_{\mathrm{A}_n(\mathbb{C})[T]} G^T + \mathrm{A}_n(\mathbb{C})[S,T] \cdot g + \mathrm{A}_n(\mathbb{C})[S,T] \cdot (S+1)$. If $P \in I \cap \mathbb{C}[S,T]$, then

$$P \text{ modulo } \mathrm{A}_n(\mathbb{C})[S,T] \cdot (S+1) \in \mathbb{C}[T] \cap (\mathrm{A}_n(\mathbb{C})[T] \cdot \mathrm{ann}_{\mathrm{A}_n(\mathbb{C})[T]} G^T + \mathrm{A}_n(\mathbb{C})[T] \cdot g).$$

So

$$I \cap \mathbb{C}[S,T] \subseteq \mathbb{C}[S,T] \cdot B_G + \mathbb{C}[S,T] \cdot (S+1).$$

As the reverse equality is obvious,

$$I \cap \mathbb{C}[S,T] = \mathbb{C}[S,T] \cdot B_G + \mathbb{C}[S,T] \cdot (S+1).$$

For δ a logarithmic derivation of fg,

$$\psi_{fFG}(\delta) = \delta - \sum_k s_k \frac{\delta \bullet f_k}{f_k} - \sum_m t_m \frac{\delta \bullet g_m}{g_m} - \frac{\delta \bullet f}{f}.$$

Under the map $\mathrm{A}_n(\mathbb{C})[S,T] \mapsto \mathrm{A}_n(\mathbb{C})[S,T]/\mathrm{A}_n(\mathbb{C})[S,T]\cdot(S+1)$,

$$\psi_{fFG}(\delta) \mapsto \delta - \sum_m t_m \frac{\delta \bullet g_m}{g_m} = \psi_G(\delta) \in \operatorname{ann}_{\mathrm{A}_n(\mathbb{C})[T]} G^T.$$

Therefore

$$I \supseteq \mathrm{A}_n(\mathbb{C})[S,T]\cdot\theta_{fFG} + \mathrm{A}_n(\mathbb{C})[S,T]\cdot g + \mathrm{A}_n(\mathbb{C})[S,T]\cdot(S+1).$$

Intersecting with $\mathbb{C}[S,T]$ and using Corollary 2.22, we deduce

$$\mathbb{C}[S,T]\cdot B_G + \mathbb{C}[S,T]\cdot(S+1) \supseteq B^g_{f'FG} + \mathbb{C}[S,T]\cdot(S+1).$$

Taking varieties finishes the proof. □

By Theorem 1 of [23], $\mathrm{V}(B_g) \subseteq (\frac{-2d+1}{d}, 0)$, g any central arrangement; by the formula (4.25) for $\mathrm{V}(B_g)$, the presence of roots $\frac{-2d+v}{d}$, $1 < v \leq n-1$ suggests g is not free. While this is not true because $\frac{-2d+v}{d}$ might not be written in lowest terms, the following outlines how such roots can measure the distance g is from being free.

Theorem 5.4. *Suppose that g is a central, reduced, tame hyperplane arrangement of rank n, v an integer such that $1 < v \leq n-1$, and $\deg(g)$ is co-prime to v. If $\frac{-2\deg(g)+v}{\deg(g)}$ is a root of the Bernstein–Sato polynomial of g, then $\mu_g \geq n - v$.*

Proof. Suppose f is a reduced, central hyperplane arrangement such that fg is free. We use the notation of the preceeding proposition and paragraphs. It suffices to prove $\deg(f) \geq n - v$.

By Proposition 2.26 (or Proposition 2.32 of [3]) if $\frac{-2\deg(g)+v}{\deg(g)}$ is a root of the Bernstein–Sato polynomial of g then $(\frac{-2\deg(g)+v}{\deg(g)}, \ldots, \frac{-2\deg(g)+v}{\deg(g)}) \in \mathrm{V}(B_G)$, where G corresponds to the factorization of g into linear terms. By Proposition 5.3,

$$\Delta_{S+1}\left(\frac{-2\deg(g)+v}{\deg(g)}, \ldots, \frac{-2\deg(g)+v}{\deg(g)}\right) \in \mathrm{V}(B^g_{fFG}) \cap \mathrm{V}(\mathbb{C}[S][T]\cdot(S+1)).$$

By Theorem 4.26, there exists an indecomposable edge X associated to the intersection lattice of fg, and an integer j_X satisfying $0 \leq j_X \leq 2\deg(g_X) + 2\deg(f_X) - 2r(X) - \deg(f_X)$ such that $\Delta_{S+1}(\frac{-2\deg(g)+v}{\deg(g)}, \ldots, \frac{-2\deg(g)+v}{\deg(g)})$ lies in the intersection of $\mathrm{V}(\mathbb{C}[S][T]\cdot(S+1))$ and

$$\Big\{ \sum_k \deg(f_{X,k}) s_k + \sum_m \deg(g_{X,m}) t_m + r(X) + \deg(f_X) + j_X = 0 \Big\}.$$

That is,

$$-\deg(f_X) + \deg(g_X)\left(\frac{-2\deg(g)+v}{\deg(g)}\right) + r(X) + \deg(f_X) + j_X = 0. \tag{5.1}$$

Since v is co-prime to $\deg(g)$, $\frac{\deg(g_X)v}{\deg(g)}$ can only be an integer if $\deg(g_X) = \deg(g)$. This implies $X = 0$ and $r(X) = n$. Rearranging (5.1) and using the upper bound on j_X we see

$$\deg(f_X) \geq r(X) - 2\deg(g_X) + \deg(g_X)\frac{2\deg(g) - v}{\deg(g)}. \tag{5.2}$$

Because $\deg(g_X) = \deg(g)$ and $X = 0$, (5.2) simplifies to

$$\deg(f) \geq n - v.$$

□

This method of argument is more versatile than the theorem suggests. In practice, information about the intersection lattice lets us drop the co-prime condition.

Example 5.5. Let $g = xyzw(x+y+z)(y-z+w)$. This example is studied in [11], Example 5.7, and [24], Example 5.8. In the latter, Saito verifies that $\frac{-2*6+2}{6}$ is a root of the Bernstein–Sato polynomial. Since proj dim $\Omega^1(\log g) = 1$ and $n = 4$, g is tame. Suppose f is a central, reduced hyperplane arrangement such that fg is free. Argue as in Theorem 5.4 until arriving at (5.1). If there is an indecomposable edge $X \neq 0$ associated to the intersection lattice of fg such that (5.1) holds, then $\deg(g_X)$ must equal 3 so that $\frac{2\deg(g_X)}{6}$ is an integer. Then g_X corresponds to the intersection of three hyperplanes of g; all such edges have rank 3 (as edges of $\mathrm{V}(g)$). So X has rank at least 3 as an edge of the intersection lattice of fg. Equation (5.2) becomes $\deg(f_X) \geq 3 - 2*3 + 3*\frac{10}{6} = 2$. On the other hand, if (5.1) is satisfied at $X = 0$, then argument of Theorem 5.4 applies and $\deg(f) \geq 2$. Hence $\mu_g \geq 2$.

Appendix A. Trace of Adjoints

Let f be free and a defining equation for a divisor Y at $\mathfrak{x}$ and $f = l_1^{d_1} \cdots l_r^{d_r}$ its unique factorization into irreducibles, up to multiplication by a unit. So any reduced defining equation f_{red} for Y at $\mathfrak{x}$ is, up to multiplication by a unit, $f_{\text{red}} = l_1 \cdots l_d$. In this section we find formulae involving the commutators of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$, which by Remark 2.2, equals $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$. These formulae are crucial to the proof of Proposition 3.6 and the precise description of the dual of $\mathscr{D}_{X,\mathfrak{x}}[S]f'F^S$. Consequently, the formulae are one of the main reasons certain Bernstein–Sato ideals have the symmetry property we used throughout the paper. These results were first proved by Castro–Jiménez and Ucha in Theorem 4.1.4 of [9]; here we include a different proof.

Definition A.1. Let f_{red} be free and $\delta_1, \ldots, \delta_n$ a basis of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$. Define a matrix $\mathrm{Ad}_{\delta i}$ whose (j,k) entry is $c_k^{i,j}$, where $c_k^{i,j} \in \mathscr{O}_{X,\mathfrak{x}}$ are determined by

$$\mathrm{ad}_{\delta_i}(\delta_j) = [\delta_i, \delta_j] = \sum_k c_k^{i,j} \delta_k.$$

Remark A.2. Note Ad_{δ_i} does not determine the map $\mathrm{ad}_{\delta_i} : \mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}}) \to \mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ since said map is not $\mathscr{O}_{X,\mathfrak{x}}$-linear. Moreover, Ad_{δ_i} depends on a choice of basis of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$.

We will eventually find, given a coordinate system, a particular basis $\delta_1, \ldots, \delta_n$ of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ so that $\mathrm{tr}\,\mathrm{Ad}_{\delta_i}$, the trace of Ad_{δ_i}, admits a nice formula. We collect some elementary facts about the interactions between $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ and $\Omega^\bullet(\log f_{\text{red}})$. Recall by Saito, cf. 1.6 of [22], the following: the inner product between $\mathrm{Der}_{X,\mathfrak{x}}(\log f_{\text{red}})$ and $\Omega^1(\log f)$ shows $\Omega^1(\log f_{\text{red}})$ is the $\mathscr{O}_{X,\mathfrak{x}}$-dual of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$; $\Omega^\bullet(\log f_{\text{red}})$ is closed under taking inner products with logarithmic vector fields; $\Omega^\bullet(\log f_{\text{red}})$ is closed under taking Lie derivatives along logarithmic vector fields of f_{red}; if f_{red} is free then $\Omega^k(\log f_{\text{red}}) = \bigwedge^k \Omega^1(\log f_{\text{red}})$.

Definition A.3. For $w \in \Omega^k(\log f_{\text{red}})$ and $\delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ let $\iota_\delta(w) \in \Omega^{k-1}(\log f_{\text{red}})$ denote the *inner product* of w and δ. Since f_{red} is free, the induced map

$$\Omega^1(\log f_{\text{red}}) \times \mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}}) \to \mathscr{O}_{X,\mathfrak{x}}$$

is a perfect pairing. Given a basis $\delta_1, \ldots, \delta_n$ of $\mathrm{Der}_{X,\mathfrak{x}}(\log f_{\text{red}})$ we may select a dual basis $\delta_1^\star, \ldots, \delta_n^\star$ of $\Omega^1(\log f_{\text{red}})$ such that

$$\iota_{\delta_i}(\delta_i^\star) = 1 \text{ and } \iota_{\delta_i}(\delta_j^\star) = 0 \text{ for } i \neq j.$$

Definition A.4. For $w \in \Omega^k(\log f_{\text{red}})$ and $\delta \in \mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ let $L_{\delta_i}(w) \in \Omega^k(\log f_{\text{red}})$ denote the *Lie derivative* of w along δ_i. Let $\delta_1, \ldots, \delta_n$ and $\delta_1^\star, \ldots, \delta_n^\star$ be as in Definition A.3. Then there exists a unique choice of $b_k^{i,j} \in \mathscr{O}_{X,\mathfrak{x}}$ such that

$$L_{\delta_i}(\delta_j^\star) = \sum_k b_k^{i,j} \delta_k^\star.$$

Define the matrix Lie_{δ_i} to have (j,k) entry $b_k^{i,j}$.

Remark A.5. Just like Ad_{δ_i}, the matrix Lie_{δ_i} does not determine the map

$$L_{\delta_i} : \Omega^1(\log f_{\mathrm{red}}) \to \Omega^1(\log f_{\mathrm{red}});$$

moreover, Lie_{δ_i} depends on the choice of basis $\delta_1, \dots, \delta_n$ of $\mathrm{Der}_{X,\mathfrak{x}}(-\log f)$ which in turn determines the basis $\delta_1^\star, \dots, \delta_n^\star$ of $\Omega^1(\log f)$.

We need the following elementary lemma. It is well known for vector fields and differential forms and can easily be shown to hold in the logarithmic case by writing a logarithmic differential form as $\frac{1}{f_{\mathrm{red}}} w$ where w is a differential form.

Lemma A.6. *Let $X, Y \in \mathrm{Der}_{X,\mathfrak{x}}(\log f_{\mathrm{red}})$. Then as maps from $\Omega^k(\log f_{\mathrm{red}}) \to \Omega^{k-1}(\log f_{\mathrm{red}})$, we have*

$$\iota_{[X,Y]} = [L_X, \iota_Y].$$

Proposition A.7. *If f_{red} is free and $\delta_1, \dots, \delta_n$ is a basis for $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$, then*

$$\mathrm{Ad}_{\delta_i} = -\,\mathrm{Lie}_{\delta_i}^T.$$

Proof. On one hand,

$$\iota_{\mathrm{ad}_{\delta_i}(\delta_j)}(\delta_t^\star) = \iota_{\sum_k c_k^{i,j}\delta_k}(\delta_t^\star) = c_t^{i,j}.$$

On the other hand,

$$[L_{\delta_i}, \iota_{\delta_j}](\delta_t^\star) = -\iota_{\delta_j}(L_{\delta_i}(\delta_t^\star)) = -\iota_{\delta_j}(\sum_k b_k^{i,t}\delta_k^\star) = -b_j^{i,t},$$

as the Lie derivative of a vector field on a constant is zero. Now use Lemma A.6. □

Since f_{red} is free, $\Omega^n(\log f_{\mathrm{red}})$ is a free, cyclic $\mathscr{O}_{X,\mathfrak{x}}$-module generated by $\delta_1^\star \wedge \cdots \wedge \delta_n^\star$. Moreover:

Proposition A.8. *Let f_{red} be free and $\delta_1, \dots, \delta_n$ be a basis for $\mathrm{Der}_{X,\mathfrak{x}}(-\log f_{\mathrm{red}})$. Then*

$$L_{\delta_i}(\delta_1^\star \wedge \cdots \wedge \delta_n^\star) = -\,\mathrm{tr}\,\mathrm{Ad}_{\delta_i}(\delta_1^\star \wedge \cdots \wedge \delta_n^\star).$$

Proof. By basic facts of Lie derivatives:

$$\begin{aligned} L_{\delta_i}(\delta_1^\star \wedge \cdots \wedge \delta_n^\star) &= \sum_j \delta_1^\star \wedge \cdots \wedge \delta_{j-1}^\star \wedge L_{\delta_i}(\delta_j^\star) \wedge \delta_{j+1}^\star \wedge \cdots \wedge \delta_n^\star \\ &= \sum_j \delta_1^\star \wedge \cdots \wedge \delta_{j-1}^\star \wedge (\sum_k b_k^{i,j}\delta_k^\star) \wedge \delta_{j+1}^\star \wedge \cdots \wedge \delta_n^\star \\ &= (\sum_k b_k^{i,k})(\delta_1^\star \wedge \cdots \wedge \delta_n^\star). \end{aligned}$$

The result follows by Proposition A.7. □

We will also need the following standard definition and proposition from differential geometry.

Definition A.9. Consider local coordinates $x_1, \dots, x_n$. Let δ be a vector field. Then $\mathrm{div}(\delta)$ is the *divergence* of δ with respect to the n-form $dx_1 \wedge \cdots \wedge dx_n$ and is defined by:

$$L_\delta(dx_1 \wedge \cdots \wedge dx_n) = \mathrm{div}(\delta)(dx_1 \wedge \cdots \wedge dx_n).$$

Proposition A.10. *In local coordinates $x_1, \cdots, x_n$, write the vector field δ as $\delta = \sum_k h_k \frac{\partial}{\partial_{x_k}}$, where $h_k \in \mathscr{O}_{X,\mathfrak{x}}$. Then $\mathrm{div}(\delta)$ with respect to $dx_1 \wedge \cdots \wedge dx_n$ satisfies the formula*

$$\mathrm{div}(\delta) = \sum_k \frac{\partial}{\partial_{x_k}} \bullet h_k.$$

Proof. Write $dx = dx_1 \wedge \cdots \wedge dx_n$. By Cartan's formula, $L_\delta(dx) = d(\iota_\delta(dx))$. Using the skew-symmetric properties of the inner product we deduce:

$$\begin{aligned} d(\iota_\delta(dx)) &= d(\sum_k (-1)^{k-1}(dx_1 \wedge \cdots \wedge \iota_\delta(dx_k) \wedge \cdots \wedge dx_n)) \\ &= d(\sum_k (-1)^{k-1} h_k (dx_1 \wedge \cdots \wedge \widehat{dx_k} \wedge \cdots \wedge dx_n)) \\ &= (\sum_k \frac{\partial}{\partial_{x_k}} \bullet h_k) dx. \end{aligned}$$

□

Consider a basis $\delta_1, \ldots, \delta_n$ of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$. Then for any choice of coordinates $x_1, \ldots, x_n$, there exists a corresponding unit $u \in \mathscr{O}_{X,\mathfrak{x}}$ such that $\delta_1^\star \wedge \cdots \wedge \delta_n^\star = \frac{u}{f_{\text{red}}} dx_1 \wedge \cdots \wedge dx_n$. See the proof of the first theorem on page 270 of [22] for justification. Clearly $u\delta_1, \ldots, \delta_n$ is still a basis of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ and since $\frac{1}{u}\delta_1^\star = (u\delta_1)^\star$, the logarithmic forms $(u\delta_1)^\star, \delta_2^\star, \ldots, \delta_n^\star$ constitute a dual basis of $\Omega^1(\log f_{\text{red}})$ satisfying:

$$(u\delta_1)^\star \wedge \delta_2^\star \wedge \cdots \wedge \delta_n^\star = \frac{1}{f_{\text{red}}} dx_1 \wedge \cdots \wedge dx_n.$$

This shows, as long as $n \geq 2$, that one can always find a basis of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ satisfying the conditions of the following definition:

Definition A.11. Let f_{red} have Euler homogeneity E at $\mathfrak{x}$. Having fixed a coordinate system $x_1, \ldots, x_n$, consider a basis $\delta_1, \ldots, \delta_n$ of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ such that $\delta_n = E$ and $\delta_1, \ldots, \delta_{n-1}$ is a basis of $\operatorname{Der}_{X,\mathfrak{x}}(-\log_0 f_{\text{red}})$. Such a basis is a *preferred basis* of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ if, in addition,

$$\delta_1^\star \wedge \cdots \wedge \delta_n^\star = \frac{1}{f_{\text{red}}} dx_1 \wedge \cdots \wedge dx_n.$$

We are finally ready to state the main formula of this section.

Proposition A.12. *Let f_{red} be free with Euler homogeneity E. Given a coordinate system $x_1, \ldots, x_n$, let $\delta_1, \ldots, \delta_n$ be a preferred basis of $\operatorname{Der}_{X,x}(-\log f_{\text{red}})$. Write $\delta_i = \sum_k h_{k,i} \frac{\partial}{\partial_{x_k}}$. Then*

(i) $\operatorname{tr} \operatorname{Ad}_{\delta_i} = -\sum_k \frac{\partial}{\partial_{z_k}} \bullet h_{k,i}$ *for $i \neq n$;*
(ii) $\operatorname{tr} \operatorname{Ad}_{\delta_n} = -\sum_k \frac{\partial}{\partial_{z_k}} \bullet h_{k,n} + 1$.

Proof. Write $dx = dx_1 \wedge \cdots \wedge dx_n$. Because $\delta_1, \cdots, \delta_n$ is a preferred basis of $\operatorname{Der}_{X,\mathfrak{x}}(-\log f_{\text{red}})$ and by standard properties of the Lie derivative

$$\begin{aligned} L_{\delta_i}(\delta_1^\star \wedge \cdots \wedge \delta_n^\star) &= L_{\delta_i}(\frac{1}{f_{\text{red}}} dx) = L_{\delta_i}(\frac{1}{f_{\text{red}}}) dx + \frac{1}{f_{\text{red}}} L_{\delta_i}(dx) \\ &= L_{\delta_i}(\frac{1}{f_{\text{red}}}) dx + (\frac{1}{f_{\text{red}}} \sum_k \frac{\partial}{\partial_{x_k}} \bullet h_{k,i}) dx. \end{aligned} \tag{A.1}$$

Note that the last equality of (A.1) follows by Proposition A.10. When $i \neq n$, $L_{\delta_i}(\frac{1}{f_{\text{red}}}) = 0$; when $i = n$, $L_{\delta_n}(\frac{1}{f_{\text{red}}}) = -\frac{1}{f_{\text{red}}}$. The result follows by the definition of a preferred basis together with Proposition A.8. □

Appendix B. Budur's Conjecture for Central, Reduced, Free Arrangements

In [7], Budur conjectured that exponentiating $\mathrm{V}(B_{F,\mathfrak{x}})$ (here $F = (f_1, \ldots, f_r)$ is collection of polynomials) gives the support of the Sabbah specialization functor, generalizing the fact that exponentiating the roots of the Bernstein–Sato polynomial gives the support of the nearby cycle functor, cf. Conjecture 2 of loc. cit. In the same paper he reduced this conjecture to proving, in language we will shortly define, that if $A - 1 \in \mathrm{V}(B_{F,\mathfrak{x}})$ then a certain $\mathscr{D}_{X,\mathfrak{x}}$-linear map ∇_A is not surjective, cf. Proposition 2 of loc. cit. For $f = f_1 \cdots f_r$ a central, reduced, and free hyperplane arrangement and $F = (f_1, \ldots, f_r)$ an arbitrary factorization of f we provide a proof here. Theorem 3.5.3 of the recently announced paper [8] gives a general proof of the conjecture by proving the claim about ∇_A for general points in the codimension one components of $\mathrm{V}(B_{F,\mathfrak{x}})$. Our method relies on the computation of $\mathrm{V}(B_{F,0})$ given in Corollary 4.27 and the behavior of ∇_A under duality, cf. Section 4 of [3].

First, let us clarify our terminology. (See also Section 3 of [3] for more details). For

$$a_1, \ldots, a_r \in \mathbb{C},$$

denote by $S - A$ the sequence $s_1 - a_1, \ldots, s_r - a_r$. Similarly, let A and $A - 1$ denote the tuple $a_1, \ldots, a_r$ and $a_1 - 1, \ldots, a_r - 1$ respectively. There is an injective $\mathscr{D}_{X,\mathfrak{x}}$-linear map

$$\nabla : \mathscr{D}_{X,\mathfrak{x}}[S]F^S \to \mathscr{D}_{X,\mathfrak{x}}[S]F^S$$

given by sending every s_k to $s_k + 1$ and identifying F^{S+1} with fF^S. This induces the $\mathscr{D}_{X,\mathfrak{x}}$-linear map

$$\nabla_A : \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S - A)\mathscr{D}_{X,\mathfrak{x}}[S]F^S} \to \frac{\mathscr{D}_{X,\mathfrak{x}}[S]F^S}{(S - (A - 1))\mathscr{D}_{X,\mathfrak{x}}[S]F^S}.$$

By Proposition 2 of [7], to prove Budur's conjecture in our setting, it suffices to prove the following:

Proposition B.1. *Let $f = f_1 \cdots f_r$ be a central, reduced, and free hyperplane arrangement where the f_k are not necessarily linear forms. Let $F = (f_1, \ldots, f_r)$. If $A - 1 \in \mathrm{V}(B_{F,0})$, then*

$$\nabla_A : \frac{\mathscr{D}_{\mathbb{C}^n,0}[S]F^S}{(S - A)\mathscr{D}_{\mathbb{C}^n,0}[S]F^S} \to \frac{\mathscr{D}_{\mathbb{C}^n,0}[S]F^S}{(S - (A - 1))\mathscr{D}_{\mathbb{C}^n,0}[S]F^S}$$

is not surjective.

Proof. Since the f_k are globally defined we may consider the global version of ∇_A. Since f is central, there is a natural $\mathbb{C}^\star$-action on $\mathrm{V}(f)$; moreover, ∇_A is equivariant with respect to this action. Therefore ∇_A is surjective at 0 if and only if it is surjective at all $\mathfrak{x} \in \mathrm{V}(f)$. So it suffices to prove ∇_A is not surjective for

$$A - 1 \in \bigcup_{j=0}^{2d-2n} \{ \left(\sum d_k s_k \right) + n + j = 0 \},$$

when f is indecomposable of rank n and degree d, cf. Corollary 4.27 and Remark 4.9.

Since f is reduced, $\mathrm{V}(B_{F,0})$ is invariant under the map φ on $\mathbb{C}[S]$ induced by $s_k \mapsto -s_k - 2$, cf. Theorem 3.16 or Proposition 8 of [18]. This map sends

$$\left\{ \left(\sum d_k s_k \right) + n + j = 0 \right\} \quad \text{to} \quad \left\{ \left(\sum d_k s_k \right) + n + (2d - 2n - j) = 0 \right\}.$$

Theorem 4.18 and Theorem 4.19 of [3] prove that the invariance of φ forces ∇_A to be surjective if and only if ∇_{-A} is surjective. So if we show ∇_A is not surjective for all

$$A - 1 \in \{ \left(\sum d_k s_k \right) + n + j = 0 \},$$

then we will have also shown ∇_{-A} is not surjective for all $-A-1 \in \{(\sum d_k s_k) + 2d - n - j = 0\}$. Thus it suffices to prove ∇_A is not surjective for

$$A - 1 \in \bigcup_{j=0}^{d-n} \{\left(\sum d_k s_k\right) + n + j = 0\}.$$

Let f' divide f, where the degree d' of f' is less than d. Just as ∇_A is induced by the $\mathscr{D}_{\mathbb{C}^n,0}$-injection $\nabla : \mathscr{D}_{\mathbb{C},0}[S]F^S \to \mathscr{D}_{\mathbb{C}^n,0}F^S$ sending each s_k to $s_k + 1$, there is an induced $\mathscr{D}_{\mathbb{C}^n,0}$-map

$$\nabla_A^{f'} : \frac{\mathscr{D}_{\mathbb{C}^n,0}[S]F^S}{(S-A)\mathscr{D}_{\mathbb{C}^n,0}[S]F^S} \to \frac{\mathscr{D}_{\mathbb{C}^n,0}[S]f'F^S}{(S-(A-1))\mathscr{D}_{\mathbb{C}^n,0}[S]f'F^S}.$$

Moreover, the non-injectivity of $\nabla_A^{f'}$ implies the non-injectivity of ∇_A. Arguing as in Section 3 of [3], we can prove a version of Theorem 3.11 of loc. cit. for $\nabla_A^{f'}$: if $\nabla_A^{f'}$ is injective, then it is surjective. By Theorem 4.19 of loc. cit., it thus suffices to prove $\nabla_A^{f'}$ is not surjective for

$$A - 1 \in \{\left(\sum d_k s_k\right) + n + d' = 0\}.$$

Now we are in the situation of Theorem 4.8, where instead of looking for

$$vB(S) \in \operatorname{ann}_{\mathscr{D}_{\mathbb{C}^n,0}[S]} f'F^S + \mathscr{D}_{\mathbb{C}^n,0}[S] \cdot g,$$

where $g = \frac{f}{f'}$, we are considering the following possibility:

$$1 \in \operatorname{ann}_{\mathscr{D}_{\mathbb{C}^n,0}[S]} f'F^S + \mathscr{D}_{\mathbb{C}^n,0}[S] \cdot g + (S-(A-1))\mathscr{D}_{\mathbb{C}^n,0}[S]. \tag{B.1}$$

Suppose, towards contradiction, (B.1) holds, i.e. $\nabla_A^{f'}$ is surjective. We argue as in Theorem 4.8, except letting $B(S)$ and v be 1, and obtain an equation resembling (4.1) except with additional terms on the right hand side from $(S-(A-1))\mathscr{D}_{\mathbb{C}^n,0}[S]$. Look at the right constant terms of this version of (4.1), evaluate each s_k at $a_k - 1$, and regard every summand as a power series. This gives an equality of elements in $\mathscr{O}_{X,0}$; denote by $\mathfrak{m}_0$ the maximal ideal of $\mathscr{O}_{X,0}$. By the argument of Theorem 4.8, the only piece of the right hand side outside of $\mathfrak{m}_0$ can come from $L_{\mathbf{0}}g$ as the relevant pieces from $P\psi_{f'F,0}(E)$ and the $(S-(A-1)\mathscr{D}_{\mathbb{C}^n,0}[S]$ terms vanished after sending each s_k to $a_k - 1$ and there are no such pieces from the $Q_j\psi_{f'F}(\delta_j)$ terms by Lemma 4.5. Certainly $g \in \mathfrak{m}_0$. Thus the entire right hand side lies in $\mathfrak{m}_0$. Since $1 \notin \mathfrak{m}_0$, our assumption that (B.1) holds is actually impossible, and the claim is proved. □

Remark B.2. (a) One can argue similarly for non-reduced f if we assume F is unmixed up to units and we check Theorem 4.18 and Theorem 4.19 of [3] for F unmixed up to units. In particular, this applies when F is a factorization into linear terms. We leave this to the reader.

(b) In this case, we obtain the expected formula (4.25) for the roots of Bernstein–Sato polynomial of an appropriate f by Remark B.2.(a) and the strategy outlined in Remark 4.28.(a). This approach does not rely on [8].

(c) The primary purpose of Theorem 3.5.3 of [8] is to analyze $\operatorname{Exp}(\operatorname{V}(B_{F,0}))$. When f is simply a central, reduced hyperplane arrangement and L is a factorization of f into linear forms, $\operatorname{Exp}(\operatorname{V}(B_{L,0})$ can be explicitly computed by Theorem 4.18 (or Maisonobe's Proposition 10 of [18]) and Corollary 2 of [7]. In this case, Budur's conjecture holds without appeal to [8]. Similar approaches work for non-reduced f and different factorizations F of f, cf. Corollary 4.19 and also Remark 6.10 of [7].

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