

ON THE MILNOR FIBER BOUNDARY OF A QUASI-ORDINARY SURFACE

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ABSTRACT. We give a recursive formula, expressed in terms of the characteristic tuples, for the Betti numbers of the boundary of the Milnor fiber of an irreducible quasi-ordinary surface S which is reduced in the sense of Lipman. The singular locus of S consists of two components, and for each component we introduce a sequence of increasingly simpler surfaces. Our recursion depends on a detailed comparison of these two sequences. In the penultimate section, we indicate how we expect pieces of these associated surfaces to glue together to reconstruct the Milnor fiber of S and its boundary.

1. INTRODUCTION

It is well-known that a quasi-ordinary hypersurface germ (of any dimension) has many strong similarities to the germ of a classical plane curve (see especially [7], but also [1], [3], [4], [6], [8], [9]). The Milnor fiber of an irreducible curve germ is constructed by plumbing its resolution graph, with the structure of the fiber corresponding to the rupture components. These in turn correspond to the essential terms of the Puiseux expansion of the curve. Something similar happens with the boundary of a quasi-ordinary surface S . A transverse slice by a hyperplane at a generic point of a singular component of S cuts out a curve germ, which though typically not irreducible, has the same Puiseux exponents in each component. The Milnor fiber of the slice can be regarded as the fiber of two fibrations, called *horizontal* and *vertical*. We are concerned with the *vertical fibration*, which is described in Section 2. As there are two components in the singular locus of S , there are two such spaces. Each space can be obtained by a relative version of the plumbing construction for a curve Milnor fiber: it is constructed by gluing standard pieces corresponding to single terms of the Puiseux expansion.

For a general treatment of Milnor fiber boundaries for non-isolated surface singularities, see [11].

Our starting point is a series defining the germ at the origin of an irreducible quasi-ordinary surface S . We assume this series contains only essential characteristic terms, each with coefficient 1:

$$(1) \quad \zeta = \sum_{j=1}^e x_1^{\lambda_{1j}} x_2^{\lambda_{2j}}.$$

(The terminology is explained in Section 2.) Such a series was called a *prototype* in [5]. Working with just this case is not a serious restriction: see our remarks in Section 2. We also assume $\lambda_{11}\lambda_{12} \neq 0$; Lipman ([8]) calls such a surface “reduced”. We note that this condition is preserved under the partial resolution procedure used in our basic construction. In the literature ζ is often

called a *branch*. The conjugates of ζ generate a function

$$(2) \quad f(x_1, x_2, y) = \prod (y - \zeta(x_1, x_2)) \quad (\text{product over all conjugates})$$

whose vanishing locus is S . This quasi-ordinary surface maps via the projection

$$\pi : (x_1, x_2, y) \rightarrow (x_1, x_2)$$

onto \mathbb{C}^2 , and the map is unramified over the complement of the coordinate axes.

In [5] we presented a recursion for calculating some basic information about the quasi-ordinary surface S , using as input the same information for two associated surfaces, which we call its *truncation* and its *derived surface*. In Section 2 we give the basic definitions, then recall our recursive setup and formulas. Our recursion can be carried out in two different ways (depending on which of the two coordinate axes is sliced, as explained in Section 2), and to draw our conclusions we need to make a precise comparison between the resulting sequences of surfaces and the bundles they carry. This elaborate but elementary comparison is carried out in Section 3.

The results of the comparison are applied in Section 4, which contains the main results of the article. Here we show (Theorem 10) that the vertical monodromies induce *diagonalizable* actions on the part of homology corresponding to eigenvalue 1. This fact, combined with the Wang sequence for each vertical fibration and our vertical monodromy formula from [5], allows us to calculate the first Betti number for the total space of each vertical fibration (Theorem 14). The two spaces have the same first Betti number, and indeed the main result of Section 3 implies that the two total spaces of vertical fibrations are “nearly” homeomorphic (i.e., homeomorphic after removing some standard pieces) *as spaces* (not as fibrations). (This last result is not proved here, but will be important in the sequel.) The boundary of the Milnor fiber is the union of these two total spaces, glued along a torus. We thus deduce the first Betti number of this boundary (Theorem 17). In particular, the first Betti number of the boundary is always even.

We make some informal remarks in Section 5 further elucidating the topological nature of the boundary, and outlining results which we intend to present in detail in forthcoming work. In Section 6, we present an example of our calculations.

2. BASIC DEFINITIONS AND THE RECURSIVE SETUP

Let $f(x_1, x_2, y)$ be the defining function of S in \mathbb{C}^3 , as in (2). The *Milnor fiber* M is the intersection of the surface $f(x_1, x_2, y) = \epsilon$ with a small ball centered at the origin, where first the radius is chosen to be sufficiently small and then ϵ is chosen to be sufficiently small. We denote the boundary of M by ∂M ; it is a closed 3-dimensional manifold (see [10], and [2] for the non-isolated case).

In the following discussion we let $i = 1$ or 2 . For convenience we also introduce a notational device: if $i = 1$ then $\bar{i} = 2$, and *vice versa*.

We define the *Milnor fiber of a transverse slice of the x_i -axis* to be the set of points (x_1, x_2, y) obtained by following, in order, three steps:

- (1) require that $\|(x_1, x_2, y)\| \leq \delta$, a sufficiently small (positive real) radius,
- (2) require that x_i be a fixed number sufficiently close to (but different from) zero,
- (3) finally require that $f(x_1, x_2, y) = \epsilon$, a number sufficiently close to (but different from) zero.

Denote this transverse Milnor fiber by F_{x_i} . By keeping x_i fixed but letting ϵ vary over a circle centered at 0, we obtain the *horizontal fibration*. Keeping ϵ fixed but letting x_i vary over a circle centered at 0, we obtain the *vertical fibration*. Thus we have a fibration over a torus. Let

$$(3) \quad h_{q,x_i} : H_q(F_{x_i}) \rightarrow H_q(F_{x_i})$$

and

$$(4) \quad v_{q,x_i} : H_q(F_{x_i}) \rightarrow H_q(F_{x_i})$$

be the respective monodromy operators, taking coefficients over \mathbb{C} ; we call them the *horizontal monodromy* and *vertical monodromy*. Let V_{x_i} be the total space of the vertical fibration; we will call it the *vertical fibration space*. It is useful at times to work with an equivalent *square ball* formulation:

$$(5) \quad V_{x_i} = \{f(x_1, x_2, y) = \epsilon\} \cap \{|x_i| = \delta, |x_j| \leq \delta\}$$

(where $|\epsilon| \ll \delta$), fibering over the circle $|x_i| = \delta$.

The graded characteristic functions

$$\mathbf{H}(i)(t) = \frac{\det(tI - h_{0,x_i})}{\det(tI - h_{1,x_i})} \quad \text{and} \quad \mathbf{V}(i)(t) = \frac{\det(tI - v_{0,x_i})}{\det(tI - v_{1,x_i})}$$

are called the *horizontal* and *vertical monodromy zeta functions*.

As we mentioned in Section 1, we are working only with a *prototype surface* in whose defining series (1) there are only essential terms, each taken with coefficient 1. To justify this restriction, we refer the reader to the discussion at the beginning of Section 2 of [7]. In a similar vein, Theorem 4.1 of our prior paper [5] says that a quasi-ordinary surface and its prototype have the same horizontal and vertical monodromy zeta functions.

In [5] we presented a recursion for calculating some basic information about the quasi-ordinary surface S , using as input the same information for two associated quasi-ordinary surfaces called its *truncation* S_\bullet and *derived surface* $S'(i)$. The recursion leads to two sequences of associated surfaces, as indicated in the following *basic diagram*:

$$(6) \quad \begin{array}{ccccccccccc} S & \longrightarrow & S'(i) & \longrightarrow & S''(i) & \longrightarrow & S^{(3)}(i) & \longrightarrow & \dots & \longrightarrow & S^{(k)}(i) & \longrightarrow & \dots & \longrightarrow & S^{(e-1)}(i) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & & & \downarrow & & & & \parallel \\ S_\bullet & & S'_\bullet(i) & & S''_\bullet(i) & & S_\bullet^{(3)}(i) & & & & S_\bullet^{(k)}(i) & & & & S_\bullet^{(e-1)}(i) \end{array}$$

The surface $S^{(k)}(i)$ is again a prototype, and it has $e - k$ essential terms, while each of the surfaces in the bottom row has a single essential term. The recursion treats the exponents of (1) in an asymmetric way; thus there are two ways to carry out this recursion (depending on whether $i = 1$ or 2), and two possible sequences of associated surfaces.

The *truncation* of S is the surface S_\bullet determined by

$$(7) \quad \zeta_\bullet = x_1^{\lambda_{11}} x_2^{\lambda_{21}},$$

i.e., by the initial term of ζ . Each vertical arrow in the basic diagram (6) represents truncation.

We write the two exponents of ζ_\bullet in lowest terms:

$$(8) \quad \lambda_{i1} = \frac{n_i}{m_i}, \quad \text{where } \gcd(n_i, m_i) = 1.$$

As we showed in [5], the degree of S_\bullet (i.e., the number of sheets for the projection π) is given by

$$d_\bullet = \text{lcm}(m_1, m_2).$$

The quantity

$$(9) \quad b_i = d_\bullet / m_{\bar{i}}$$

occurs repeatedly in the calculations in this paper, and it has a geometric significance: although the surface S_\bullet is irreducible, its transverse slice $x_i = \text{constant}$ consists of b_i irreducible curves, for each of which the number of sheets (under projection to the line $x_i = \text{constant}$) is $m_{\bar{i}}$. Another quantity which appears is

$$(10) \quad c_\bullet = \gcd(n_1, n_2).$$

We would like to know a simple geometric interpretation of this number, which (along with d_\bullet) is one of the two basic ingredients of our formula in Theorem 12.

We define r_i and s_i to be the smallest nonnegative integers so that

$$(11) \quad \begin{bmatrix} m_{\bar{i}} & n_{\bar{i}} \\ r_i & s_i \end{bmatrix}$$

has determinant 1. The *derived surface* $S'(i)$ is the quasi-ordinary surface determined by the series

$$(12) \quad \zeta'(i) = \sum_{j=1}^{e-1} x_1^{\lambda'_{1j}(i)} x_2^{\lambda'_{2j}(i)},$$

where the new exponents are computed by these formulas:

$$(13) \quad \begin{aligned} \lambda'_{\bar{i}j}(i) &= m_{\bar{i}}(\lambda_{\bar{i},j+1} - \lambda_{\bar{i}1} + d_\bullet \lambda_{\bar{i}1}), \\ \lambda'_{ij}(i) &= b_i(\lambda_{i,j+1} - \lambda_{i1} + d_\bullet \lambda_{i1}) + b_i r_i \lambda_{i1} \lambda'_{\bar{i}j}(i). \end{aligned}$$

We introduce the following shorthand: for a finite sequence $\mathbf{s} = \{s_j\}$, let

$$(14) \quad T(\mathbf{s}) = \{s_{j+1} - s_1 + d_\bullet s_1\}.$$

(Note that this reduces the length of the sequence by one.) Then the formulas in (13) say that

$$(15) \quad \begin{aligned} \boldsymbol{\lambda}'_{\bar{i}}(i) &= m_{\bar{i}} T(\boldsymbol{\lambda}_{\bar{i}}), \\ \boldsymbol{\lambda}'_i(i) &= b_i T(\boldsymbol{\lambda}_i) + b_i r_i \lambda_{i1} \boldsymbol{\lambda}'_{\bar{i}}(i). \end{aligned}$$

As indicated in our basic diagram (6), the derivation process may be iterated. The surface $S^{(k)}(i)$ will be called the k^{th} *derived surface*. Working inductively, suppose that $S^{(k)}(i)$ has defining series

$$(16) \quad \zeta^{(k)}(i) = \sum_{j=1}^{e-k} x_1^{\lambda_{1j}^{(k)}(i)} x_2^{\lambda_{2j}^{(k)}(i)}.$$

Write its leading exponents in lowest terms:

$$(17) \quad \lambda_{i1}^{(k)}(i) = \frac{n_i^{(k)}(i)}{m_i^{(k)}(i)}, \quad \lambda_{\bar{i}1}^{(k)}(i) = \frac{n_{\bar{i}}^{(k)}(i)}{m_{\bar{i}}^{(k)}(i)}.$$

Then the degree of its truncation $S_{\bullet}^{(k)}(i)$ is

$$(18) \quad d_{\bullet}^{(k)}(i) = \text{lcm}(m_1^{(k)}(i), m_2^{(k)}(i)).$$

As before we define

$$(19) \quad b_i^{(k)} = d_{\bullet}^{(k)}(i)/m_i^{(k)}(i)$$

and

$$(20) \quad c_{\bullet}^{(k)}(i) = \text{gcd}(n_1^{(k)}(i), n_2^{(k)}(i)),$$

and again we define $r_i^{(k)}$ and $s_i^{(k)}$ to be the smallest nonnegative integers so that

$$(21) \quad \det \begin{bmatrix} m_i^{(k)}(i) & n_i^{(k)}(i) \\ r_i^{(k)} & s_i^{(k)} \end{bmatrix} = 1.$$

Then $S^{(k+1)}(i)$ is the quasi-ordinary surface determined by the series

$$(22) \quad \zeta^{(k+1)}(i) = \sum_{j=1}^{e-k-1} x_1^{\lambda_{1j}^{(k+1)}(i)} x_2^{\lambda_{2j}^{(k+1)}(i)}$$

where the new exponents are computed by these formulas:

$$(23) \quad \begin{aligned} \lambda_{ij}^{(k+1)}(i) &= m_i^{(k)}(i) \left(\lambda_{i,j+1}^{(k)}(i) - \lambda_{i1}^{(k)}(i) + d_{\bullet}^{(k)}(i) \lambda_{i1}^{(k)}(i) \right), \\ \lambda_{ij}^{(k+1)}(i) &= b_i^{(k)} \left(\lambda_{i,j+1}^{(k)}(i) - \lambda_{i1}^{(k)}(i) + d_{\bullet}^{(k)}(i) \lambda_{i1}^{(k)}(i) \right) + b_i^{(k)} r_i^{(k)} \lambda_{i1}^{(k)}(i) \lambda_{ij}^{(k+1)}(i). \end{aligned}$$

Again this is made more palatable by a shorthand: for a finite sequence $\mathbf{s} = \{s_j\}$, let

$$(24) \quad T^{(k)}(\mathbf{s}) = \{s_{j+1} - s_1 + d_{\bullet}^{(k)}(i) s_1\}.$$

Then the formulas in (23) say that

$$(25) \quad \begin{aligned} \lambda_i^{(k+1)}(i) &= m_i^{(k)}(i) T^{(k)}(\lambda_i^{(k)}(i)), \\ \lambda_i^{(k+1)}(i) &= b_i^{(k)} T^{(k)}(\lambda_i^{(k)}(i)) + b_i^{(k)} r_i^{(k)} \lambda_{i1}^{(k)}(i) \lambda_i^{(k+1)}(i). \end{aligned}$$

For the surface S , let d denote its degree. Let $\chi(i)$ denote the Euler characteristic of the transverse Milnor fiber F_{x_i} , and as above let $\mathbf{H}(i)$, and $\mathbf{V}(i)$ denote its horizontal and vertical monodromy zeta functions. For the truncation S_{\bullet} , let d_{\bullet} , $\chi_{\bullet}(i)$, $\mathbf{H}_{\bullet}(i)$, and $\mathbf{V}_{\bullet}(i)$ denote the same quantities. For the derived surface $S'(i)$, denote these quantities by $d'(i)$, $\chi'(i)$, $\mathbf{H}'(i)$, and $\mathbf{V}'(i)$. Theorem 4.3 of [5] gives us the following formulas:

(a)

$$d = d_{\bullet} d'(i)$$

(b)

$$\chi_{\bullet}(i) = m_i b_i + n_i b_i - m_i n_i (b_i)^2 = d_{\bullet} + d_{\bullet} \lambda_{i1} - (d_{\bullet})^2 \lambda_{i1}$$

(c)

$$\chi(i) = d'(\chi_{\bullet}(i) - b_i) + b_i \chi'(i) = d' \chi_{\bullet}(i) + b_i (\chi'(i) - d')$$

(d)

$$\mathbf{H}_{\bullet}(i)(t) = \frac{(t^{d_{\bullet}} - 1)(t^{n_i b_i} - 1)}{(t^{n_i d_{\bullet}} - 1)^{b_i}}$$

(e)

$$\mathbf{H}(i)(t) = \frac{\mathbf{H}_\bullet(t^{d'}) (\mathbf{H}'(t))^{b_i}}{(t^{d'} - 1)^{b_i}}$$

(f)

$$\mathbf{V}_\bullet(i)(t) = \frac{(t-1)^{d_\bullet}}{(t^{n_i b_i / c_\bullet} - 1)^{c_\bullet (d_\bullet - 1)}}$$

(g)

$$\mathbf{V}(i)(t) = \frac{(\mathbf{V}_\bullet(t))^{d'} \mathbf{V}'(t^{b_i})}{(t^{b_i} - 1)^{d'}}$$

Note that (a) implies that $d'(1) = d'(2)$; in the subsequent formulas this quantity is simply denoted by d' .

In a subsequent section, we will apply formulas (f) and (g) to the k^{th} derived surface $S^{(k)}(i)$ rather than to S , and we record the needed versions of those formulas here. Letting $d^{(k)}(i)$ and $\mathbf{V}^{(k)}(i)$ denote the degree and the vertical monodromy zeta function of this surface, and letting $d_\bullet^{(k)}(i)$ and $\mathbf{V}_\bullet^{(k)}(i)$ denote the same quantities for its truncation $S_\bullet^{(k)}(i)$, we have these formulas:

$$(26) \quad \mathbf{V}_\bullet^{(k)}(i)(t) = \frac{(t-1)^{d_\bullet^{(k)}(i)}}{\left(t^{n_i^{(k)}(i) b_i^{(k)} / c_\bullet^{(k)}(i)} - 1\right)^{c_\bullet^{(k)}(i) (d_\bullet^{(k)}(i) - 1)}} \quad (\text{for } k = 0 \text{ to } e-1),$$

$$(27) \quad \mathbf{V}^{(k)}(i)(t) = \frac{\left(\mathbf{V}_\bullet^{(k)}(i)(t)\right)^{d^{(k+1)}(i)} \mathbf{V}^{(k+1)}(i) \left(t^{b_i^{(k)}}\right)}{\left(t^{b_i^{(k)}} - 1\right)^{d^{(k+1)}(i)}} \quad (\text{for } k = 0 \text{ to } e-2).$$

3. COMPARING THE TWO RECURSIONS

In the previous section we presented a derivation process leading to the basic diagram of surfaces in (6). There are two such diagrams, depending on our choice of $i = 1$ or 2 . This section is devoted to a detailed comparison of the surfaces in the two diagrams.

Recall that $d^{(k)}(i)$ and $d_\bullet^{(k)}(i)$ denote, respectively, the degrees of the surfaces $S^{(k)}(i)$ and $S_\bullet^{(k)}(i)$.

Theorem 1. *The degrees of the surfaces in the two basic diagrams are the same:*

$$(1) \quad d^{(k)}(1) = d^{(k)}(2) \text{ for each } k = 1, \dots, e-1.$$

$$(2) \quad d_\bullet^{(k)}(1) = d_\bullet^{(k)}(2) \text{ for each } k = 1, \dots, e-1.$$

Proof. We first observe that the formulas which compute $d_\bullet^{(k)}(i)$ depend only on the first k terms of the series (1). Thus we obtain the same value for $d_\bullet^{(k)}(i)$ if we replace this series by the shortened series

$$(28) \quad \tilde{\zeta} = \sum_{j=1}^k x_1^{\lambda_{1j}} x_2^{\lambda_{2j}}.$$

Letting \tilde{S} denote the surface defined by this shortened series, the three surfaces $S_{\bullet}^{(k)}(i)$, $\tilde{S}_{\bullet}^{(k)}(i)$, and $\hat{S}^{(k)}(i)$ are the same. Thus if we know that $d^{(k)}(1) = d^{(k)}(2)$ for each prototype surface, we infer that $d_{\bullet}^{(k)}(1) = d_{\bullet}^{(k)}(2)$ as well. We now prove the first equality by induction on k . The base case $d'(1) = d'(2)$ has already been observed. The inductive step is accomplished by applying formula (a) of the previous section to the surface $S^{(k)}(i)$; this tells us that

$$d^{(k)}(i) = d_{\bullet}^{(k)}(i) d^{(k+1)}(i).$$

□

In view of the theorem, we can simplify notation by writing $d_{\bullet}^{(k)}$ instead of $d_{\bullet}^{(k)}(1)$ or $d_{\bullet}^{(k)}(2)$. By repeated application of (a), we infer that the degree of our surface S is given by

$$(29) \quad d = d_{\bullet} d'_{\bullet} d_{\bullet}^{(2)} \cdots d_{\bullet}^{(e-1)}.$$

Corollary 2. $d_{\bullet}^{(k)} = \text{lcm}(m_1^{(k)}(1), m_2^{(k)}(1)) = \text{lcm}(m_1^{(k)}(2), m_2^{(k)}(2))$

Proof. Refer to formula (18). □

Now consider the first formula of (13):

$$(30) \quad \lambda'_{i_j}(1) = m_{\bar{i}}(\lambda_{\bar{i}, j+1} - \lambda_{\bar{i}1} + d_{\bullet} \lambda_{\bar{i}1}).$$

One can interpret it as a formula to be applied to the plane curve with Puiseux series

$$(31) \quad \sum_{j=1}^e x_{\bar{i}}^{\lambda_{\bar{i}j}},$$

thus obtaining a derived curve with Puiseux series

$$(32) \quad \sum_{j=1}^{e-1} x_{\bar{i}}^{\lambda'_{i_j}(1)}.$$

This derivation process for a curve was used in [5], and we need to understand some aspects of it here. To begin, observe that the curve defined by (31) is one of the irreducible components of the transverse slice of S by $x_i = 1$; the other components have the same essential exponents but differing coefficients.

Consider the denominator $m_{\bar{i}}^{(k)}(i)$ of $\lambda_{\bar{i}1}^{(k)}(i)$, an exponent appearing in the series (22) for the k^{th} derived surface, equivalently a denominator appearing in the k^{th} derived curve of the transverse slice by $x_i = 1$. It can also be computed from series (1) without going through the derivation process, as the following theorem shows.

Theorem 3. *The exponent $\lambda_{\bar{i}k}$ may be written as a fraction with denominator*

$$(33) \quad m_{\bar{i}} m'_{\bar{i}}(i) m_{\bar{i}}^{(2)}(i) \cdots m_{\bar{i}}^{(k-1)}(i)$$

and with numerator relatively prime to $m_{\bar{i}}^{(k-1)}(i)$.

Proof. Write these exponents as follows:

$$(34) \quad (\lambda_{\bar{i}k}, \lambda_{\bar{i}2}, \lambda_{\bar{i}3}, \dots, \lambda_{\bar{i}e}) = \left(\frac{\alpha_1}{\beta_1}, \frac{\alpha_2}{\beta_1 \beta_2}, \frac{\alpha_3}{\beta_1 \beta_2 \beta_3}, \dots, \frac{\alpha_e}{\beta_1 \beta_2 \cdots \beta_e} \right),$$

where each numerator α_i is relatively prime to β_i . By definition we have $\beta_1 = m_{\bar{i}}$. Applying the top formula of (13) and observing that $m_{\bar{i}}\lambda_{\bar{i}1} = \beta_1\lambda_{\bar{i}1}$ is an integer, we see that

$$(35) \quad (\lambda'_{\bar{i}1}(i), \lambda'_{\bar{i}2}(i), \dots, \lambda'_{\bar{i}, e-1}(i)) = \left(\frac{\alpha'_2}{\beta_2}, \frac{\alpha'_3}{\beta_2\beta_3}, \dots, \frac{\alpha'_e}{\beta_2\beta_3 \dots \beta_e} \right),$$

where again each α'_i is relatively prime to β_i . This shows that $\beta_2 = m'_{\bar{i}}(i)$. Similarly, next apply the top formula of (23) with $k = 1$, observing that $m_{\bar{i}}^{(1)}(i)\lambda'_{\bar{i}1} = \beta_2\lambda'_{\bar{i}1}$ is an integer, to see that

$$(36) \quad (\lambda_{\bar{i}1}^{(2)}(i), \lambda_{\bar{i}2}^{(2)}(i), \dots, \lambda_{\bar{i}, e-2}^{(2)}(i)) = \left(\frac{\alpha_3^{(2)}}{\beta_3}, \frac{\alpha_4^{(2)}}{\beta_3\beta_4}, \dots, \frac{\alpha_e^{(2)}}{\beta_3\beta_4 \dots \beta_e} \right),$$

with each $\alpha_i^{(2)}$ relatively prime to β_i , and deduce that $\beta_3 = m_{\bar{i}}^{(2)}(i)$. Continue in this fashion to conclude that in general $\beta_k = m_{\bar{i}}^{(k-1)}(i)$. \square

The quantity in (33) has a geometric meaning. Consider again the surface \tilde{S} defined by the shortened series (28). A component of its transverse slice has Puiseux series

$$(37) \quad \sum_{j=1}^k x_{\bar{i}}^{\lambda_{\bar{i}j}}$$

and this curve has degree $m_{\bar{i}} m'_{\bar{i}}(i) m_{\bar{i}}^{(2)}(i) \dots m_{\bar{i}}^{(k-1)}(i)$.

Theorem 4. *The degree of \tilde{S} is $d = d_{\bullet} d'_{\bullet} d_{\bullet}^{(2)} \dots d_{\bullet}^{(k-1)}$, and this quantity divides*

$$(38) \quad \left(m_1 m'_1(2) m_1^{(2)}(2) \dots m_1^{(k-1)}(2) \right) \cdot \left(m_2 m'_2(1) m_2^{(2)}(1) \dots m_2^{(k-1)}(1) \right).$$

Proof. The first equation is simply (29) applied to \tilde{S} . The other statement is (5.9.4) of [8]. Here is the essence of Lipman's argument. As explained in our introduction, one obtains the defining function for the quasi-ordinary surface \tilde{S} by taking a product over all conjugates of the branch (28). One can obtain all such conjugates by independently taking all possible conjugates for both transverse slice curves (31) and forming all possible combinations. This will give an equation of the degree given in (38). There is likely to be redundancy, however, since two different choices of conjugates for the curves may lead to the same branch of the surface. Thus one obtains a power of the defining function. \square

To continue our comparison of the two possible diagrams in (6), we make an elementary observation in linear algebra, recording it as Lemma 5. Given a matrix

$$(39) \quad U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \quad \text{with } U_{22} \neq 0,$$

denote its determinant by Δ and let

$$(40) \quad \text{sw}(U) = \begin{bmatrix} \Delta/U_{22} & U_{12}/U_{22} \\ -U_{21}/U_{22} & 1/U_{22} \end{bmatrix},$$

a matrix with determinant U_{11}/U_{22} . (We have chosen notation to suggest the word "swap.") Let $x(1), x(2), y(1), y(2)$ be indeterminates.

Lemma 5. *These two equations are equivalent:*

$$(41) \quad \begin{bmatrix} x(1) \\ y(2) \end{bmatrix} = U \begin{bmatrix} x(2) \\ y(1) \end{bmatrix} \Leftrightarrow \begin{bmatrix} x(1) \\ y(1) \end{bmatrix} = \text{sw}(U) \begin{bmatrix} x(2) \\ y(2) \end{bmatrix}.$$

Furthermore $\det(\text{sw}(U)) = 1$ if and only if $U_{11} = U_{22}$. □

As before, we use shorthand notation for finite sequences, e.g.

$$(42) \quad \begin{aligned} \boldsymbol{\lambda}_{\tilde{i}} &= \{\lambda_{\tilde{i}j}\}_{j=1}^e, \\ \boldsymbol{\lambda}'_{\tilde{i}}(i) &= \{\lambda'_{\tilde{i}j}(i)\}_{j=1}^{e-1}, \\ \boldsymbol{\lambda}_{\tilde{i}}^{(k)}(i) &= \{\lambda_{\tilde{i}j}^{(k)}(i)\}_{j=1}^{e-k}, \\ &\text{etc.} \end{aligned}$$

Theorem 6. *For each k from 1 to $e - 1$, there exists an $\text{SL}(2, \mathbb{Z})$ matrix $M^{(k)}$ such that*

$$\begin{bmatrix} \boldsymbol{\lambda}_1^{(k)}(1) \\ \boldsymbol{\lambda}_2^{(k)}(1) \end{bmatrix} = M^{(k)} \begin{bmatrix} \boldsymbol{\lambda}_1^{(k)}(2) \\ \boldsymbol{\lambda}_2^{(k)}(2) \end{bmatrix}.$$

Proof. Write out explicitly the equations in (15), using first the choice $(i, \tilde{i}) = (1, 2)$ and then the choice $(i, \tilde{i}) = (2, 1)$:

$$(43) \quad \begin{aligned} \boldsymbol{\lambda}'_2(1) &= m_2 T(\boldsymbol{\lambda}_2) \\ \boldsymbol{\lambda}'_1(1) &= b_1 T(\boldsymbol{\lambda}_1) + b_1 r_1 \lambda_{11} \boldsymbol{\lambda}'_2(1) \\ \boldsymbol{\lambda}'_1(2) &= m_1 T(\boldsymbol{\lambda}_1) \\ \boldsymbol{\lambda}'_2(2) &= b_2 T(\boldsymbol{\lambda}_2) + b_2 r_2 \lambda_{21} \boldsymbol{\lambda}'_1(2). \end{aligned}$$

Eliminating $T(\boldsymbol{\lambda}_1)$ and $T(\boldsymbol{\lambda}_2)$, we find this system of two equations:

$$(44) \quad \begin{aligned} \boldsymbol{\lambda}'_1(1) &= \frac{b_1}{m_1} \boldsymbol{\lambda}'_1(2) + b_1 r_1 \lambda_{11} \boldsymbol{\lambda}'_2(1) \\ \boldsymbol{\lambda}'_2(2) &= \frac{b_2}{m_2} \boldsymbol{\lambda}'_2(1) + b_2 r_2 \lambda_{21} \boldsymbol{\lambda}'_1(2) \end{aligned}$$

and write it as a matrix equation

$$(45) \quad \begin{bmatrix} \boldsymbol{\lambda}'_1(1) \\ \boldsymbol{\lambda}'_2(2) \end{bmatrix} = U' \begin{bmatrix} \boldsymbol{\lambda}'_1(2) \\ \boldsymbol{\lambda}'_2(1) \end{bmatrix}$$

in which

$$(46) \quad U' = \begin{bmatrix} b_1/m_1 & b_1 r_1 \lambda_{11} \\ b_2 r_2 \lambda_{21} & b_2/m_2 \end{bmatrix}.$$

Observe that $b_1/m_1 = d_{\bullet}/(m_1 m_2) = b_2/m_2$. Thus by Lemma 5, the matrix $M' = \text{sw}(U')$ has determinant 1.

To show that M' is the required matrix for the statement of the theorem when $k = 1$, we need to establish that its entries are integers. The bottom right entry of M' is the integer $m_2/b_2 = (m_1 m_2)/d_{\bullet}$. By (9), we see that $b_1 m_2 = b_2 m_1$, and together with (8) this implies that

the top right entry is the integer $n_1 r_1$; similarly the bottom left entry is $-n_2 r_2$. Again using (9) and (8), we see that the determinant of U' is

$$(47) \quad \begin{aligned} \Delta &= \frac{b_1}{m_1} \cdot \frac{b_2}{m_2} - b_1 b_2 r_1 r_2 \lambda_{11} \lambda_{21} \\ &= (d_\bullet)^2 (1 - m_1 m_2 r_1 r_2 \lambda_{11} \lambda_{21}) \\ &= (d_\bullet)^2 (1 - n_1 n_2 r_1 r_2). \end{aligned}$$

Since b_2 divides d_\bullet , the top diagonal entry $m_2 \Delta / b_2$ of M' is an integer.

We have thus established the base case in an inductive proof of the theorem. Hypothesize inductively that we have found a matrix

$$(48) \quad U^{(k)} = \begin{bmatrix} U_{11}^{(k)} & U_{12}^{(k)} \\ U_{21}^{(k)} & U_{22}^{(k)} \end{bmatrix}$$

for which

$$(49) \quad \begin{bmatrix} \lambda_1^{(k)}(1) \\ \lambda_2^{(k)}(2) \end{bmatrix} = U^{(k)} \begin{bmatrix} \lambda_1^{(k)}(2) \\ \lambda_2^{(k)}(1) \end{bmatrix},$$

having identical diagonal entries, and such that

$$(50) \quad M^{(k)} = \text{sw}(U^{(k)})$$

has integer entries. To simplify notations further, we suppress all superscripts (k) , and we replace each superscript $(k+1)$ by a plus sign. Thus, for example, we write equation (49) as

$$(51) \quad \begin{bmatrix} \lambda_1(1) \\ \lambda_2(2) \end{bmatrix} = U \begin{bmatrix} \lambda_1(2) \\ \lambda_2(1) \end{bmatrix},$$

and we aim to prove that there is an appropriate matrix U^+ for which

$$(52) \quad \begin{bmatrix} \lambda_1^+(1) \\ \lambda_2^+(2) \end{bmatrix} = U^+ \begin{bmatrix} \lambda_1^+(2) \\ \lambda_2^+(1) \end{bmatrix}.$$

In the second equation of (25), let $(i, \tilde{i}) = (1, 2)$; with our simplified notation, this says that

$$(53) \quad \lambda_1^+(1) = b_1 T(\lambda_1(1)) + b_1 r_1 \lambda_{11}(1) \lambda_2^+(1).$$

(Note that b_1 and r_1 mean $b_1^{(k)}$ and $r_1^{(k)}$, etc.)

Apply the inductive hypothesis, and then use the first equation of (25) twice:

$$(54) \quad \begin{aligned} \lambda_1^+(1) &= b_1 T(U_{11} \lambda_1(2) + U_{12} \lambda_2(1)) + b_1 r_1 \lambda_{11}(1) \lambda_2^+(1) \\ &= b_1 U_{11} T(\lambda_1(2)) + b_1 U_{12} T(\lambda_2(1)) + b_1 r_1 \lambda_{11}(1) \lambda_2^+(1) \\ &= \frac{b_1}{m_1(2)} U_{11} \lambda_1^+(2) + \frac{b_1}{m_2(1)} U_{12} \lambda_2^+(1) + b_1 r_1 \lambda_{11}(1) \lambda_2^+(1) \\ &= \frac{b_1}{m_1(2)} U_{11} \lambda_1^+(2) + \left(\frac{b_1}{m_2(1)} U_{12} + b_1 r_1 \lambda_{11}(1) \right) \lambda_2^+(1). \end{aligned}$$

An identical calculation using the opposite choice $(i, \tilde{i}) = (2, 1)$ establishes that

$$(55) \quad \lambda_2^+(2) = \frac{b_2}{m_2(1)} U_{22} \lambda_2^+(1) + \left(\frac{b_2}{m_1(2)} U_{21} + b_2 r_2 \lambda_{21}(2) \right) \lambda_1^+(2).$$

These calculations show that U^+ exists and give a recursion for its entries:

$$(56) \quad U^+ = \begin{bmatrix} U_{11}^+ & U_{12}^+ \\ U_{21}^+ & U_{22}^+ \end{bmatrix} = \begin{bmatrix} \frac{b_1}{m_1(2)} U_{11} & \frac{b_1}{m_2(1)} U_{12} + b_1 r_1 \lambda_{11}(1) \\ \frac{b_2}{m_1(2)} U_{21} + b_2 r_2 \lambda_{21}(2) & \frac{b_2}{m_2(1)} U_{22} \end{bmatrix}.$$

By the inductive hypothesis we know that $U_{11} = U_{22}$. The second part of Theorem 1 and equation (19) imply that

$$(57) \quad b_1 m_2(1) = b_2 m_1(2);$$

thus the diagonal entries of U^+ are equal. In fact the recursive formula for the bottom right entry, together with (19), give us an explicit formula for these diagonal entries:

$$(58) \quad U_{22}^+ = \frac{b_2}{m_2} \cdot \frac{b'_2}{m'_2(1)} \cdot \frac{b_2^{(2)}}{m_2^{(2)}(1)} \cdots \frac{b_2^{(k)}}{m_2^{(k)}(1)} \\ = \frac{d_{\bullet}^{(0)}(i) \cdot d_{\bullet}^{(1)}(i) \cdot d_{\bullet}^{(2)}(i) \cdots d_{\bullet}^{(k)}(i)}{m_1 \cdot m'_1(1) \cdot m_1^{(2)}(1) \cdots m_1^{(k)}(1) \cdot m_2 \cdot m'_2(1) \cdot m_2^{(2)}(1) \cdots m_2^{(k)}(1)}.$$

(In this display, we have momentarily restored all superscripts, but will continue to suppress them in what follows.) Finally we must show that the entries of

$$(59) \quad M^+ = \begin{bmatrix} M_{11}^+ & M_{12}^+ \\ M_{21}^+ & M_{22}^+ \end{bmatrix} = \text{sw}(U^+)$$

are integers.

By (40), M_{22}^+ is the reciprocal of the quantity in (58), and Theorem 4 says that this is an integer. To show that the other entries of M^+ are integers, we first establish the following recursive formulas for its entries:

$$(60) \quad M_{11}^+ = m_1(2) b_2 s_1 s_2 M_{11} - d_{\bullet} \left(r_1 s_2 \frac{n_2(2)}{m_2(2)} + r_2 s_1 \frac{n_1(1)}{m_1(1)} \right)$$

$$(61) \quad M_{12}^+ = s_1 m_1(2) M_{12} + r_1 n_1(2)$$

$$(62) \quad M_{21}^+ = s_2 m_2(1) M_{21} - r_2 n_2(1)$$

$$(63) \quad M_{22}^+ = \frac{m_2(1)}{b_2} M_{22} = \frac{m_1(2)}{b_1} M_{22}$$

Formula (63) is immediate from the diagonal (and equal) entries of (56).

To derive formula (61), we first combine the leading component of the vector equation (49) with formula (17) to obtain

$$(64) \quad \lambda_{11}(1) = \frac{n_1(1)}{m_1(1)} = U_{11} \frac{n_1(2)}{m_1(2)} + U_{12} \frac{n_2(1)}{m_2(1)}.$$

Here is our derivation of formula (61):

$$\begin{aligned}
M_{12}^+ &= \frac{U_{12}^+}{U_{22}^+} = \frac{\frac{b_1}{m_2(1)} U_{12} + b_1 r_1 \lambda_{11}(1)}{U_{22}^+} && \text{by (56)} \\
&= \frac{\frac{b_1}{m_2(1)} U_{12} + b_1 r_1 \left(U_{11} \frac{n_1(2)}{m_1(2)} + U_{12} \frac{n_2(1)}{m_2(1)} \right)}{U_{22}^+} && \text{by (64)} \\
&= \frac{\frac{b_1}{m_2(1)} U_{12} (1 + r_1 n_2(1)) + b_1 r_1 U_{11} \frac{n_1(2)}{m_1(2)}}{U_{22}^+} \\
&= \frac{b_1 s_1 U_{12}}{U_{22}^+} + \frac{r_1 n_1(2) \frac{b_1 U_{11}}{m_1(2)}}{U_{22}^+} && \text{by (21)} \\
&= \frac{b_1 s_1 \bar{U}_{12}}{\frac{b_1}{m_1(2)} U_{22}} + r_1 n_1(2) && \text{by (57), using that } U_{11}^+ = U_{22}^+ \\
&= s_1 m_1(2) M_{12} + r_1 n_1(2)
\end{aligned}$$

A completely parallel derivation yields formula (62).

The derivation of formula (63) is more elaborate. Formula (64) yields

$$(65) \quad \frac{m_2(1)m_1(2)n_1(1)}{m_1(1)} M_{22} = m_2(1)n_1(2) + M_{12} m_1(2)n_2(1)$$

and a completely parallel process yields its counterpart

$$(66) \quad \frac{m_1(2)m_2(1)n_2(2)}{m_2(2)} M_{22} = -M_{21} m_2(1)n_1(2) + m_1(2)n_2(1).$$

We next observe, using (21) twice, that

$$1 = \det \begin{bmatrix} m_1(2) & n_1(2) \\ r_2 & s_2 \end{bmatrix} \cdot \det \begin{bmatrix} m_2(1) & n_2(1) \\ r_1 & s_1 \end{bmatrix}.$$

Thus, using first formulas (61) and (62), and then formulas (65) and (66),

$$\begin{aligned}
&M_{12}^+ M_{21}^+ + 1 \\
&= (s_1 m_1(2) M_{12} + r_1 n_1(2)) (s_2 m_2(1) M_{21} - r_2 n_2(1)) \\
&\quad + (s_2 m_1(2) - r_2 n_1(2)) (s_1 m_2(1) - r_1 n_2(1)) \\
&= m_1(2) m_2(1) s_1 s_2 (M_{12} M_{21} + 1) \\
&\quad - r_1 s_2 (-n_1(2) m_2(1) M_{21} + m_1(2) n_2(1)) - r_2 s_1 (n_2(1) m_1(2) M_{12} + m_2(1) n_1(2)) \\
&= m_1(2) m_2(1) s_1 s_2 (M_{12} M_{21} + 1) - m_1(2) m_2(1) M_{22} \left(r_1 s_2 \frac{n_2(2)}{m_2(2)} + r_2 s_1 \frac{n_1(1)}{m_1(1)} \right).
\end{aligned}$$

Thus

$$\begin{aligned}
M_{11}^+ &= \frac{M_{12}^+ M_{21}^+ + 1}{M_{22}^+} \\
&= \frac{m_1(2)m_2(1)s_1s_2(M_{12}M_{21} + 1)}{\frac{m_2(1)}{b_2}M_{22}} \\
&\quad - \frac{m_1(2)m_2(1)M_{22}\left(r_1s_2\frac{n_2(2)}{m_2(2)} + r_2s_1\frac{n_1(1)}{m_1(1)}\right)}{\frac{m_1(2)}{b_1}M_{22}} \\
&= m_1(2)b_2s_1s_2M_{11} - m_2(1)b_1\left(r_1s_2\frac{n_2(2)}{m_2(2)} + r_2s_1\frac{n_1(1)}{m_1(1)}\right) \\
&= m_1(2)b_2s_1s_2M_{11} - d_\bullet\left(r_1s_2\frac{n_2(2)}{m_2(2)} + r_2s_1\frac{n_1(1)}{m_1(1)}\right).
\end{aligned}$$

We have already remarked that M_{22}^+ is an integer. Formulas (61) and (62) show that M_{12}^+ and M_{11}^+ are integers, since all the quantities appearing on the right are integers. In view of Corollary 2, formula (60) shows that M_{11}^+ is an integer. \square

In Theorem 1, we have proved that the quantities $d_\bullet^{(k)}(i)$ attached to the surfaces $S_\bullet^{(k)}(i)$ of our basic diagram (6) are the same, no matter whether we let $i = 1$ or 2 . We will conclude this section by proving that the same thing is true for the quantities $c_\bullet^{(k)}(i)$. To do this, we need the following elementary number-theoretic lemma.

Lemma 7. *Suppose the four rational numbers $\frac{n_1(1)}{m_1(1)}$, $\frac{n_2(1)}{m_2(1)}$, $\frac{n_2(2)}{m_2(2)}$, $\frac{n_1(2)}{m_1(2)}$ are related as follows:*

$$\begin{bmatrix} \frac{n_1(1)}{m_1(1)} \\ \frac{n_2(1)}{m_2(1)} \end{bmatrix} = M \begin{bmatrix} \frac{n_2(2)}{m_2(2)} \\ \frac{n_1(2)}{m_1(2)} \end{bmatrix}$$

where $M \in \text{SL}(2, \mathbb{Z})$.

Suppose furthermore that

$$\text{lcm}(m_1(1), m_2(1)) = \text{lcm}(m_2(2), m_1(2)).$$

Then

$$\text{gcd}(n_1(1), n_2(1)) = \text{gcd}(n_2(2), n_1(2)).$$

Proof. We can multiply each column vector by the least common multiple of its denominators, preserving the equation while not changing the greatest common divisor of its numerator; this reduces us to the case where all four numbers are integers. In this case, one argues as follows: since the entries of M are integers, the gcd of the vector on the right divides both entries of the vector on the left, and thus divides its gcd. Since M is in $\text{SL}(2, \mathbb{Z})$, one can reverse the argument, using M^{-1} . \square

Theorem 8. *For each $k = 1, \dots, e-1$, we have $c_\bullet^{(k)}(1) = c_\bullet^{(k)}(2)$.*

Proof. Apply Lemma 7 using the four rational numbers $\frac{n_1^{(k)}(1)}{m_1^{(k)}(1)}$, $\frac{n_2^{(k)}(1)}{m_2^{(k)}(1)}$, $\frac{n_2^{(k)}(2)}{m_2^{(k)}(2)}$, $\frac{n_1^{(k)}(2)}{m_1^{(k)}(2)}$. Equation (18) tells us that the least common multiples of the denominator pairs are $d_\bullet^{(k)}(1)$ and

$d_{\bullet}^{(k)}(2)$, and Theorem 1 asserts that they are equal. Equation (20) tells us that the greatest common divisors of the numerator pairs are $c_{\bullet}^{(k)}(1)$ and $c_{\bullet}^{(k)}(2)$, which are likewise equal. \square

4. TOPOLOGICAL RESULTS

We now embark on a study of the topology of the vertical fibration space V_{x_i} . We find it convenient to use the square ball formulation as in display (5). Recall from (4) that v_{q,x_i} denotes its monodromy in dimension q ; we are going to analyze v_{1,x_i} .

We will write v for the diffeomorphism of F_{x_i} coming from the vertical fibration. It is shown in [5] that F_{x_i} is naturally a union of pieces F_r (for $r = 1, \dots, e$), with each piece arising from the r^{th} stage of the resolution of the transverse slice. (These pieces are usually not connected.) The boundary between F_r and F_{r+1} is a set of cycles denoted by \mathcal{C}_r , and otherwise the pairs of pieces are disjoint. There is also a set of boundary cycles \mathcal{C}_e on the last piece F_e ; note that \mathcal{C}_e is the boundary of F_{x_i} . We show in [5] that v preserves these pieces, permuting the elements of each \mathcal{C}_r , and that the action on each F_r is isotopic to a finite action. At the very end of the proof of Theorem 4.3 (8), we establish the following key fact.

Theorem 9. *The action of v_{1,x_i} on the cycles in each \mathcal{C}_r is transitive.* \square

The next result will provide the basis for computing the Betti numbers of the Milnor fiber boundary from the vertical monodromies.

Theorem 10. *The Jordan blocks of the 1-eigenspace of $v_{1,x_i} : H_1(F_{x_i}) \rightarrow H_1(F_{x_i})$ have size 1.*

Proof. There is a weight filtration on $H_1(F_{x_i}; \mathbb{C})$ defined by

$$\begin{aligned} W_0 &= \text{image of } H_1(\sqcup_{r=1}^e \mathcal{C}_r) \text{ in } H_1(F_{x_i}), \\ W_1 &= \text{image of } H_1(\sqcup_{r=1}^e F_r) \text{ in } H_1(F_{x_i}), \\ W_2 &= H_1(F_{x_i}). \end{aligned}$$

The radical R of the intersection pairing on $H_1(F_{x_i})$ is the image of $H_1(\partial F_{x_i}) = H_1(\mathcal{C}_e)$ under inclusion, and thus is contained in W_0 . The intersection pairing induces a perfect pairing on the quotients of the weight filtration, for which W_0/R is dual to W_2/W_1 and for which W_1/W_0 is self-dual. This pairing is compatible with the action of v_{1,x_i} . Since v acts finitely on each F_r , the restriction of v_{1,x_i} acts finitely on W_1 , hence is diagonalizable. On the other hand, by Theorem 9, v_{1,x_i} acts transitively on the cycles in \mathcal{C}_r , so the generator of the fixed cycles inside the space generated by \mathcal{C}_r is $\sum_s C_{r,s}$ (summing over all the elements of \mathcal{C}_r , with a consistent orientation). This cycle is trivial in homology, since it bounds the union of components F_i , $i \leq r$. Therefore there are no non-trivial fixed cycles under the action of v_{1,x_i} on W_0 . Because v_{1,x_i} is diagonalizable on W_0 , there is no 1-eigenspace on W_0 or on W_0/R . By duality, the 1-eigenspace of $H_1(F_{x_i})$ also does not meet W_2/W_1 . Therefore the 1-eigenspace is contained in W_1 , where v_{1,x_i} is diagonalizable. \square

For more details about surface diffeomorphisms, see [12] p. 269 ff., on which the previous argument is modeled.

Corollary 11. *Let $\xi(i)$ denote the dimension of the 1-eigenspace of v_{1,x_i} . Then the multiplicity of $t - 1$ in the monodromy zeta function $\mathbf{V}(i)$ is $1 - \xi(i)$.*

Proof. This follows from Theorem 10 and the fact that the vertical fibration space is connected. \square

In the next three theorems we will deduce a formula for the first Betti number of the vertical fibration space. Recall from formulas (18) and (20) the quantities $d_{\bullet}^{(k)}(i)$ and $c_{\bullet}^{(k)}(i)$. For the surfaces $S^{(k)}(i)$ and $S_{\bullet}^{(k)}(i)$, we denote by $\xi^{(k)}(i)$ and $\xi_{\bullet}^{(k)}(i)$ the respective dimensions of the 1-eigenspaces of vertical monodromy.

Theorem 12. *For $k = 0$ through $e - 2$, the dimension of the 1-eigenspace of the vertical monodromy of $S^{(k)}(i)$ is given by*

$$\xi^{(k)}(i) = d^{(k+1)}(i)(c_{\bullet}^{(k)}(i) - 1)(d_{\bullet}^{(k)}(i) - 1) + \xi^{(k+1)}(i).$$

In particular (taking $k = 0$)

$$\xi(i) = d'(c_{\bullet} - 1)(d_{\bullet} - 1) + \xi'(i).$$

At the other extreme,

$$\xi^{(e-1)}(i) = (c_{\bullet}^{(e-1)}(i) - 1)(d_{\bullet}^{(e-1)}(i) - 1).$$

Proof. Theorem 9, Theorem 10, and Corollary 11 apply to all the quasi-ordinary surfaces of our basic diagram (6). Thus the multiplicity of $t - 1$ in the vertical monodromy zeta function $\mathbf{V}^{(k)}(i)$ is $1 - \xi^{(k)}(i)$; similarly the multiplicity of $t - 1$ in $\mathbf{V}_{\bullet}^{(k)}(i)$ is $1 - \xi_{\bullet}^{(k)}(i)$. Formula (27) tells us that

$$1 - \xi^{(k)}(i) = d^{(k+1)}(i)(1 - \xi_{\bullet}^{(k)}(i)) + 1 - \xi^{(k+1)}(i) - d^{(k+1)}(i),$$

while formula (26) implies that

$$(67) \quad 1 - \xi_{\bullet}^{(k)}(i) = d_{\bullet}^{(k)}(i) - c_{\bullet}^{(k)}(i)(d_{\bullet}^{(k)}(i) - 1).$$

Together they imply the first statement of the theorem. Formula (67) may also be applied with $k = e - 1$; since $S^{(e-1)}(i)$ and $S_{\bullet}^{(e-1)}(i)$ are the same surface, this gives the last statement in the theorem. \square

We now relate these quantities for the two possible sequences in our basic diagram (6).

Theorem 13. *For each k from 0 to $e - 1$, we have $\xi^{(k)}(1) = \xi^{(k)}(2)$.*

Proof. Theorems 1 and 8 tell us that $d^{(k)}(1) = d^{(k)}(2)$, $d_{\bullet}^{(k)}(1) = d_{\bullet}^{(k)}(2)$, and $c_{\bullet}^{(k)}(1) = c_{\bullet}^{(k)}(2)$. Thus the last statement of Theorem 12 implies that $\xi^{(e-1)}(1) = \xi^{(e-1)}(2)$. The first statement of Theorem 12 then tells us, recursively, that $\xi^{(k)}(1) = \xi^{(k)}(2)$ for $k = e - 2, e - 3, \dots, 0$. \square

Now let $h_1(i) = \dim H_1(V_{x_i})$ be the first Betti number of the vertical fibration space.

Theorem 14. $h_1(i) = \xi(i) + 1$.

Proof. Working with homology groups over \mathbb{C} , we consider a portion of the Wang sequence of the vertical fibration:

$$(68) \quad H_2(V_{x_i}) \rightarrow H_1(F_{x_i}) \xrightarrow{v_{1,x_i}-1} H_1(F_{x_i}) \rightarrow H_1(V_{x_i}) \rightarrow H_0(F_{x_i}) \xrightarrow{v_{0,x_i}-1} H_0(F_{x_i}).$$

(For the Wang sequence, see Lemma 8.4 of [10].) The map $v_{0,x_i} - 1$ is trivial and $H_0(F_x) \cong \mathbb{C}$. Furthermore the kernel of $v_{1,x_i} - 1$ is the 1-eigenspace. Taking an alternating sum of ranks, we find that

$$\xi(i) - \mu + \mu - h_1(i) + 1 = 0.$$

□

Corollary 15. *The first Betti numbers of the vertical fibration spaces are equal:*

$$h_1(1) = h_1(2).$$

□

Henceforth we simply write ξ instead of $\xi(1)$ or $\xi(2)$. Recall that M denotes the Milnor fiber of our quasi-ordinary surface S and that ∂M denotes its boundary.

Theorem 16. *The boundary of the Milnor fiber is the union of the two vertical fibration spaces along a common torus:*

$$(69) \quad \partial M = V_{x_1} \cup_{T^2} V_{x_2}.$$

Proof. Using the square ball formulation of (5), one obtains the boundary of M by letting either $|x_1| = \delta$ or $|x_2| = \delta$. These pieces are the vertical fibration spaces. Their intersection, which lies inside $|x_1| = |x_2| = \delta$, consists of a finite number of tori. Theorem 9, applied with $r = e$, implies that in fact there is just one torus. □

Theorem 17. *The first Betti number of ∂M is given by*

$$h_1(\partial M) = 2\xi.$$

Proof. Consider a portion of the Mayer-Vietoris sequence:

$$H_1(T^2) \xrightarrow{i} H_1(V_{x_1}) \oplus H_1(V_{x_2}) \rightarrow H_1(\partial M) \xrightarrow{\partial} H_0(T^2) \rightarrow H_0(V_{x_1}) \oplus H_0(V_{x_2})$$

The last map is injective, and thus $\partial = 0$. If one composes the map

$$T^2 \rightarrow V_{x_1} \times V_{x_2}$$

with the product of projections onto the bases of these fibrations,

$$V_{x_1} \times V_{x_2} \rightarrow S^1 \times S^1,$$

one obtains a finite covering space map. Thus the induced map on homology

$$H_1(T^2) \rightarrow H_1(S^1) \oplus H_1(S^1)$$

is an isomorphism; hence i is injective. Thus $h_1(\partial M) = h_1(1) + h_1(2) - 2 = 2\xi$ by Theorem 14. □

5. NOTES

Our comments here are somewhat informal; we will present careful proofs elsewhere. The Milnor fiber of an irreducible quasi-ordinary surface germ, or at least a good model of it, is essentially a relative version of a singular plane curve. Both the Milnor fiber and its boundary can be viewed in terms of families, over products of circles, of curve Milnor fibers. Moreover, these families are constructed from parametrized versions of the plumbing construction for irreducible curves, in which pieces of the Milnor fiber are defined iteratively in terms of Puiseux exponents. The corresponding ingredients in our case are the characteristic exponents and the Milnor fibers of single-term truncations of the quasi-ordinary parametrization, obtained after partial resolution. Although this paper deals exclusively with surfaces, we believe it contains all the ingredients needed for analysis of any quasi-ordinary hypersurface.

In this paper, our comparison between the two derived surfaces $S^{(k)}(1)$ and $S^{(k)}(2)$ has been numerical: we have given a detailed description of how certain invariants of the two surfaces are related. In ongoing work, however, we have discovered that the comparison goes deeper. In what follows, we will describe the deeper relationship between the two derived surfaces, and explain the greater importance of the matrix $M^{(k)}$ of display (50).

We have seen that the two vertical fibration spaces have the same Betti numbers. Moreover, it turns out that the vertical fibration spaces are related in a simple topological way. Consider the vertical fibration spaces for the truncation S_{\bullet} . From each bundle we remove d_{\bullet} thickened circles (solid tori). We will show that these punctured vertical fibration spaces are isomorphic (as spaces, not as bundles). Then we will iterate the construction. The punctured vertical fibration spaces of the k^{th} truncations $S_{\bullet}^{(k)}(i)$ are again isomorphic. Copies of these basic pieces are glued together along tori to form the total vertical fibration spaces, analogous to the classical construction of the Milnor fiber of a curve. We need $d_{\bullet}^{(k)}$ copies of the $(k-1)^{\text{st}}$ construction to glue on to the k^{th} truncation. Thus the two vertical fibration spaces are isomorphic, once $d = d_{\bullet} d'_{\bullet} d_{\bullet}^{(2)} \dots d_{\bullet}^{(e-1)}$ disjoint solid tori are removed from each one. This fact, and the next theorem, will not be proved in this article, but play an important role in its sequel. Writing

$$M^{(k)} = \begin{bmatrix} M_{11}^{(k)} & M_{12}^{(k)} \\ M_{21}^{(k)} & M_{22}^{(k)} \end{bmatrix},$$

we claim that the mapping

$$\rho : (x_1, x_2, z) \rightarrow (x_1^{M_{11}^{(k)}} x_2^{M_{21}^{(k)}}, x_1^{M_{12}^{(k)}} x_2^{M_{22}^{(k)}}, z)$$

restricts to a homeomorphism between prototypes of $S^{(k)}(1)$ and $S^{(k)}(2)$, and this map is invertible away from the set $\{x_1 x_2 = 0\}$. Note that the projections of these spaces to \mathbb{C}_{x_1, x_2}^2 are unramified over the axes. Thus the locus of points where ρ is not defined consists of $d^{(k)}$ topological 4-balls. Removing these balls from each space, we thus obtain the following conclusion.

Theorem 18. *The punctured Milnor fibers of the surfaces $S^{(k)}(1)$ and $S^{(k)}(2)$ are homeomorphic.* \square

Looking at the boundaries of these Milnor fibers, we conclude that they are homeomorphic away from $d^{(k)}$ disjoint thickened circles. Furthermore, the analysis that leads to a proof of Theorem

18 shows that, in the decomposition (69) of each of these Milnor fibers, the two vertical fibration spaces V_{x_1} and V_{x_1} are homeomorphic away from these thickened circles.

Proofs of these statements should pave the way for analyzing the Milnor fiber of a quasi-ordinary hypersurface of arbitrary dimension.

6. AN EXAMPLE

Consider the surface with branch

$$\zeta = x_1^{2/7} x_2^{4/5} + x_1^{5/14} x_2 + x_1^2 x_2^{19/10}.$$

Here $e = 3$. Following the layout of our basic diagram (6), we display the branches of the associated surfaces when $i = 1$:

$$\begin{array}{lll} \zeta = x_1^{2/7} x_2^{4/5} + x_1^{5/14} x_2 + x_1^2 x_2^{19/10} & \zeta'(1) = x_1^{705/2} x_2^{141} + x_1^{373} x_2^{291/2} & \zeta''(1) = x_1^{1451} x_2^{573/2} \\ \zeta_{\bullet} = x_1^{2/7} x_2^{4/5} & \zeta'_{\bullet}(1) = x_1^{705/2} x_2^{141} & \zeta''_{\bullet}(1) = x_1^{1451} x_2^{573/2} \end{array}$$

When $i = 2$, the branches of the surfaces are as follows:

$$\begin{array}{lll} \zeta = x_1^{2/7} x_2^{4/5} + x_1^{5/14} x_2 + x_1^2 x_2^{19/10} & \zeta'(2) = x_1^{141/2} x_2^{987} + x_1^{82} x_2^{2259/2} & \zeta''(2) = x_1^{305} x_2^{606303/2} \\ \zeta_{\bullet} = x_1^{2/7} x_2^{4/5} & \zeta'_{\bullet}(2) = x_1^{141/2} x_2^{987} & \zeta''_{\bullet}(2) = x_1^{305} x_2^{606303/2} \end{array}$$

In both cases, the degrees of these surfaces are:

$$\begin{array}{lll} d = 140 & d' = 4 & d'' = 2 \\ d_{\bullet} = 35 & d'_{\bullet} = 2 & d''_{\bullet} = 2 \end{array}$$

We also have:

$$\begin{array}{lll} c_{\bullet} = 2 & c'_{\bullet} = 141 & c''_{\bullet} = 1 \end{array}$$

Observe that

$$M^{(1)} \begin{bmatrix} 141/2 \\ 987 \end{bmatrix} = \begin{bmatrix} -23 & 2 \\ -12 & 1 \end{bmatrix} \begin{bmatrix} 141/2 \\ 987 \end{bmatrix} = \begin{bmatrix} 705/2 \\ 141 \end{bmatrix} \quad \text{and} \quad M^{(1)} \begin{bmatrix} 82 \\ 2259/2 \end{bmatrix} = \begin{bmatrix} 373 \\ 291/2 \end{bmatrix}$$

in accordance with Theorem 6. Similarly, with the auxiliary matrix $U^{(1)}$ we have

$$U^{(1)} \begin{bmatrix} 141/2 \\ 141 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 141/2 \\ 141 \end{bmatrix} = \begin{bmatrix} 705/2 \\ 987 \end{bmatrix} \quad \text{and} \quad U^{(1)} \begin{bmatrix} 82 \\ 291/2 \end{bmatrix} = \begin{bmatrix} 373 \\ 2259/2 \end{bmatrix}.$$

The second auxiliary matrix $U^{(2)}$, calculated from $U^{(1)}$ by the recursion of (56), is

$$U^{(2)} = \begin{bmatrix} \frac{2}{1} \cdot 1 & \frac{2}{1} \cdot 2 + 2 \cdot 0 \cdot \frac{705}{2} \\ \frac{1}{2} \cdot 12 + 1 \cdot 1 \cdot 987 & \frac{1}{1} \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 993 & 1 \end{bmatrix}$$

and thus

$$M^{(2)} = \text{sw}(U^{(2)}) = \begin{bmatrix} -3971 & 4 \\ -993 & 1 \end{bmatrix}.$$

Observe that

$$M^{(2)} \begin{bmatrix} 305 \\ 606303/2 \end{bmatrix} = \begin{bmatrix} 1451 \\ 573/2 \end{bmatrix}.$$

Using the formulas of Theorem 12, we find these dimensions of the 1-eigenspaces of the vertical monodromy:

$$\begin{aligned}\xi'' &= (c''_{\bullet} - 1)(d''_{\bullet} - 1) = (1 - 1)(2 - 1) = 0 \\ \xi' &= d''(c'_{\bullet} - 1)(d'_{\bullet} - 1) + \xi'' = 2(141 - 1)(2 - 1) + 0 = 280 \\ \xi &= d'(c_{\bullet} - 1)(d_{\bullet} - 1) + \xi' = 4(2 - 1)(35 - 1) + 280 = 416\end{aligned}$$

Thus each of the two vertical fibration spaces has first Betti number 417, and the first Betti number of the boundary of the Milnor fiber is 832.

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