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ON REALIZATIONS OF SOME PLANE ALGEBROID CURVES

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ABSTRACT. In this work we study plane algebroid curves whose Milnor and Tjurina numbers differ by two. Since this difference is characterized by exact Kähler differentials, and this case is already totally classified, one may ask about a realization for such a curve. Tensoring the local ring by a suitable algebra, we describe its universal central extension, and characterize a type of representation for it.

INTRODUCTION

Let C be an algebroid plane curve defined by an irreducible power series f and let $r := \mu - \tau$ be the difference between the Milnor and Tjurina numbers of C. Zariski proved that $r = 0$ if and only if C is monomial, i.e., it is of the form $Y^n = X^m$, or equivalently, $f(X,Y) = X^m - Y^n$ where $\gcd(m,n) = 1$. Later on, A. Hefez and V. Bayer in [2] totally classified plane algebroid curves with $r = 1$ and thouroughly characterized the ones with $r = 2$.

Algebroid plane curves with $r = 2$ were studied in many different ways. For instance, in [28] Zariski described their moduli space of equisingularity classes along with the study of its quasi-compactness. The semigroup of some of such singularities also appeared in the literature within a different context. They were related to smooth non-hyperelliptic projective curves having Weierstrass points of maximal weight. Such curves were characterized by T. Kato in [18] as those admitting a double cover of an elliptic curve, that's why they – and actually the semigroups as well – are nowadays referred as *bielliptic*. F. Torres generalized Kato's result in [24] introducing the language of both γ-hyperelliptic curves and γ-hyperelliptic semigroups.

A key point in the study of algebroid plane curves in terms of r is the well known result proved by Zariski in [27]:

$$r = \dim_{\mathbb{C}}(\Omega_{R/\mathbb{C}}/dR) \tag{1}$$

where R is the local ring of C. This connection with exact Kähler differentials, enables us to give another strand to the study of plane algebroid curves with $r = 2$, that is, one may ask about a realization for such a curve, tensoring its local ring by a suitable algebra.

More precisely, let $\mathfrak{g}$ be a simple complex finite dimensional Lie algebra, and $\widehat{\mathfrak{L}}$ be the universal central extension of the algebra

$$\mathfrak{L} := \mathfrak{g} \otimes_{\mathbb{C}} R \tag{2}$$

Consider the natural exact sequence

$$0 \longrightarrow \mathfrak{a} \longrightarrow \widehat{\mathfrak{L}} \longrightarrow \mathfrak{L} \longrightarrow 0,$$

where $\mathfrak{a}$ is central in $\widehat{\mathfrak{L}}$. By Kassel's Theorem [15], we have that

$$\mathfrak{a} = \Omega_{R/\mathbb{C}}/dR \tag{3}$$

So, combining (1) and (3), we get the main subject of this work: we study the universal central extension of (the local ring of) an algebroid plane curve with $r = 2$, or equivalently, for which

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the vector space of cocycles is 2-dimensional. For simplicity, we choose $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C})$ in (2), and we also analyze the construction of representations for $\widehat{\mathfrak{L}}$ in the same way as Wakimoto in [25, 26].

It is well-known that Lie algebras, especially finite-dimensional ones, turn up naturally in a wide range of fields in Mathematics and Physics. As far as finite dimensional semisimple algebras over an algebraically closed field go, their structure and representation theory is completely understood (see for instance [11]). This is summed up in one of its highest forms by the Borel-Weil theorem [11, p.392-393], although the representation theory had already been completed in the 1930s by H. Weyl.

The study of infinite dimensional Lie algebras is much newer, relatively speaking. It has started with the work of V. Kac and R. Moody in the late 1960s on the mathematical front, and by various physicists in the context of current algebras. Even though it is not nearly as complete a theory as the finite dimensional case, it has found many stunning applications in Physics and Mathematics over the last 50 years and remains a source of much interplay between them. For more on those connections, see, for instance, [16] and [13]

Extensions like $\widehat{\mathfrak{L}}$, by their turn, have also been studied in several different manners. One of which is when R is the ring of regular functions on a smooth and projective complex curve with a finite number of points removed. Such an extension generalizes a Kac-Moody algebra since for $\mathbb{P}^1$ with two punctures, $R \simeq \mathbb{C}[t, t^{-1}]$.

So, from a geometric perspective, one can say that the underlying curve in V. Kac and R. V. Moody's [16, 17, 19] is $\mathbb{P}^1 \setminus \{0, \infty\}$. From this point of view, two generalizations are quite natural, and, in fact, were actually made. One may remove more than two points from the same curve, i.e., $\mathbb{P}^1$, or one may change the (smooth) curve to one of higher genus, keeping two points left aside; or even combine both approaches. For instance, this was done by I. M. Krichever and S. P. Novikov in [14], M. Bremner in [3], and by B. Cox, F. Futorny along with the first named author in [4]. The reason of considering punctured curves, rather than the original ones, is because, if so, the loop algebra obtained has nontrivial universal central extension.

In a similar vein, it is natural to extend this study to local rings of singularities of curves. Actually, as will be clear throughout these lines, it is more convenient to deal with its completion instead. So, in this sense, the subject of this article stated above stands for another possible variation: we deal with singular curves whose normal models are of genus zero, if considered as one-punctured curves. Differently from the smooth case, for singular curves, the removal of one point is enough to get nontrivial cocycles.

Still in this line of arguments, when R is the local ring of an algebroid curve, then it would be interesting to analyze when $\mathfrak{a}$ is nontrivial, so that one could ask about its universal extension. From the perspective of getting a free field realization of such an R, the case $r = 1$ produces a one-dimensional central extension, which would be somewhat similar to the Heisenberg Lie algebra, since the cocycles are numbers. So our choice of $r = 2$ also corresponds to the first case for which it is worth carrying out such a discussion.

Now we summarize the content of this work. In order to do so, let R be the local ring of an algebroid plane curve with $r = 2$, and $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ be the special linear Lie algebra of order 2. Let also $\mathfrak{L} := \mathfrak{g} \otimes_{\mathbb{C}} R$ be the loop algebra and $\widehat{\mathfrak{L}}$ its universal central extension. In Section 1 we give a description of $\widehat{\mathfrak{L}}$ in terms of its operations and cocycles in Theorem 1.3. It is possible to get somewhat simple expressions for the cocycles and also give a proof without expressing variables in terms of the parameter of the curve or using its equation either (the reader should note how hard would be the work if so). This is due to formula (8), a helpful relation for differentials on curves, which essentially owes to the proof of [2, Prp. 2]. Differently from the articles mentioned above, although starting from an infinite dimensional Lie algebra, here we get a finite dimensional space of elements which produce non-trivial cocycles. Once the central extension is described, a natural question is whether its structure can be stated in terms of superalgebras as was done, for instance, in [3, Thm 4.6] for the elliptic case. So in Theorem 1.4 we show that $\widehat{\mathfrak{L}}$ admits a natural projection onto a non-trivial superalgebra.

In Section 2, we study the Heisenberg Lie algebra associated to R. The aim is to give it a Fock like representation, which we do within the simplest possible way, i.e., killing one of the (two) generators of the space of cocycles and making the other play the role of the unit of the universal enveloping algebra to get the Weyl type algebra. Such a choice determines the realization space which happens to be the ring of polynomials in two variables. We are also able to express the actions of the generators of the Heisenberg algebra either by multiplication or partial derivatives. This is the content of Theorem 2.1.

In Section 3, we study a realization of the loop algebra $\mathfrak{L}$. The strategy is to induce the one from the case where $R = \mathbb{C}[[t]]$ and make double indices play the role of sum. This is the content of Theorem 3.1. At the end of the section we introduce some notation, which may appear as too formal, just for the sake of presenting the results in a similar manner as the most known case, i.e., the double punctured projective line.

In Section 4 we characterize Wakimoto type free field realizations of $\widehat{\mathfrak{L}}$ in terms of the ones of $\mathfrak{L}$ in Theorem 4.1. The result should be compared against [10, Thm 13.1.1]. Similarly, the representation space is the tensor product of the ones of the Heisenberg and Loop algebras. On the other hand, we did not obtain a realization determined (via partial derivative by these actions (a and b) via partial derivatives. Rather, we make use of third one (named c) defined recursivily. The problem of establishing relations between these actions seems to be a good question to pursue, though not really easy, in our opinion.

Finally, we just make few remarks on the whole topic of the present work. As said above, the study of universal central extensions, Heisenberg and loop algebras, free field realizations has long played a core role in the general theme of Lie algebras and their natural relation to describe certain phenomena in Physics and the connection with geometry is recent. The reference which guided us in this work, i.e., B. Cox, F. Futorny and the first named author's [4], was mentioned by recent articles in different ways: [6, 7, 8, 9, 12, 20, 21, 22, 23]. For instance, the articles [6] and [7] go in similar directions as ours, in the sense that both address in more detail the algebra structure of the Lie algebras rather than their representations. On the other hand, [9] and [8] discuss cornerstones of the representation theory of infinite dimensional Lie algebras, namely Fock spaces and free field realizations, like [4] did, for Lie algebras built out of algebraic curves with a finite number of deleted points. The main difference between the present article and the aforementioned ones is that in our case the algebraic curve we start from is singular. We think that any insight about the topic is worthwhile and, to the best of our knowledge, this is the first attempt to deal with the subject for singularities. The study of punctured projective singular curves is a natural subject to be considered in the near future.

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1. The Universal Central Extension

Let C be an algebroid plane curve defined by an irreducible power series f, and let R be its local ring, that is,

$$R = \frac{\mathbb{C}[[X, Y]]}{\langle F \rangle}$$

One defines its Milnor and Tjurina numbers as

$$\mu_C := \dim_{\mathbb{C}} \frac{\mathbb{C}[[X,Y]]}{\langle F_X, F_Y\rangle} \qquad\qquad \tau_C := \dim_{\mathbb{C}} \frac{\mathbb{C}[[X,Y]]}{\langle F, F_X, F_Y\rangle}$$

and sets the following invariant of the curve

$$r = r_C := \mu_C - \tau_C$$

In this section we describe the universal central extension (of the local ring) of an algebroid plane curve for which $r = 2$. To begin with, we just recall the following classification result.

Theorem 1.1 (V. Bayer, A. Hefez)**.** *Let C be an irreducible algebroid plane curve, and let R be its local ring. Then $r = 2$ if and only if C is analytically equivalent to a curve whose local ring*

$$R = \mathbb{C}[[x,y]]$$

has one of the following forms:

(I) *the parametrization is given by*

$$x = t^n \qquad\qquad y = t^m + t^\lambda$$

with $n < m < \lambda$ and the following two possibilities

(a) *$n = 4$, $m = 6$ and λ is odd; or*

(b) *$\gcd(n,m) = 1$ and $\lambda = (n-1)m - 3n$.*

(II) *the parametrization is given by*

$$x = t^n \qquad\qquad y = t^m + t^\lambda + f(t)$$

with $n < m < \lambda$, $\gcd(n,m) = 1$, and the following two possibilities

(a) *$n = 4$, $m > 8$, $\lambda = 2m - 8$ and*

$$f(t) = \frac{3m-8}{2m} t^{3m-16} + ct^{3m-12}$$

(b) *$n \geq 5$, $m > 2n/(n-3)$, $\lambda = (n-2)m - 2n$ and*

$$f(t) = ct^{(n-1)m-3n}$$

where $c \in \mathbb{C}$ is a constant.

Proof. See [2, Thms. 12 and 17]. □

The parameter λ which appears above plays a central role in the theory of Kähler differentials of plane algebroid curves. It is called the *Zariski invariant* of C and can be described, and actually computed, in the following way. Given any non-monomial curve C with local ring $R = \mathbb{C}[[x,y]]$ and parametrization $x = t^n$, $y = t^m + t^\lambda + f(t)$, with $n < m < \lambda < v(f(t))$, then

$$\lambda := \min(v(\Omega_{R/\mathbb{C}}) \setminus v(dR)) - n + 1 \tag{4}$$

where v is induced by the discrete valuation map of the integral closure $\overline{R}$.

For simplicity, we will refer to the curves of Theorem 1.1 above as being of *kind* (I) or (II), and name the subcases as (I.a), (I.b), (II.a) and (II.b). With this in mind, we have the following.

Remark 1.2. The semigroup of values of curves of kind (I.a) have a special importance in the theory of Weierstrass points of projective curves. In fact, the symmetric ones correspond to semigroups of maximal weight. T. Kato proved in [18] that a non-hyperelliptic curve having a Weierstrass point (with semigroup) of maximal weight admits a double cover of an elliptic curve, i.e., it is bielliptic. F. Torres generalized Kato's result in [24] introducing the notion of

γ-hyperelliptic curves as those admitting a double cover of a curve of genus γ and also defining γ-hyperelliptic semigroups. Within this approach, one can find in [5, Thm. 1.4] a purely numerical converse for Kato's result.

Theorem 1.3. *Let C be an irreducible plane algebroid curve with $r = 2$ with local ring $R = \mathbb{C}[[x, y]]$. Let $\mathfrak{g} := \mathfrak{sl}_2(\mathbb{C})$ be the special linear Lie algebra of order 2. Set*

$$\mathfrak{L} := \mathfrak{g} \otimes_{\mathbb{C}} R.$$

Then there are $\xi, \eta \in \Omega_{R/\mathbb{C}}$ such that the universal central extension of $\mathfrak{L}$ is

$$\widehat{\mathfrak{L}} = \mathfrak{L} \oplus_{\mathbb{C}} (\mathbb{C}\,\overline{\xi} \oplus \mathbb{C}\,\overline{\eta})$$

and the bracket operation on $\widehat{\mathfrak{L}}$, computed on generators, is of the form

$$[A \otimes f + \overline{\alpha}\,,\, B \otimes g + \overline{\beta}] = [A, B] \otimes fg + (A, B)c(f, g),$$

where $A, B \in \mathfrak{g}$, $f, g \in R$ are monomials, $\alpha, \beta \in \Omega_{R/\mathbb{C}}$, (A, B) the Killing form and

$$c(f,g) = \begin{cases} -\overline{\xi} & \text{if } f = x \text{ and } g = y \\ \overline{\xi} & \text{if } f = y \text{ and } g = x \\ -i\,\overline{\eta} & \text{if } f = x^i \text{ and } g = x^k y \text{ with } i + k = 2 \text{ and } i > 0 \\ k\,\overline{\eta} & \text{if } f = x^i y \text{ and } g = x^k \text{ with } i + k = 2 \text{ and } k > 0 \\ 0 & \text{otherwise,} \end{cases}$$

if C is of kind (I), or

$$c(f,g) = \begin{cases} -\overline{\xi} & \text{if } f = x \text{ and } g = y \\ \overline{\xi} & \text{if } f = y \text{ and } g = x \\ -l\,\overline{\eta} & \text{if } f = xy^j \text{ and } g = y^l \text{ with } j + l = 2 \text{ and } l > 0 \\ j\,\overline{\eta} & \text{if } f = y^j \text{ and } g = xy^l \text{ with } j + l = 2 \text{ and } j > 0 \\ 0 & \text{otherwise.} \end{cases}$$

if C is of kind (II).

Proof. Whatever is the finite dimensional Lie algebra $\mathfrak{g}$ or the commutative and associative $\mathbb{C}$-algebra R, we have that the universal central extension of $\mathfrak{L} = \mathfrak{g} \otimes R$ can be given by $\widehat{\mathfrak{L}} = \mathfrak{L} \oplus_{\mathbb{C}} \mathfrak{a}$ and Kassel's Theorem states that

$$\mathfrak{a} = \Omega_{R/\mathbb{C}}/dR.$$

In order to define the bracket in $\widehat{\mathfrak{L}}$, we first recall that it is central. So, computing on generators, we have

$$[A \otimes f + \overline{\alpha}, B \otimes g + \overline{\beta}] = [A \otimes f, B \otimes g], \tag{5}$$

where $A, B \in \mathfrak{g}, f, g \in R$ and $\alpha, \beta \in \Omega_{R/\mathbb{C}}$. Now Kassel's Theorem says as well that

$$[A \otimes f, B \otimes g] = [A, B] \otimes fg + (A, B)\overline{fdg}, \tag{6}$$

where (A, B) is the Killing form on $\mathfrak{g}$. So, to get the statement of the theorem, it is just a matter of describing $\mathfrak{a}$, and computing the classes $\overline{fdg}$ in each case.

Since $r = 2$, by (1) it suffices to expressing these classes in terms of two suitable differentials. Set

$$\xi := mydx - nxdy \tag{7}$$

and recall the following Zariski's statements in [28, Sec III.3]: (i) ξ is a nonexact differential of minimal value, so by (4), we have that $v(\xi) = \lambda + n - 1$; and (ii) if $\omega \in \Omega_{R/\mathbb{C}}$ is such that

$v(\omega)+1 \notin \mathrm{S} := v(R)$, then ω is not exact, so, in particular ω is nonexact if $v(\omega)=c-2$ where c is the conductor of the numerical semigroup S, i.e, the smallest element of S for which all larger naturals are in S as well. We now analyze each case.

So, if C is as in (I.a), by (i), $\xi = 6ydx - 4xdy$ is a nonexact differential with $v(\xi)=\lambda+3$ and it is easily seen that the conductor is $c=\lambda+9$. So, it suffices to find $\eta \in \Omega_{R/\mathbb{C}}$ such that $v(\eta)=c-2=\lambda+7$, which will be linearly independent with ξ because it has different value, and it will be nonexact owing to (ii). Thus, just take $\eta = x\xi$ and note that

$$v(\eta)=v(x)+v(\xi)=4+(3+\lambda)=\lambda+7$$

as desired. Now, it can be easily read off from [2] that the remaining cases all correspond to curves for which the conductor is given by the formula $c=(n-1)m-n+1$. Hence if C is like (I.b), by (i), $v(\xi)=(n-1)m-2n-1$; thus, to obtain $v(\eta)=c-2$ just take $\eta=x\xi$. If C is as in (II.a) take $\eta=y\xi$ since $v(\eta)=3m-5=c-2$, and if C is like (II.b) take, again, $\eta=y\xi$ since $v(\eta)=(n-1)m-n-1=c-2$.

In order to compute the cocycles $\overline{fdg}$ we may take f and g as being monomials. So write

$$f=x^iy^j, \qquad g=x^ky^l$$

and set $p:=i+k$ and $q:=j+l$. We claim that if ξ is as in (7) and $p,q\geq 1$ then

$$fdg=\frac{kq-lp}{mq+np}\,x^{p-1}y^{q-1}\xi \quad \mathrm{mod}\, dR. \tag{8}$$

In fact, call a the constant which appears in (8) above. Then

$$\begin{aligned} d\left(\frac{k-ma}{p}x^py^q\right) &= \frac{k-ma}{p}\left(px^{p-1}y^qdx+qx^py^{q-1}dy\right)\\ &=(k-ma)x^{p-1}y^qdx+\frac{kq-mqa}{p}x^py^{q-1}dy\\ &=(k-ma)x^{p-1}y^qdx+\frac{lp+npa}{p}x^py^{q-1}dy\\ &=(kx^{p-1}y^qdx+lx^py^{q-1}dy)-(max^{p-1}y^qdx-nax^py^{q-1}dy)\\ &=fdg-ax^{p-1}y^{q-1}\xi \end{aligned}$$

so the claim follows.

Therefore it suffices to deal with fdg as in (8) if $p,q\geq 1$. But if, for instance, $q=0$ then $j=l=0$, which yields

$$fdg=x^id(x^k)=kx^{i+k-1}dx=d(kx^{i+k}/(i+k))$$

so fdg is exact. By a similar argument, it is also exact if $p=0$.

For the remaining possibilities, call, for short, $\omega := fdg$ and write

$$\omega=\frac{kq-lp}{mq+np}\,x^{p-1}y^{q-1}\xi. \tag{9}$$

Recall from [28] that any differential with order at least $c-1$ is exact. Hence, for ω to be nonexact we must have $v(\omega)\leq c-2$.

If C is of the form (I.a), then $c=\lambda+9$, so by (9) we must have

$$v(\omega)=4(p-1)+6(q-1)+\lambda+3\leq\lambda+7$$

for ω not to vanish in $\mathfrak{a}$, which yields

$$4p+6q\leq 14 \tag{10}$$

Therefore, from (10) only two possibilities are left.

If $p = q = 1$, by the formula (9) we get

$$\omega = \begin{cases} -(1/10)\xi & \text{if } \ i = l = 1 \\ (1/10)\xi & \text{if } k = j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and if $p = 2$ and $q = 1$ we get

$$\omega = \begin{cases} -(i/14)\eta & \text{if } i > 0 \text{ and } l = 1 \\ (k/14)\eta & \text{if } k > 0 \text{ and } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

So just rename ξ as 10ξ and η as 14η.

If C is as in (I.b), then $c = (n-1)m - n + 1$ and we must consider

$$v(\omega) = n(p-1) + m(q-1) + (n-1)m - 2n - 1 \leq (n-1)m - n - 1$$

for ω not to vanish in $\mathfrak{a}$, which yields

$$n(p-2) + m(q-1) \leq 0 \tag{11}$$

Therefore, from (11), only two possibilities are left.

If $p = q = 1$, by the formula (9) we get

$$\omega = \begin{cases} -(1/(m+n))\xi & \text{if } \ i = l = 1 \\ (1/(m+n))\xi & \text{if } k = j = 1 \\ 0 & \text{otherwise} \end{cases}$$

and if $p = 2$ and $q = 1$ we get

$$\omega = \begin{cases} -(i/(m+2n))\eta & \text{if } i > 0 \text{ and } l = 1 \\ (k/(m+2n))\eta & \text{if } k > 0 \text{ and } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

So just rename ξ as $(m+n)\xi$ and η as $(m+2n)\eta$.

If C is of the form (II.a), then $c = 3m - 3$ and we must consider

$$v(\omega) = 4(p-1) + m(q-1) + 2m - 5 \leq 3m - 5$$

for ω not to vanish in $\mathfrak{a}$, which yields

$$4(p-1) + m(q-2) \leq 0 \tag{12}$$

Therefore, from (12) only three possibilities are left.

(i) $p = q = 1$, which coincides with the prior case.

(ii) $p = 1$ and $q = 2$, from which we get

$$\omega = \begin{cases} -(l/(2m+4))\eta & \text{if } \ l > 0 \text{ and } i = 1 \\ (j/(2m+4))\eta & \text{if } j > 0 \text{ and } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

so just rename ξ as $(m+n)\xi$ and η as $(2m+4)\eta$.

(iii) $p \geq 2$ and $q = 1$, from which we get

$$\omega = \frac{k - lp}{m + np}\, x^{p-1}\xi$$

and thus

$$v(\omega) = v(x^{p-1}\xi) = 4(p-1) + 2m - 5 = 4(p-2) + 2m - 1$$

Since $p \geq 2$ we have that $4(p-2) + 2m \in \mathrm{S}$ and hence ω is exact.

If C is of the form (II.b), then $c = (n-1)m - n + 1$, and we must consider

$$v(\omega) = n(p-1) + m(q-1) + (n-2)m - n - 1 \leq (n-1)m - n - 1$$

for ω not to vanish in $\mathfrak{a}$, which yields

$$n(p-1) + m(q-2) \leq 0 \tag{13}$$

Therefore, from (13) only three possibilities are left.

(i) $p = q = 1$, identical to the previous case.

(ii) $p = 1$ and $q = 2$, from which we get

$$\omega = \begin{cases} -(l/(2m+n))\eta & \text{if } \ l > 0 \text{ and } i = 1 \\ (j/(2m+n))\eta & \text{if } j > 0 \text{ and } k = 1 \\ 0 & \text{otherwise} \end{cases}$$

so just rename ξ as $(m+n)\xi$ and η as $(2m+n)\eta$.

(iii) $p \geq 2$ and $q = 1$; from which we get

$$\omega = \frac{k - lp}{m + np}\, x^{p-1}\xi$$

and thus

$$v(\omega) = v(x^{p-1}\xi) = n(p-1) + (n-2)m - n - 1 = n(p-2) + (n-2)m - 1.$$

Since $p \geq 2$ we have that $n(p-2) + (n-2)m \in \mathrm{S}$ and hence ω is exact. $\square$

The above result allows us to conclude that the universal central extension $\widehat{\mathfrak{L}}$ projects onto a superalgebra whose multiplication is induced by the bracket operation in $\mathfrak{L}$. For the remainder of this work, R and $\mathfrak{g}$ are as in the prior theorem, and so is the convention for x, y, ξ, η and $\mathfrak{a}$.

Theorem 1.4. *There exists a natural epimorphism $\widehat{\mathfrak{L}} \longrightarrow \mathfrak{L}'$, where $\mathfrak{L}'$ is a nontrivial superalgebra.*

Proof. Consider $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}^0 + \widehat{\mathfrak{L}}^1$ as a sum of $\mathbb{C}$-vector spaces, where

$$\widehat{\mathfrak{L}}^0 = \Big(\mathfrak{g} \otimes \mathbb{C}[[x^2, y]]\Big) \oplus \mathbb{C}\eta \qquad \widehat{\mathfrak{L}}^1 = \Big(\mathfrak{g} \otimes \mathbb{C}[[x^2, y]]x\Big) \oplus \mathbb{C}\xi$$

if C is of kind (I), and also

$$\widehat{\mathfrak{L}}^0 = \Big(\mathfrak{g} \otimes \mathbb{C}[[x, y^2]]\Big) \oplus \mathbb{C}\eta \qquad \widehat{\mathfrak{L}}^1 = \Big(\mathfrak{g} \otimes \mathbb{C}[[x, y^2]]y\Big) \oplus \mathbb{C}\xi$$

if C is of kind (II).

If so, we claim that $[\widehat{\mathfrak{L}}^0, \widehat{\mathfrak{L}}^0] \subset \widehat{\mathfrak{L}}^0$, $[\widehat{\mathfrak{L}}^0, \widehat{\mathfrak{L}}^1] \subset \widehat{\mathfrak{L}}^1$ and $[\widehat{\mathfrak{L}}^1, \widehat{\mathfrak{L}}^1] \subset \widehat{\mathfrak{L}}^0$ in any case. Indeed, we first consider curves of kind (I). Set $M_0 := \mathbb{C}[[x^2, y]]$, $M_1 := \mathbb{C}[[x^2, y]]x$, and take $A, B \in \mathfrak{g}$, $f, g \in R$. Note that

$$AB \otimes fg \in \begin{cases} \widehat{\mathfrak{L}}^0 & \text{if } f, g \in M_0 \\ \widehat{\mathfrak{L}}^1 & \text{if } \ f \in M_0 \text{ and } g \in M_1 \\ \widehat{\mathfrak{L}}^0 & \text{if } \ f, g \in M_1 \end{cases} \tag{14}$$

So, by (6), it suffices to show that (14) holds replacing $AB\otimes fg$ by $\overline{fdg}$. Since $0\in\widehat{\mathfrak{L}}^0\cap\widehat{\mathfrak{L}}^1$, it is enough dealing with f,g for which $\overline{fdg}\neq 0$. But by the previous theorem, the only nonzero cocycles, up to sign (or switch since $\overline{fdg}=-\overline{gdf}$), are

$$\begin{aligned}\overline{xdy}&=-\xi\in\widehat{\mathfrak{L}}^1\\ \overline{xdxy}&=-\eta\in\widehat{\mathfrak{L}}^0\\ \overline{x^2dy}&=-2\eta\in\widehat{\mathfrak{L}}^0\end{aligned}\tag{15}$$

and the claim follows for curves (I) since $x\in M_1$, $y\in M_0$, $xy\in M_1$ and $x^2\in M_0$.

Analogously, if C is of kind (II) we consider $M_0:=\mathbb{C}[[x,y^2]]$, $M_1:=\mathbb{C}[[x,y^2]]y$. By the prior theorem, in this case, the only nonzero cocycles, up to sign or switch, are

$$\begin{aligned}\overline{xdy}&=-\xi\in\widehat{\mathfrak{L}}^1\\ \overline{xydy}&=-\eta\in\widehat{\mathfrak{L}}^0\\ \overline{xdy^2}&=-2\eta\in\widehat{\mathfrak{L}}^0\end{aligned}$$

and the claim follows for curves (II) as $x\in M_0$, $y\in M_1$, $xy\in M_1$ and $y^2\in M_0$.

Now consider the ideal $M:=\langle M_0\cap M_1\rangle\subset R$, and set $R'=R/M$. Let

$$\widehat{\mathfrak{L}}':=\big(\mathfrak{g}\otimes R'\big)\oplus\big(\mathbb{C}\xi\oplus\mathbb{C}\eta\big)$$

be a central extension of $\mathfrak{g}\otimes R'$ such that

$$[A\otimes\overline{h},B\otimes\overline{g}]:=[A,B]\otimes\overline{hg}+(A,B)\overline{hdg}$$

where $A,B\in\mathfrak{g}$ and $h,g\in R$. We claim that this bracket operation is well defined. In fact, to check this, it suffices to show that the induced bilinear map

$$\begin{array}{ccc}M\times M&\longrightarrow&\mathfrak{a}\\ (h,g)&\longmapsto&\overline{hdg}\end{array}$$

vanishes. In order to see so, first consider C of kind (I). Set

$$M_0':=\mathbb{C}[[X^2,Y]],\quad M_1':=\mathbb{C}[[X^2,Y]]X,$$

and write $R'=\mathbb{C}[[X,Y]]/\langle M_0'\cap M_1',F\rangle$ where $F\in\mathbb{C}[[X,Y]]$ is such that $R=\mathbb{C}[[X,Y]]/\langle F\rangle$. Asssume $h\in M_0\cap M_1$ and write $h=\overline{H}$ where

$$H=a_{00}+a_{10}X+a_{01}Y+a_{20}X^2+a_{11}XY+a_{02}Y^2+\ldots\quad\in\mathbb{C}[[X,Y]]$$

Since $h\in M_0$, then $H\in M_0'$ and there exist $b_{00},b_{01},b_{20},b_{02}\in\mathbb{C}$ such that

$$(a_{00}-b_{00})+a_{10}X+(a_{01}-b_{01})Y+(a_{20}-b_{20})X^2+a_{11}XY+(a_{02}-b_{02})Y^2+\ldots\in\langle F\rangle\tag{16}$$

Since $h\in M_1$, then $H\in M_1'$ and there exist $b_{10},b_{11}\in\mathbb{C}$ such that

$$a_{00}+(a_{10}-b_{10})X+a_{01}Y+a_{20}X^2+(a_{11}-b_{11})XY+a_{02}Y^2+\ldots\in\langle F\rangle\tag{17}$$

Now, according to [2], we have that

$$F=(X^2+X^3)^2+XY^{(a+3)/2}$$

with $a>6$ and odd if C is of kind (I.a), and

$$F=X^n-Y^m+X^{n-2}Y^{m-3}$$

with $n>2$ and $m>4$ if C is of kind (I.b). So in both cases we conclude that $a_{10}=a_{11}=0$ by (16) and that $a_{00}=a_{01}=a_{20}=a_{02}=0$ by (17). Therefore, if $h,g\in M$ then one may write $h=\overline{H}$ and $g=\overline{G}$ such that $H,G\in\mathbb{C}[[X,Y]]$ with homogeneous decomposition of the form

$$H=H_3+H_4+\ldots\qquad G=G_3+G_4+\ldots$$

and hence

$$\begin{aligned}\overline{hdg} = \overline{\overline{H}d\,\overline{G}} &= \overline{H(x,y)dG(x,y)} \\ &= \overline{H_3(x,y)dG_3(x,y)} + \overline{H_3(x,y)dG_4(x,y)} + \overline{H_4(x,y)dG_3(x,y)} + \ldots\end{aligned}$$

just recalling that by construction $\overline{X} = x$ and $\overline{Y} = y$. Now, from equations (15), we see that $\overline{H_i(x,y)dG_j(x,y)} = 0$ for any pair of homogeneous polynomials $H_i(X,Y), G_j(X,Y)$ of degrees, respecpectively, i and j with $i, j \geq 3$. So the well definition of the bracket operation follows for cuves of kind (I). It follows for the ones of kind (II) by similar arguments along with the fact that any such a curve is of the form

$$F = X^n - Y^m + X^{m-3}Y^{m-2} + \sum_{i=2}^{2+\lfloor m/n \rfloor} X^{n-2}Y^{m-i}$$

with $n \geq 4$and $m > 2n/(n-3)$ owing to [2] as well. So the decomposition

$$\widehat{\mathfrak{L}}' = \Big(\big(\, \mathfrak{g} \otimes \overline{M_0} \, \big) \oplus \mathbb{C}\eta \Big) \oplus \Big(\big(\, \mathfrak{g} \otimes \overline{M_1} \, \big) \oplus \mathbb{C}\xi \Big)$$

yields a nontrivial $\mathbb{Z}_2$-grading for $\widehat{\mathfrak{L}}'$, and the epimorphism $\widehat{\mathfrak{L}} \to \widehat{\mathfrak{L}}'$ is defined just modding out R by M. □

2. The Heisenberg Type Lie Algebra

The aim of this section is getting a representation for a central extension of R. So consider the $\mathbb{C}$-vector space R as a commutative Lie algebra. We define its *Heisenberg type Lie algebra* $\mathfrak{R}$ as the central extension

$$0 \longrightarrow \mathbb{C}\xi \oplus \mathbb{C}\eta \longrightarrow \mathfrak{R} \longrightarrow R \longrightarrow 0.$$

This Lie algebra is totally described by the monomials $b_{mn} := x^m y^n$, with $m, n \in \mathbb{N}$, the central elements $\overline{\xi}$ and $\overline{\eta}$, and, from Theorem 1.3, the following relations

$$[b_{ij}, b_{kl}] = \begin{cases} -\overline{\xi} & \text{if } i = 1,\ j = 0,\ k = 0 \text{ and } l = 1 \\ \overline{\xi} & \text{if } i = 0,\ j = 1,\ k = 1 \text{ and } l = 0 \\ -i\,\overline{\eta} & \text{if } j = 0,\ l = 1,\ i + k = 2 \text{ and } i > 0 \\ k\,\overline{\eta} & \text{if } j = 1,\ l = 0,\ i + k = 2 \text{ and } k > 0 \\ 0 & \text{otherwise} \end{cases}$$

if C is of kind (I), or

$$[b_{ij}, b_{kl}] = \begin{cases} -\overline{\xi} & \text{if } i = 1,\ j = 0,\ k = 0 \text{ and } l = 1 \\ \overline{\xi} & \text{if } i = 0,\ j = 1,\ k = 1 \text{ and } l = 0 \\ -l\,\overline{\eta} & \text{if } i = 1,\ k = 0,\ j + l = 2 \text{ and } l > 0 \\ j\,\overline{\eta} & \text{if } i = 0,\ k = 1,\ j + l = 2 \text{ and } j > 0 \\ 0 & \text{otherwise} \end{cases}$$

if C is of kind (II). Furthermore,

$$[b_{mn}, \xi] = 0, \qquad\qquad [b_{mn}, \eta] = 0.$$

in both cases. Since the brackets (and hence the arguments) are similar, we will deal just with the first case.

Recall that the universal enveloping algebra of $\mathfrak{R}$ is given by

$$\mathfrak{U}(\mathfrak{R}) := \frac{\bigoplus_{n\geq 0}\mathfrak{R}^{\otimes n}}{\langle r\otimes s - s\otimes r - [r,s]\rangle_{r,s\in\mathfrak{R}}}.$$

Note that $\mathfrak{U}(\mathfrak{R})$ is an associative algebra with a unit, and, by general principles, any representation of $\mathfrak{R}$ is automatically a $\mathfrak{U}(\mathfrak{R})$-module. Besides, $\mathfrak{U}(\mathfrak{R})$ is generated by b_{mn}, $\overline{\eta}$, $\overline{\xi}$ with relations

$$b_{10}b_{01} - b_{01}b_{10} = -\overline{\xi} \tag{18}$$

$$b_{10}b_{11} - b_{11}b_{10} = -\overline{\eta} \tag{19}$$

$$b_{20}b_{01} - b_{01}b_{20} = -2\overline{\eta} \tag{20}$$

$$b_{ij}b_{kl} - b_{kl}b_{ij} = 0 \quad \text{otherwise} \tag{21}$$

$$b_{mn}\,\overline{\xi} - \overline{\xi}\,b_{mn} = 0 \tag{22}$$

$$b_{mn}\,\overline{\eta} - \overline{\eta}\,b_{mn} = 0 \tag{23}$$

for every $m,n\in\mathbb{N}$.

Theorem 2.1. *Let $\pi := \mathbb{C}[b_{10},b_{01}]$. Then the assignments to $\mathrm{End}(\pi)$ given by*

$$\begin{array}{ccccccc} \overline{\xi} & \mapsto & 0 & \qquad & \overline{\eta} & \mapsto & 1 \\ b_{10} & \mapsto & b_{10} & & b_{01} & \mapsto & b_{01} \\ b_{11} & \mapsto & \dfrac{\partial}{\partial b_{10}} & & b_{20} & \mapsto & -2\dfrac{\partial}{\partial b_{01}} \end{array}$$

$$b_{mn} \mapsto 0 \quad \textit{otherwise}$$

give rise to a Fock like representation of the Heisenberg type Lie algebra $\mathfrak{R}$ if C is of kind (I).

Proof. We start by defining the *Weyl type algebra* as

$$\widetilde{\mathfrak{R}} := \frac{\mathfrak{U}(\mathfrak{R})}{\langle\,\overline{\xi}\,,\,1-\overline{\eta}\,\rangle}$$

where $\overline{\xi},\overline{\eta}\in\mathfrak{R}^{\otimes 1}=\mathfrak{R}$ and $1\in\mathfrak{R}^{\otimes 0}=\mathbb{C}$. Note that here $1\in\mathfrak{R}^{\otimes 0}$ is the unit of $\mathfrak{U}(R)$ while $1_R = b_{00}$ is not a unit even in $\mathfrak{R}^{\otimes 1}$ since any multiplication between elements in $\mathfrak{R}^{\otimes 1}$ is in $\mathfrak{R}^{\otimes 2}$. Their action also differ in π, 1 acts as the identity while b_{00} acts trivially. Consider the subalgebra $\widetilde{\mathfrak{R}}'$ generated by all b_{mn} different from b_{10},b_{01}. It is a commutative subalgebra, thus it has a trivial one-dimensional representation (actually, $\widetilde{\mathfrak{R}}'$ is a maximal commutative subalgebra of $\widetilde{\mathfrak{R}}$). The Fock representation π of $\widetilde{\mathfrak{R}}$ is the induced representation from the one of $\widetilde{\mathfrak{R}}'$, i.e.,

$$\pi := \mathrm{Ind}_{\widetilde{\mathfrak{R}}'}^{\widetilde{\mathfrak{R}}}\mathbb{C} = \widetilde{\mathfrak{R}}\otimes_{\widetilde{\mathfrak{R}}'}\mathbb{C}.$$

Let $\widetilde{\mathfrak{R}}''$ be the subalgebra of $\widetilde{\mathfrak{R}}$ generated by b_{10},b_{01}. By Poincaré-Birkhoff-Witt Theorem we have that $\widetilde{\mathfrak{R}}\cong\widetilde{\mathfrak{R}}''\otimes\widetilde{\mathfrak{R}}'$. Thus

$$\pi = \widetilde{\mathfrak{R}}\otimes_{\widetilde{\mathfrak{R}}'}\mathbb{C}\cong\widetilde{\mathfrak{R}}''\otimes\widetilde{\mathfrak{R}}'\otimes_{\widetilde{\mathfrak{R}}'}\mathbb{C}\cong\widetilde{\mathfrak{R}}''\otimes_{\mathbb{C}}\mathbb{C}\cong\widetilde{\mathfrak{R}}''$$

Therefore

$$\pi\cong\widetilde{\mathfrak{R}}''\cong\mathbb{C}[b_{10},b_{01}],$$

where the last isomorphism comes from the construction of $\widetilde{\mathfrak{R}}$ along with the equation (18) above. Under this isomorphism, the generators b_{10},b_{01} act on π by multiplication. We will find the action of the b_{mn} other than b_{10},b_{01}. Note that if they also differ from b_{11} and b_{20}, they may act as 0 owing to (21). So it remains to find the action of b_{11},b_{20}, which will be done by induction using the relations mentioned above. By the construction of $\widetilde{\mathfrak{R}}$ along with (22) and

(23) we have that $b_{mn}\cdot 1=0$ for every $b_{mn}\neq b_{10},b_{01}$ and where $1\in\mathbb{C}[b_{10},b_{01}]$. Thus, from (19), we get

$$\begin{aligned} b_{11}\cdot b_{10} &= b_{11}b_{10}\cdot 1 = b_{10}b_{11}\cdot 1+\overline{\eta}\cdot 1\\ &= b_{10}\cdot(b_{11}\cdot 1)+1\cdot 1\\ &= b_{10}\cdot 0+1=1 \end{aligned}$$

Now assume, by induction, that $b_{11}\cdot b_{10}^{k-1}=(k-1)b_{10}^{k-2}$. Thus

$$\begin{aligned} b_{11}\cdot b_{10}^{k} &= b_{11}b_{10}\cdot b_{10}^{k-1} = b_{10}b_{11}\cdot b_{10}^{k-1}+1\cdot b_{10}^{k-1}\\ &= b_{10}\cdot(k-1)b_{10}^{k-2}+b_{10}^{k-1}=kb_{10}^{k-1}. \end{aligned}$$

Since there is no relation (and hence restriction) of b_{11} with respect to b_{01}, we have that b_{11} acts as $\partial/\partial b_{10}$ in $\mathbb{C}[b_{10},b_{01}]$. Similarly, using (20), one may conclude that b_{20} acts as $-2\partial/\partial b_{01}$ in $\mathbb{C}[b_{10},b_{01}]$. □

The reader can check that the same result holds for curves of kind (II) replacing $\partial/\partial b_{10}$ by $-\partial/\partial b_{01}$ and $b_{20}\mapsto -2\partial/\partial b_{01}$ by $b_{02}\mapsto 2\partial/\partial b_{10}$.

3. The Action of the Loop Type Algebra

The aim of this section is giving to the *loop type algebra* $\mathfrak{L}$ a representation into a suitable space. In order to do so, let E,F,H be the standard basis of $\mathfrak{g}$, that is,

$$E:=\begin{pmatrix}0&1\\0&0\end{pmatrix}\qquad F:=\begin{pmatrix}0&0\\1&0\end{pmatrix}\qquad H:=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

which are such that

$$[E,F]=H\qquad [H,E]=2E\qquad [H,F]=-2F \tag{24}$$

and, for the sake of simplicity, set also

$$e_{ij}:=E\otimes x^iy^j\qquad f_{ij}:=F\otimes x^iy^j\qquad h_{ij}:=H\otimes x^iy^j$$

which are the basis elements of $\mathfrak{L}$.

With this in mind we have the following result for curves of kind (I). A similar, though harder to express, result holds to curves of kind (II) as well.

Theorem 3.1. *Let $W:=\mathbb{C}[[t_l]]_{l\in\mathbb{N}}$ be the ring of formal power series in infinitely many variables. Then the assignments into $\mathrm{End}(W)$ given by*

$$\begin{aligned} e_{ij} &\longmapsto \sum_{k=0}^{j}\binom{j}{k}\frac{\partial}{\partial t_{N(i,j,k)}}\\ h_{ij} &\longmapsto -2\sum_{k=0}^{j}\binom{j}{k}\Bigg(\sum_{q-p=N(i,j,k)} t_p\frac{\partial}{\partial t_q}\Bigg)\\ f_{ij} &\longmapsto -\sum_{k=0}^{j}\binom{j}{k}\Bigg(\sum_{r-p-q=N(i,j,k)} t_pt_q\frac{\partial}{\partial t_r}\Bigg) \end{aligned}$$

where

$$N(i,j,k)=ni+mk+\lambda(j-k)$$

give rise to a representation of the loop type algebra $\mathfrak{L}=\mathfrak{g}\otimes R$, where R is a local ring of a curve of kind (I).

Proof. Note that R is a subalgebra of $\mathbb{C}[[t]]$, so $\mathfrak{L}$ is a sub Lie algebra of $\mathfrak{g}\otimes\mathbb{C}[[t]]$. So the result follows from the fact that

$$x^i y^j = \sum_{k=0}^{j} \binom{j}{v} t^{N(i,j,k)}$$

and that the assignments into $\mathrm{End}(W)$ given by

$$\begin{array}{rcl} e_i := E\otimes t^i & \longmapsto & \dfrac{\partial}{\partial t_i} \\ h_i := H\otimes t^i & \longmapsto & -2\displaystyle\sum_{q-p=i} t_p \frac{\partial}{\partial t_q} \\ f_i := F\otimes t^i & \longmapsto & -\displaystyle\sum_{r-p-q=i} t_p t_q \frac{\partial}{\partial t_r} \end{array}$$

give rise to a representation of $\mathfrak{g}\otimes\mathbb{C}[[t]]$ into $\mathrm{End}(W)$, which is essentially the representation of the original loop algebra of $\mathfrak{g}$.

At any rate, to check so, identifying source and target in the above assignments, one has to show that, in $\mathrm{End}(W)$, the following relations hold:

(A) $[e_i, f_j] = h_{i+j}$
(B) $[h_i, e_j] = 2e_{i+j}$
(C) $[h_i, f_j] = -2f_{i+j}$

which hold in $\mathfrak{g}\otimes\mathbb{C}[[t]]$ due to (24). So take $P\in W$. To prove (A), write

$$\begin{aligned} [e_i, f_j](P) &= e_i f_j(P) - f_j e_i(P) \\ &= \frac{\partial}{\partial t_i}\left(-\sum_{r-p-q=j} t_p t_q \frac{\partial P}{\partial t_r}\right) + \sum_{r-p-q=j} t_p t_q \frac{\partial}{\partial t_r}\Big(\frac{\partial P}{\partial t_i}\Big) \\ &= \sum_{r-p-q=j} -\Big(\frac{\partial(t_p t_q)}{\partial t_i}\frac{\partial P}{\partial t_r} + t_p t_q \frac{\partial^2 P}{\partial t_i \partial t_r}\Big) + t_p t_q \frac{\partial^2 P}{\partial t_r \partial t_i} \\ &= -\sum_{r-p-q=j}\Big(t_p \frac{\partial t_q}{\partial t_i} + t_q \frac{\partial t_p}{\partial t_i}\Big)\frac{\partial P}{\partial t_r} \\ &= -\sum_{r-p-q=j}\Big(t_p \delta_{q,i} + t_q \delta_{p,i}\Big)\frac{\partial P}{\partial t_r} \\ &= -\left(\sum_{r-p-i=j} t_p \frac{\partial P}{\partial t_r} + \sum_{r-i-q=j} t_q \frac{\partial P}{\partial t_r}\right) \\ &= -2\sum_{r-p=i+j} t_p \frac{\partial P}{\partial t_r} = h_{i+j}(P) \end{aligned}$$

so (A) follows from the generality of P. To prove (B), write

$$\begin{aligned}
[h_i, e_j](P) &= h_i e_j(P) - e_j h_i(P) \\
&= -2 \sum_{q-p=i} t_p \frac{\partial}{\partial t_q}\Big(\frac{\partial P}{\partial t_j}\Big) - \frac{\partial}{\partial t_j}\left(-2 \sum_{q-p=i} t_p \frac{\partial P}{\partial t_q}\right) \\
&= -2 \sum_{q-p=i} \left(t_p \frac{\partial^2 P}{\partial t_q \partial t_j} - \frac{\partial t_p}{\partial t_j}\frac{\partial P}{\partial t_q} - t_p \frac{\partial^2 P}{\partial t_j \partial t_q}\right) \\
&= 2 \frac{\partial P}{\partial t_{i+j}} = 2 e_{i+j}(P)
\end{aligned}$$

and (B) follows as well. To prove (C), write

$$\begin{aligned}
[h_i, f_j](P) &= h_i f_j(P) - f_j h_i(P) \\
&= 2 \sum_{q-p=i} t_p \frac{\partial}{\partial t_q}\left(\sum_{r-c-d=j} t_c t_d \frac{\partial P}{\partial t_r}\right) - \sum_{r-c-d=j} t_c t_d \frac{\partial}{\partial t_r}\left(2 \sum_{q-p=i} t_p \frac{\partial P}{\partial t_q}\right) \\
&= 2 \sum_{q-p=i} t_p \left(\sum_{r-c-d=j} \frac{\partial t_c t_d}{\partial t_q}\frac{\partial P}{\partial t_r} + t_c t_d \frac{\partial^2 P}{\partial t_q t_r}\right) \\
&\qquad - 2 \sum_{r-c-d=j} t_c t_d \left(\sum_{q-p=i} \frac{\partial t_p}{\partial t_r}\frac{\partial P}{\partial t_q} + t_p \frac{\partial^2 P}{\partial t_r t_q}\right) \\
&= 2 \sum_{q-p=i} t_p \Big(\sum_{r-q-d=j} t_d + \sum_{r-c-q=j} t_c \Big) \frac{\partial P}{\partial t_r} - 2 \sum_{q-c-d=i+j} t_c t_d \frac{\partial P}{\partial t_q} \\
&= 4 \sum_{r-c-p=i+j} t_p t_c \frac{\partial P}{\partial t_r} - 2 \sum_{q-c-d=i+j} t_c t_d \frac{\partial P}{\partial t_q} \\
&\qquad = 2 \sum_{r-c-p=i+j} t_p t_c \frac{\partial P}{\partial t_r} \\
&\qquad = 2 f_{i+j}(P)
\end{aligned}$$

and the claim follows. □

Just for the sake of keeping the standard notation of the literature, one may express the actions for e_{ij}, h_{ij} and f_{ij} above using generating functions. So set

$$e(z,u) := \sum_{i,j\in\mathbb{N}} e_{ij} z^{-i-1} u^{-j-1}$$

and the same for h and f. Set also

$$a_{ij} := \sum_{k=0}^{j} \binom{j}{k} \frac{\partial}{\partial t_{N(i,j,k)}} \qquad a'_{ij} := -2 \sum_{k=0}^{j} \binom{j}{k} \Big(\sum_{q-p=N(i,j,k)} t_p \frac{\partial}{\partial t_q}\Big)$$

$$a''_{ij} := -\sum_{k=0}^{j} \binom{j}{k} \Big(\sum_{r-p-q=N(i,j,k)} t_p t_q \frac{\partial}{\partial t_r}\Big)$$

and

$$a(z,u)=\sum_{i,j\in\mathbb{N}} a_{ij} z^{-i-1}u^{-j-1} \qquad a'(z,u)=\sum_{i,j\in\mathbb{N}} a'_{ij} z^{-i-1}u^{-j-1}$$

$$a''(z,u)=\sum_{i,j\in\mathbb{N}} a''_{ij} z^{-i-1}u^{-j-1}$$

Then one may rephrase Theorem 3.1 saying that the assignments into $\operatorname{End}(W)$ given by

$$\begin{array}{ccc} e(z,u) & \longmapsto & a(z,u) \\ h(z,u) & \longmapsto & a'(z,u) \\ f(z,u) & \longmapsto & a''(z,u) \end{array}$$

give rise to a representation of the loop type algebra $\mathfrak{L}=\mathfrak{g}\otimes R$, where R is a local ring of a curve of kind (I).

4. On Wakimoto Type Free Field Realizations

In this section, we discuss Wakimoto type free field realizations for the universal central extension $\widehat{\mathfrak{L}}$. In order to do so, we will basically combine Theorems 2.1 and 3.1. So the actions these results yield may be extended to a broader space, namely, the tensor prouct of the representation spaces, i.e., $W\otimes\pi$. We may abuse notation and keep the same symbols for both new actions. For instance, we have

$$\begin{array}{cccc} a_{ij}: & W\otimes\pi & \longrightarrow & W\otimes\pi \\ & P\otimes p & \longmapsto & a_{ij}\cdot P\otimes p \end{array} \qquad \begin{array}{cccc} b_{ij}: & W\otimes\pi & \longrightarrow & W\otimes\pi \\ & P\otimes p & \longmapsto & P\otimes b_{ij}\cdot p \end{array}$$

We also fix the following standard notation

$$b(z,u):=\sum_{i,j\in\mathbb{N}} b_{ij} z^{-i-1}u^{-j-1}$$

and if $\{c_{ij}\}_{i,j\in\mathbb{N}}$ is any collection of endomorphisms of $W\otimes\pi$ we set

$$c(z,u):=\sum_{i,j\in\mathbb{N}} c_{ij} z^{-i-1}u^{-j-1}$$

With this in mind, we have the following result.

Theorem 4.1. *Let $V:=\mathbb{C}[[t_l]]_{l\in\mathbb{N}}\otimes\mathbb{C}[b_{10},b_{01}]$. The assignments into $\operatorname{End}(V)$ given by*

$$\begin{array}{ccl} e(z,u) & \mapsto & a(z,u) \\ h(z,u) & \mapsto & a'(z,u)+b(z,u), \\ f(z,u) & \mapsto & a''(z,u)+c(z,u) \end{array}$$

where the endomorphisms c_{ij} satisfy the recursive relation

$$c_{(k,l)+(i,j)}:=\left[\sum_{k=0}^{j}\binom{j}{k}\left(\sum_{q-p=N(i,j,k)} t_p\frac{\partial}{\partial t_q}\right)-\frac{b_{ij}}{2}\;,\;c_{kl}\right]$$

along with the conditions

$$\left[\frac{\partial}{\partial t_n}\;,\;c_{01}\right]=\frac{\partial}{\partial b_{10}} \qquad \left[\frac{\partial}{\partial t_n}\;,\;c_{11}\right]=-4 \qquad \left[\frac{\partial}{\partial t_{2n}}\;,\;c_{01}\right]=-8$$

and

$$\left[\sum_{k=0}^{j}\binom{j}{k}\frac{\partial}{\partial t_{N(i,j,k)}},c_{kl}\right]=0$$

otherwise, give a Wakimoto type free field realization of the universal central extension $\widehat{\mathfrak{L}}$ of $\mathfrak{g}\otimes R$, where R is the local ring of a curve of kind (I).

Proof. We start by working on $\widehat{\mathfrak{L}}$. We have that

$$\begin{aligned}
[e_{ij}, f_{kl}] &= [E\otimes x^iy^j, F\otimes x^ky^l] \\
&= [E\otimes b_{ij}, F\otimes b_{kl}] \\
&= [E,F]\otimes b_{i+k,j+l} + (E,F)[b_{ij}, b_{kl}] \\
&= H\otimes b_{i+k,j+l} + (E,F)[b_{ij}, b_{kl}] \\
&= h_{i+k,j+l} + (E,F)[b_{ij}, b_{kl}]
\end{aligned}$$

where (E,F) is the Killing form. Similarly,

$$\begin{aligned}
[h_{ij}, e_{kl}] &= 2e_{i+k,j+l} + (H,E)[b_{ij}, b_{kl}] \\
[h_{ij}, f_{kl}] &= -2f_{i+k,j+l} + (H,F)[b_{ij}, b_{kl}]
\end{aligned}$$

Now

$$\begin{aligned}
[e(z,u), f(w,v)] &= \Big[\sum_{i,j\in\mathbb{N}} e_{ij}z^{-i-1}u^{-j-1}, \sum_{k,l\in\mathbb{N}} f_{kl}w^{-k-1}v^{-l-1}\Big] \\
&= \sum_{i,j,k,l\in\mathbb{N}} [e_{ij}, f_{kl}]z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1} \\
&= \sum_{i,j,k,l\in\mathbb{N}} \Big(h_{i+k,j+l} + (E,F)[b_{ij}, b_{kl}]\Big) z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1}
\end{aligned}$$

Thus, using the fact that $(E,F)=4$, we get

$$[e(z,u), f(w,v)] = \sum_{i,j,k,l\in\mathbb{N}} \Big(h_{i+k,j+l} + 4[b_{ij}, b_{kl}]\Big) z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1} \tag{25}$$

Similarly, using the Killing forms $(H,E)=0$ and $(H,F)=0$, we have

$$[h(z,u), e(w,v)] = \sum_{i,j,k,l\in\mathbb{N}} 2e_{i+k,j+l}\, z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1} \tag{26}$$

$$[h(z,u), f(w,v)] = \sum_{i,j,k,l\in\mathbb{N}} -2f_{i+k,j+l}\, z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1} \tag{27}$$

We will have to check that the three formulas above hold in $\mathrm{End}(V)$, in which the assignments of the theorem turn the formula (25) into

$$\begin{aligned}
[a(z,u), a''(w,v) + c(w,v)] = &\sum_{i,j,k,l\in\mathbb{N}} \Big(a'_{i+k,j+l} + b_{i+k,j+l}\Big) z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1} \\
&+ \sum_{i,j,k,l\in\mathbb{N}} 4[b_{ij}, b_{kl}]\ z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1}
\end{aligned}$$

On the other hand

$$\begin{aligned}
[a(z,u), a''(w,v) + c(w,v)] &= [a(z,u), a''(w,v)] + [a(z,u), c(w,v)] \\
&= \sum_{i,j,k,l\in\mathbb{N}} a'_{i+k,j+l}\, z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1} \\
&\qquad + [a(z,u), c(w,v)]
\end{aligned}$$

where the last equality comes from Theorem 3.1. So this yields the condition

$$[a(z,u),c(w,v)] = \sum_{i,j,k,l\in\mathbb{N}} \Big(b_{i+k,j+l} + 4[b_{ij},b_{kl}]\Big) z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1} \tag{28}$$

Again, in $\mathrm{End}(V)$, formula (26) becomes

$$[a'(z,u)+b(z,u),a(w,v)] = \sum_{i,j,k,l\in\mathbb{N}} 2a_{i+k,j+l}\, z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1}$$

but, on the other hand, we have

$$\begin{aligned}[a'(z,u)+b(z,u),a(w,v)] &= [a'(z,u),a(w,v)] + [b(z,u),a(w,v)]\\ &= \sum_{i,j,k,l\in\mathbb{N}} 2a_{i+k,j+l}\, z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1}\end{aligned}$$

due to Theorem 3.1 and the fact that $[b(z,u),a(w,v)] = 0$ since a and b act in different parts of the tensor product so they commute. Therefore, (26) holds in $\mathrm{End}(V)$.

Finally, in $\mathrm{End}(V)$, formula (27) corresponds to

$$\begin{aligned}[a'(z,u)+b(z,u),a''(w,v)+c(u,v)] = &\sum_{i,j,k,l\in\mathbb{N}} -2a''_{i+k,j+l}\ z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1}\\ &+\sum_{i,j,k,l\in\mathbb{N}} -2c_{i+k,j+l}\ z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1}\end{aligned}$$

but we also have

$$\begin{aligned}[a'(z,u)+b(z,u),a''(w,v)+c(w,v)] &= [a'(z,u),a''(w,v)] + [a'(z,u),c(w,v)]\\ &\quad + [b(z,u),a''(w,v)] + [b(z,u),c(w,v)]\\ &= \sum_{i,j,k,l\in\mathbb{N}} -2a''_{i+k,j+l}\, z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1}\\ &\quad + [a'(z,u),c(w,v)] + [b(z,u),c(w,v)]\end{aligned}$$

owing once more to Theorem 3.1 and the fact that a'' and b commute since they act separately. So combining both equations we get the condition

$$[a'(z,u),c(w,v)] + [b(z,u),c(w,v)] = \sum_{i,j,k,l\in\mathbb{N}} -2c_{i+k,j+l}\ z^{-i-1}u^{-j-1}w^{-k-1}v^{-l-1} \tag{29}$$

Now (28) and (29) yield the following relations

$$\begin{aligned}[a_{ij},c_{kl}] &= b_{i+k,j+l} + 4[b_{ij},b_{kl}]\\ [a'_{ij}+b_{ij},c_{kl}] &= -2c_{i+k,j+l}\end{aligned}$$

and the result follows from equations (18) to (23) along with Theorem 2.1. □

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