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REMARKS ON THE GAUDIN MODEL MODULO p

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In memory of Egbert Brieskorn (1936–2013)

Abstract. We discuss the Bethe ansatz in the Gaudin model on the tensor product of finite-dimensional $\mathfrak{sl}_2$-modules over the field $\mathbb{F}_p$ with p elements, where p is a prime number. We define the Bethe ansatz equations and show that if $(t_1^0, \dots, t_k^0)$ is a solution of the Bethe ansatz equations, then the corresponding Bethe vector is an eigenvector of the Gaudin Hamiltonians. We characterize solutions $(t_1^0, \dots, t_k^0)$ of the Bethe ansatz equations as certain two-dimensional subspaces of the space of polynomials $\mathbb{F}_p[x]$. We consider the case when the number of parameters k equals 1. In that case we show that the Bethe algebra, generated by the Gaudin Hamiltonians, is isomorphic to the algebra of functions on the scheme defined by the Bethe ansatz equation. If $k = 1$ and in addition the tensor product is the product of vector representations, then the Bethe algebra is also isomorphic to the algebra of functions on the fiber of a suitable Wronski map.

1. Introduction

The Gaudin model is a certain collection of commuting linear operators on the tensor product $V = \otimes_{i=1}^n V_i$ of representations of a Lie algebra $\mathfrak{g}$. The operators are called the Gaudin Hamiltonians. The Bethe ansatz is a method used to construct common eigenvectors and eigenvalues of the Gaudin operators. One looks for an eigenvector in a certain form $W(t)$, where $W(t)$ is a V-valued function of some parameters $t = (t_1, \dots, t_k)$. One introduces a system of equations on the parameters, called the Bethe ansatz equations, and shows that if t^0 is a solution of the system, then the vector $W(t^0)$ is an eigenvector of the Gaudin Hamiltonians, see for example [B, G, FFR, MV1, MV2, MTV1, MTV4, RV, SchV, SV1, V1, V2, V3]. The Gaudin model has strong relations with the Schubert calculus and real algebraic geometry, see for example [MTV2, MTV3, So].

All that is known in the case when the Lie algebra $\mathfrak{g}$ is defined over the field $\mathbb{C}$ of complex numbers. In the paper we consider the case of the field $\mathbb{F}_p$ with p elements, where p is a prime number, cf. [SV3]. We carry out the first steps of the Bethe ansatz, the deeper parts of the Gaudin model over a finite field remain to be developed. We consider the case of the Lie algeba $\mathfrak{sl}_2$, where the notations and constructions are shorter and simpler.

It is known that over $\mathbb{C}$, the Gaudin model is a semi-classical limit of the KZ differential equations of conformal field theory, and the construction of the multidimensional hypergeometric solutions of the KZ differential equations lead, in that limit, to the Bethe ansatz construction of eigenvectors of the Gaudin Hamiltonians, see [RV]. The $\mathbb{F}_p$-analogs of the hypergeometric solutions of the KZ differential equations were constructed recently in [SV3], see also [V5]. Thus the constructions of this paper may be thought of as a semi-classical limit of the constructions in [SV3].

In Section 2 we define the Bethe ansatz equations and show that if $(t_1^0, \dots, t_k^0)$ is a solution of the Bethe ansatz equations, then the corresponding Bethe vector is an eigenvector of the Gaudin

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Hamiltonians. In Section 3 we characterize solutions $(t_1^0, \dots, t_k^0)$ of the Bethe ansatz equations as certain two-dimensional subspaces of the space of polynomials $\mathbb{F}_p[x]$. In Section 4 we consider the case in which the number k of the parameters equals 1. In that case we show that the Bethe algebra, generated by the Gaudin Hamiltonians, is isomorphic to the algebra of functions on the scheme defined by the Bethe ansatz equation, see Theorem 4.2. If $k = 1$ and in addition the tensor product is the product of vector representations, then the Bethe algebra is also isomorphic to the algebra of functions on the fiber of a suitable Wronski map, see Corollary 4.9.

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2. $\mathfrak{sl}_2$ Gaudin model

2.1. $\mathfrak{sl}_2$ Gaudin model over $\mathbb{C}$. Let e, f, h be the standard basis of the complex Lie algeba $\mathfrak{sl}_2$ with $[e, f] = h$, $[h, e] = 2e$, $[h, f] = -2f$. The element

$$\Omega = e \otimes f + f \otimes e + \frac{1}{2} h \otimes h \ \in\ \mathfrak{sl}_2 \otimes \mathfrak{sl}_2 \tag{2.1}$$

is called the Casimir element. Given n, for $1 \leqslant i < j \leqslant n$ let $\Omega^{(i,j)} \in (U(\mathfrak{sl}_2))^{\otimes n}$ be the element equal to Ω in the i-th and j-th factors and to 1 in other factors. Let $z^0 = (z_1^0, \dots, z_n^0) \in \mathbb{C}^n$ have distinct coordinates. For $s = 1, \dots, n$ introduce

$$H_s(z^0) = \sum_{l \neq s} \frac{\Omega^{(s,l)}}{z_s^0 - z_l^0} \ \in\ (U(\mathfrak{sl}_2))^{\otimes n}, \tag{2.2}$$

the *Gaudin Hamiltonians*, see [G]. For any s, l, we have

$$\big[H_s(z^0), H_l(z^0)\big] = 0, \tag{2.3}$$

and for any $x \in \mathfrak{sl}_2$ and s we have

$$[H_s(z^0), x \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \dots \otimes 1 \otimes x] = 0. \tag{2.4}$$

Let $V = \otimes_{i=1}^n V_i$ be a tensor product of $\mathfrak{sl}_2$-modules. The commutative subalgebra of $\mathrm{End}(V)$ generated by the Gaudin Hamiltonians $H_i(z^0)$, $i = 1, \dots, n$, and the identity operator Id is called the *Bethe algebra* of V. If $W \subset V$ is a subspace invariant with respect to the Bethe algebra, then the restriction of the Bethe algebra to W is called the Bethe algebra of W, denoted by $\mathcal{B}(W)$.

The general problem is to describe the Bethe algebra, its common eigenvectors and eigenvalues.

2.2. Irreducible $\mathfrak{sl}_2$-modules. For a nonnegative integer i denote by L_i the irreducible $i+1$-dimensional module with basis $v_i, fv_i, \dots, f^i v_i$ and action $h.f^k v_i = (i - 2k) f^k v_i$ for $k = 0, \dots, i$; $f.f^k v_i = f^{k+1} v_i$ for $k = 0, \dots, i-1$, $f.f^i v_i = 0$; $e.v_i = 0$, $e.f^k v_i = k(i - k + 1) f^{k-1} v_i$ for $k = 1, \dots, i$.

For $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{\geqslant 0}$, denote $|m| = m_1 + \dots + m_n$ and $L^{\otimes m} = L_{m_1} \otimes \dots \otimes L_{m_n}$. For $J = (j_1, \dots, j_n) \in \mathbb{Z}^n_{\geqslant 0}$, with $j_s \leqslant m_s$ for $s = 1, \dots, n$, the vectors

$$f_J v_m := f^{j_1} v_{m_1} \otimes \dots \otimes f^{j_n} v_{m_n} \tag{2.5}$$

form a basis of $L^{\otimes m}$. We have

$$f.f_J v_m \ = \ \sum_{s=1}^n f_{J+1_s} v_m, \qquad h.f_J v_m = (|m| - 2|J|) f_J v_m,$$

$$e.f_J v_m = \sum_{s=1}^n j_s (m_s - j_s + 1) f_{J-1_s} v_m.$$

For $\lambda \in \mathbb{Z}$, introduce the weight subspace $L^{\otimes m}[\lambda] = \{ v \in L^{\otimes m} \mid h.v = \lambda v\}$ and the singular weight subspace $\mathrm{Sing} L^{\otimes m}[\lambda] = \{ v \in L^{\otimes m}[\lambda] \mid h.v = \lambda v,\ e.v = 0\}$. We have the weight decomposition $L^{\otimes m} = \oplus_{k=0}^{|m|} L^{\otimes m}[|m|-2k]$. Denote

$$\mathcal{I}_k = \{J \in \mathbb{Z}^n_{\geqslant 0} \mid |J| = k,\ j_s \leqslant m_s,\ s = 1,\dots,n\}.$$

The vectors $(f_J v)_{J\in\mathcal{I}_k}$ form a basis of $L^{\otimes m}[|m|-2k]$.

By (2.4), the Bethe algebra $\mathcal{B}(L^{\otimes m})$ preserves each of the subspaces $L^{\otimes m}[|m|-2k]$ and $\mathrm{Sing} L^{\otimes m}[|m|-2k]$. If $w \in L^{\otimes m}$ is a common eigenvector of the Bethe algebra, then for any $x \in \mathfrak{sl}_2$ the vector $x.w$ is also an eigenvector with the same eigenvalues. These observations show that in order to describe $\mathcal{B}(L^{\otimes m})$, its eigenvectors and eigenvalues it is enough to describe for all k the algebra $\mathcal{B}(\mathrm{Sing} L^{\otimes m}[|m|-2k])$, its eigenvectors and eigenvalues.

2.3. **Bethe ansatz on** $\mathrm{Sing} L^{\otimes m}\big[|m|-2k\big]$ **over** $\mathbb{C}$. Given $k, n \in \mathbb{Z}_{>0}$, $m = (m_1,\dots,m_n) \in \mathbb{Z}^n_{>0}$. Let $z^0 = (z^0_1,\dots,z^0_n) \in \mathbb{C}^n$ have distinct coordinates. The system of the *Bethe ansatz equations* is the system of equations

$$\sum_{j\neq i} \frac{2}{t_i - t_j} - \sum_{s=1}^n \frac{m_s}{t_i - z^0_s} = 0, \qquad i = 1,\dots,k, \tag{2.6}$$

on $t = (t_1,\dots,t_k) \in \mathbb{C}^k$. If $(t^0_1,\dots,t^0_k, z^0_1,\dots,z^0_n) \in \mathbb{C}_p^{k+n}$ has distinct coordinates, denote

$$\lambda_s(t^0,z^0) = \sum_{l\neq s} \frac{m_s m_l/2}{z^0_s - z^0_l} - \sum_{i=1}^k \frac{m_s}{z^0_s - t^0_i}, \qquad s = 1,\dots,n. \tag{2.7}$$

For any function or differential form $F(t_1,\dots,t_k)$, denote

$$\mathrm{Sym}_t[F(t_1,\dots,t_k)] = \sum_{\sigma\in S_k} F(t_{\sigma_1},\dots,t_{\sigma_k}), \quad \mathrm{Ant}_t[F(t_1,\dots,t_k)] = \sum_{\sigma\in S_k} (-1)^{|\sigma|} F(t_{\sigma_1},\dots,t_{\sigma_k}).$$

For $J = (j_1,\dots,j_n) \in \mathcal{I}_k$ define the *weight function*

$$W_J(t,z) = \frac{1}{j_1!\dots j_n!}\,\mathrm{Sym}_t\left[\prod_{s=1}^n \prod_{i=1}^{j_s} \frac{1}{t_{j_1+\dots+j_{s-1}+i} - z_s}\right]. \tag{2.8}$$

For example,

$$W_{(1,0,\dots,0)} = \frac{1}{t_1 - z_1}, \qquad W_{(2,0,\dots,0)} = \frac{1}{t_1 - z_1}\frac{1}{t_2 - z_1},$$
$$W_{(1,1,0,\dots,0)} = \frac{1}{t_1 - z_1}\frac{1}{t_2 - z_2} + \frac{1}{t_2 - z_1}\frac{1}{t_1 - z_2}.$$

The function

$$W_{k,n,m}(t,z) = \sum_{J\in\mathcal{I}_k} W_J(t,z) f_J v_m \tag{2.9}$$

is the $L^{\otimes m}[|m|-2k]$-valued *vector weight function*.

Theorem 2.1 ([RV, B], cf. [SV1])**.** *If $(t^0,z^0) = (t^0_1,\dots,t^0_k, z^0_1,\dots,z^0_n)$ is a solution of the Bethe ansatz equations (2.6), then the vector $W_{k,n,m}(t^0,z^0)$ lies in $\mathrm{Sing} L^{\otimes m}[|m|-2k]$ and is an eigenvector of the Gaudin Hamiltonians, moreover,*

$$H_i(z^0).W_{k,n,m}(t^0,z^0) = \lambda_i(t^0,z^0) W_{k,n,m}(t^0,z^0), \qquad i = 1,\dots,n. \tag{2.10}$$

The eigenvector $W_{k,n,m}(t^0, z^0)$ is called the *Bethe eigenvector*. On the Bethe eigenvectors see, for example, [SchV, MV1, MV2, V1, V2, V3].

The fact that $W_{k,n,m}(t^0, z^0)$ in Theorem 2.1 lies in $\mathrm{Sing} L^{\otimes m}[|m| - 2k]$ may be reformulated as follows. For any $J \in \mathcal{I}_{k-1}$, we have

$$\sum_{s=1}^{n} (j_s + 1)(m_s - j_s) W_{J+\mathbf{1}_s}(t^0, z^0) = 0, \tag{2.11}$$

where we set $W_{J+\mathbf{1}_s}(t^0, z^0) = 0$ if $J + \mathbf{1}_s \notin \mathcal{I}_k$.

2.4. **Proof of Theorem 2.1.** We sketch the proof following [SV1]. The intermediate statements in this proof will be used later when constructing eigenvectors of the Bethe algebra over $\mathbb{F}_p$. The proof is based on the following cohomological relations.

Given $k, n \in \mathbb{Z}_{>0}$ and a multi-index $J = (j_1, \dots, j_n)$ with $|J| \leqslant k$, introduce a differential form

$$\begin{aligned} \eta_J = \frac{1}{j_1! \cdots j_n!} \operatorname{Ant}_t \Big[&\frac{d(t_1 - z_1)}{t_1 - z_1} \wedge \dots \wedge \frac{d(t_{j_1} - z_1)}{t_{j_1} - z_1} \wedge \frac{d(t_{j_1+1} - z_2)}{t_{j_1+1} - z_2} \wedge \dots \\ &\wedge \frac{d(t_{j_1+\dots+j_{n-1}+1} - z_n)}{t_{j_1+\dots+j_{n-1}+1} - z_n} \wedge \dots \wedge \frac{d(t_{j_1+\dots+j_n} - z_n)}{t_{j_1+\dots+j_n} - z_n} \Big], \end{aligned}$$

which is a logarithmic differential form on $\mathbb{C}^n \times \mathbb{C}^k$ with coordinates z, t. If $|J| = k$, then for any $z^0 \in \mathbb{C}^n$ we have on $\{z^0\} \times \mathbb{C}^k$ the identity

$$\eta_J \big|_{\{z^0\} \times \mathbb{C}^k} = W_J(t, z^0) dt_1 \wedge \dots \wedge dt_k. \tag{2.12}$$

Example 2.1. For $k = n = 2$ we have

$$\begin{aligned} \eta_{(2,0)} &= \frac{d(t_1 - z_1)}{t_1 - z_1} \wedge \frac{d(t_2 - z_1)}{t_2 - z_1}, \\ \eta_{(1,1)} &= \frac{d(t_1 - z_1)}{t_1 - z_1} \wedge \frac{d(t_2 - z_2)}{t_2 - z_2} - \frac{d(t_2 - z_1)}{t_2 - z_1} \wedge \frac{d(t_1 - z_2)}{t_1 - z_2}. \end{aligned}$$

Introduce the logarithmic differential 1-forms

$$\begin{aligned} \alpha &= \sum_{1 \leqslant i < j \leqslant n} \frac{m_i m_j}{2} \frac{d(z_i - z_j)}{z_i - z_j} + \sum_{1 \leqslant i < j \leqslant k} 2 \frac{d(t_i - t_j)}{t_i - t_j} - \sum_{s=1}^{n} \sum_{i=1}^{k} m_s \frac{d(t_i - z_s)}{t_i - z_s}, \\ \alpha' &= \sum_{1 \leqslant i < j \leqslant k} 2 \frac{d(t_i - t_j)}{t_i - t_j} - \sum_{s=1}^{n} \sum_{i=1}^{k} m_s \frac{d(t_i - z_s)}{t_i - z_s}. \end{aligned}$$

We shall use the following algebraic identities for logarithmic differential forms.

Theorem 2.2 ([SV1])**.** *We have*

$$\alpha' \wedge \eta_J = \sum_{s=1}^{n} (j_s + 1)(m_s - j_s) \eta_{J+\mathbf{1}_s}, \tag{2.13}$$

for any J with $|J| = k - 1$, and

$$\alpha \wedge \sum_{J \in \mathcal{I}_k} \eta_J f_J v_m = \sum_{i<j} \Omega^{(i,j)} \frac{d(z_i - z_j)}{z_i - z_j} \wedge \sum_{|J|=k} \eta_J f_J v_m. \tag{2.14}$$

Proof. Identity (2.13) is Theorem 6.16.2 in [SV1] for the case of the Lie algebra $\mathfrak{sl}_2$. Identity (2.14) is Theorem 7.5.2" in [SV1] for the case of the Lie algebra $\mathfrak{sl}_2$. □

If (t^0, z^0) is a solution of the Bethe ansatz equations, then $\alpha'|_{(t^0,z^0)} = 0$ and formulas (2.13), (2.12) give (2.11). Similarly, if (t^0, z^0) is a solution of the Bethe ansatz equations, then

$$\alpha|_{(t^0,z^0)} = \sum_{s=1}^{n} \lambda_s(t^0, z^0) dz_s$$

and formulas (2.14) and (2.12) give (2.18). Theorem 2.1 is proved.

2.5. **Bethe ansatz on** $\mathrm{Sing}L^{\otimes m}\big[|m| - 2k\big]$ **over** $\mathbb{F}_p$**.** Given

$$k, n \in \mathbb{Z}_{>0}, \quad m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0},$$

let p be a prime number. Consider the Lie algebra $\mathfrak{sl}_2$ as an algebra over the field $\mathbb{F}_p$ and the $\mathfrak{sl}_2$-modules L_{m_s}, $s = 1, \dots, n$, over $\mathbb{F}_p$. Let $z^0 = (z^0_1, \dots, z^0_n) \in \mathbb{F}^n_p$ have distinct coordinates. The Gaudin Hamiltonians $H_s(z^0)$ of formula (2.2) define commuting $\mathbb{F}_p$-linear operators on the $\mathbb{F}_p$-vector space $L^{\otimes m} = \otimes^n_{s=1} L_{m_s}$. By formula (2.4) the Gaudin Hamiltonians preserve the $\mathbb{F}_p$-subspaces $\mathrm{Sing}L^{\otimes m}[|m| - 2k]$ and we may study eigenvectors of the Gaudin Hamiltonians on a subspace $\mathrm{Sing}L^{\otimes m}[|m| - 2k]$.

Consider the *system of Bethe ansatz equations*

$$\sum_{j\neq i} \frac{2}{t_i - t_j} - \sum_{s=1}^{n} \frac{m_s}{t_i - z^0_s} = 0, \qquad i = 1, \dots, k, \tag{2.15}$$

as a system of equations on $t = (t_1, \dots, t_k) \in \mathbb{F}^k_p$. If $(t^0_1, \dots, t^0_k, z^0_1, \dots, z^0_n) \in \mathbb{F}^{k+n}_p$ has distinct coordinates, denote

$$\lambda_s(t^0, z^0) = \sum_{l\neq s} \frac{m_s m_l/2}{z^0_s - z^0_l} - \sum_{i=1}^{k} \frac{m_s}{z^0_s - t^0_i} \ \in \mathbb{F}_p, \qquad s = 1, \dots, n. \tag{2.16}$$

Theorem 2.3. *Let p be a prime number and $p > |m|$. Let $t^0 \in \mathbb{F}^k_p$ be a solution of the Bethe ansatz equations (2.15). Then the vector $W_{k,n,m}(t^0, z^0)$ is well-defined and lies in the subspace $\mathrm{Sing}L^{\otimes m}[|m| - 2k]$, that is, the equations*

$$\sum_{s=1}^{n} (j_s + 1)(m_s - j_s) W_{J+\mathbf{1}_s}(t^0, z^0) = 0, \tag{2.17}$$

hold, also the vector $W_{k,n,m}(t^0, z^0)$ satisfies the equations

$$H_s(z^0).W_{k,n,m}(t^0, z^0) = \lambda_s(t^0, z^0) W_{k,n,m}(t^0, z^0), \qquad s = 1, \dots, n. \tag{2.18}$$

Proof. The proof of Theorem 2.3 is the same as the proof of Theorem 2.1 since identities (2.13) and (2.14) hold over half integers and can be projected to $\mathbb{F}_p$. □

3. Two-dimensional spaces of polynomials

3.1. **Two-dimensional spaces of polynomials over** $\mathbb{C}$**.** For a function $g(x)$ denote $g' = \frac{dg}{dx}$. For functions $g(x), h(x)$ define the *Wronskian*

$$\mathrm{Wr}(g(x), h(x)) = g'(x)h(x) - g(x)h'(x).$$

Theorem 3.1 ([SchV], cf. [MV2])**.** *Let $k \in \mathbb{Z}_{>0}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$. Let $(t^0, z^0) \in \mathbb{C}^{k+n}$ have distinct coordinates. Denote*

$$y(x) = \prod_{i=1}^{k} (x - t^0_i), \qquad T(x) = \prod_{s=1}^{n} (x - z^0_s)^{m_s}. \tag{3.1}$$

We have the following two statements.

(i) *If* (t^0, z^0) *is a solution of the Bethe ansatz equations* (2.6)*, then* $k \leqslant |m|+1$, $2k \neq |m|+1$, *and there exists a polynomial* $\tilde{y}(x) \in \mathbb{C}[x]$ *of degree* $|m|+1-k$ *such that*

$$\mathrm{Wr}(\tilde{y}(x), y(x)) = T(x). \tag{3.2}$$

(ii) *If there exists a polynomial* $\tilde{y}(x)$ *satisfying equation* (3.2)*, then* $k \leqslant |m|+1$, $2k \neq |m|+1$ *and* (t^0, z^0) *is a solution of the Bethe ansatz equations* (2.6).

Proof. We will use the proof below in the proof of the p-version of Theorem 3.1. Equation (3.2) is a first order differential equation with respect to $\tilde{y}(x)$. Then

$$\left(\frac{\tilde{y}(x)}{y(x)}\right)' = \frac{T(x)}{y(x)^2} \tag{3.3}$$

and

$$\tilde{y}(x) = y(x)\int \frac{T(x)}{y(x)^2}dx = y(x)\int \frac{T(x)}{\prod_{i=1}^k (x-t_i^0)^2}dx. \tag{3.4}$$

We have the unique presentation $T(x) = Q(x)\prod_{i=1}^k (x-t_i^0)^2 + R(x)$ with $P(x), Q(x) \in \mathbb{C}[x]$ such that $Q(x) = 0$ if $2k > |m|$ and $Q(x) = a_{|m|-2k}x^{|m|-2k} + \cdots + a_0$ is of degree $|m|-2k$ otherwise; $\deg R(x) < 2k$. We have the unique presentation

$$\frac{R(x)}{\prod_{i=1}^k (x-t_i^0)^2} = \sum_{i=1}^k \left(\frac{a_{i,2}}{(x-t_i^0)^2} + \frac{a_{i,1}}{x-t_i^0}\right), \tag{3.5}$$

where

$$a_{i,2} = \frac{T(x)}{\prod_{j\neq i}^k (x-t_j^0)^2}\Big|_{x=t_i^0}, \qquad a_{i,1} = \frac{d}{dx}\left(\frac{T(x)}{\prod_{j\neq i}^k (x-t_j^0)^2}\right)\Big|_{x=t_i^0}. \tag{3.6}$$

We have

$$\frac{d}{dx}\left(\frac{T(x)}{\prod_{j\neq i}^k (x-t_j^0)^2}\right)\Big|_{x=t_i^0} = \left(\sum_{s=1}^n \frac{m_s}{t_i^0 - z_s^0} - \sum_{j\neq i}\frac{2}{t_i^0 - t_j^0}\right)\frac{T(t_i^0)}{\prod_{j\neq i}^k (t_i^0 - t_j^0)^2}.$$

Since (t^0, z^0) has distinct coordinates we conclude that $a_{i,1} = 0$ for $i = 1, \dots, k$, if and only if (t^0, z^0) is a solution of (2.6).

Let (t^0, z^0) be a solution of (2.6). By formula (3.4) we have

$$\tilde{y}(x) = \prod_{i=1}^k (x-t_i^0)\left(c - \sum_{i=1}^k \frac{a_{i,2}}{x-t_i^0}\right), \quad \text{if } 2k > |m|, \tag{3.7}$$

$$\tilde{y}(x) = \prod_{i=1}^k (x-t_i^0)\left(\frac{a_{|m|-2k}}{|m|-2k+1}x^{|m|-2k+1} + \cdots + a_0 x + c - \sum_{i=1}^k \frac{a_{i,2}}{x-t_i^0}\right),$$

if $2k \leqslant |m|$, where $c \in \mathbb{C}$ is an arbitrary number. In each of the two cases we may choose c so that $\deg \tilde{y}(x) \neq \deg y(x)$. Using the identity

$$\mathrm{Wr}(x^\alpha, x^\beta) = (\alpha - \beta)x^{\lambda+\beta-1} \tag{3.8}$$

we obtain in this case that

$$\deg \tilde{y}(x) + \deg y(x) = |m| + 1. \tag{3.9}$$

Hence $k \leqslant |m|+1$ and $k \neq |m|+1-k$. The first part of the theorem is proved.

Let there exist a polynomial $\tilde{y}(x)$ satisfying equation (3.2). Adding to $\tilde{y}(x)$ the polynomial $y(x)$ with a suitable coefficient if necessary we may assume that $\deg \tilde{y}(x) \neq \deg y(x)$. Then (3.9) implies $k \leqslant |m|+1$ and $k \neq |m|+1-k$.

By formula (3.3) we have

$$\Big(\frac{\tilde{y}(x)}{y(x)}\Big)' = a_{|m|-2k}x^{|m|-2k} + \cdots + a_0 + \sum_{i=1}^{k}\Big(\frac{a_{i,2}}{(x-t_i^0)^2} + \frac{a_{i,1}}{x-t_i^0}\Big) \ \text{ if } \ 2k \leqslant |m| \tag{3.10}$$

and

$$\Big(\frac{\tilde{y}(x)}{y(x)}\Big)' = \sum_{i=1}^{k}\Big(\frac{a_{i,2}}{(x-t_i^0)^2} + \frac{a_{i,1}}{x-t_i^0}\Big) \ \text{ if } \ 2k > |m|. \tag{3.11}$$

The function $\frac{\tilde{y}(x)}{y(x)}$ has a unique decomposition into the sum of a polynomial and simple fractions. The term by term derivative of that decomposition equals the right-hand side of (3.10) or (3.11). Hence all of coefficients $a_{i,1}$ must be zero. Hence the roots of $y(x)$ satisfy the Bethe ansatz equations. □

Remark. This construction assigns to a solution (t^0, z^0) of the Bethe ansatz equations the two-dimensional subspace $\langle \tilde{y}(x), y(x)\rangle$ of the space of polynomials $\mathbb{C}[x]$ such that $\deg y(x) = k, \deg \tilde{y}(x) = |m|-k+1$, $\mathrm{Wr}(y(x), \tilde{y}(x)) = T(x)$. That subspace is a point of the Grassmannian of two-dimensional subspaces of $\mathbb{C}[x]$.

3.2. Two-dimensional spaces of polynomials over $\mathbb{F}_p$.

Theorem 3.2. *Let $k \in \mathbb{Z}_{>0}$, $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$. Let $p > |m|+1$, $p > n+k$. Let $(t^0, z^0) \in \mathbb{F}_p^{k+n}$ have distinct coordinates. Denote*

$$y(x) = \prod_{i=1}^{k}(x - t_i^0), \qquad T(x) = \prod_{s=1}^{n}(x - z_s^0)^{m_s} \ \in \mathbb{F}_p[x]. \tag{3.12}$$

We have the following two statements.

(i) *If (t^0, z^0) is a solution of the Bethe ansatz equations (2.15), then $k \leqslant |m|+1$, $2k \neq |m|+1$, and there exists a polynomial $\tilde{y}(x) \in \mathbb{F}_p[x]$ of degree $|m|+1-k$ such that*

$$\mathrm{Wr}(\tilde{y}(x), y(x)) = T(x). \tag{3.13}$$

(ii) *If there exists a polynomial $\tilde{y}(x) \in \mathbb{F}_p[x]$ satisfying equation (3.13), then $k \leqslant |m|+1$, $2k \neq |m|+1$ and (t^0, z^0) is a solution of the Bethe ansatz equations (2.15).*

Proof.

Lemma 3.3. *Let p be a prime number. Let $d_1, \dots, d_k \in \mathbb{Z}_{>0}$ with $d_i \leqslant 2$ for all i. Let $t_1^0, \dots, t_k^0 \in \mathbb{F}_p$ be distinct and $T(x) \in \mathbb{F}_p[x]$. Then there exists a unique presentation*

$$\frac{T(x)}{\prod_{i=1}^{k}(x-t_i^0)^{d_i}} = Q(x) + \sum_{i=1}^{k}\sum_{j=1}^{d_i}\frac{a_{i,j}}{(x-t_i^0)^j}, \tag{3.14}$$

where $Q(x) \in \mathbb{F}_p[x]$ and

$$a_{i,j} = \frac{d^{j-1}}{dx^{j-1}}\left(\frac{T(x)}{\prod_{l\neq i}^{k}(x-t_l^0)^{d_l}}\right)\Big|_{x=t_i^0}. \tag{3.15}$$

Proof. The uniqueness is clear. Let us show the existence. Lift $t_1^0, \dots, t_k^0$, $T(x)$ to $t_1^1, \dots, t_k^1 \in \mathbb{Z}$, $T^1(x) \in \mathbb{Z}[x]$. We have

$$\frac{T^1(x)}{\prod_{i=1}^k (x - t_i^1)^{d_i}} = Q^1(x) + \sum_{i=1}^k \sum_{j=1}^{d_i} \frac{a_{i,j}^1}{(x - t_i^1)^j}, \tag{3.16}$$

where $Q^1(x) \in \mathbb{Z}[x]$ and

$$a_{i,j}^1 = \frac{d^{j-1}}{dx^{j-1}} \left(\frac{T^1(x)}{\prod_{j \neq i}^k (x - t_j^1)^{d_j}} \right) \Bigg|_{x = t_i^0}. \tag{3.17}$$

It is easy to see that for $j = 1, 2$ and all i the coefficient $a_{i,j}^1$ has a well-defined projection to $\mathbb{F}_p$. By projecting (3.16) to $\mathbb{F}_p$ we obtain a presentation of (3.14). $\square$

The proof of Theorem 3.2 is based on Lemma 3.3 and is analogous to the proof of Theorem 3.1. If (t^0, z^0) is a solution of (2.15), then

$$\left(\frac{\tilde{y}(x)}{y(x)} \right)' = \frac{T(x)}{\prod_{i=1}^k (x - t^0)^2} = Q(x) + \sum_{i=1}^k \frac{a_{i,2}}{(x - t_i^0)^2},$$

where $a_{i,2}$ are given by (3.15); $Q(x) \in \mathbb{F}_p[x]$, $Q(x) = 0$ if $2k > |m|$ and

$$Q(x) = a_{|m|-2k} x^{|m|-2k} + \dots + a_0$$

is of degree $|m| - 2k + 1$ if $2k \leqslant |m|$, see Section 3.1.

If $2k \leqslant |m|$, then

$$\tilde{y}(x) = \prod_{i=1}^k (x - t_i^0) \left(\frac{a_{|m|-2k}}{|m| - 2k + 1} x^{|m|-2k+1} + \dots + a_0 x - \sum_{i=1}^k \frac{a_{i,2}}{x - t_i^0} \right)$$

is a polynomial of degree $|m| - k + 1$ satisfying (3.2). Notice that the polynomial in the brackets is well-defined since $p > |m| + 1$. If $2k > |m|$, then

$$\tilde{y}(x) = -\prod_{i=1}^k (x - t_i^0) \left(\sum_{i=1}^k \frac{a_{i,2}}{x - t_i^0} \right)$$

is a polynomial satisfying (3.13) of degree $< k$. Formula (3.8) and inequality $p > k + n$ imply (3.9). The first part of Theorem 3.2 is proved.

Let there exist a polynomial $\tilde{y}(x)$ satisfying equation (3.13). Adding to $\tilde{y}(x)$ a suitable polynomial of the form $c(x^p) y(x)$ for some $c(x) \in \mathbb{F}_p[x]$ if necessary, we may assume that $\deg \tilde{y}(x) - \deg y(x) \not\equiv 0 \bmod p$. Then (3.9) holds, $k \leqslant |m| + 1$ and $k \neq |m| + 1 - k$.

By formula (3.3) and Lemma 3.3 we have

$$\left(\frac{\tilde{y}(x)}{y(x)} \right)' = a_{|m|-2k} x^{|m|-2k} + \dots + a_0 + \sum_{i=1}^k \left(\frac{a_{i,2}}{(x - t_i^0)^2} + \frac{a_{i,1}}{x - t_i^0} \right) \quad \text{if } 2k \leqslant |m| \tag{3.18}$$

and

$$\left(\frac{\tilde{y}(x)}{y(x)} \right)' = \sum_{i=1}^k \left(\frac{a_{i,2}}{(x - t_i^0)^2} + \frac{a_{i,1}}{x - t_i^0} \right) \quad \text{if } 2k > |m|. \tag{3.19}$$

The function $\frac{\tilde{y}(x)}{y(x)}$ has a unique decomposition into the sum of a polynomial and simple fractions. The term by term derivative of that decomposition equals the right-hand side of (3.18) or (3.19). Hence all of coefficients $a_{i,1}$ must be zero. Hence the roots of $y(x)$ satisfy the Bethe ansatz equations. $\square$

Remark. This construction assigns to a solution (t^0, z^0) of the Bethe ansatz equations (2.15) the two-dimensional subspace $\langle \tilde{y}(x), y(x)\rangle$ of the space of polynomials $\mathbb{F}_p[x]$ such that

$$\deg y(x) = k, \deg \tilde{y}(x) = |m| - k + 1, \quad \mathrm{Wr}(y(x), \tilde{y}(x)) = T(x).$$

That subspace is a point of the Grassmannian of two-dimensional subspaces in $\mathbb{F}_p[x]$.

4. Example: the case $k = 1$

4.1. **Gaudin model on** $\mathrm{Sing}L^{\otimes m}[|m| - 2]$**.** Let $m = (m_1, \dots, m_n) \in \mathbb{Z}^n_{>0}$ and $p > |m| + 1$. Consider the Gaudin model on $\mathrm{Sing}L^{\otimes m}[|m| - 2]$ over $\mathbb{F}_p$. That means that $k = 1$ in the notations of the previous sections. A basis of $L^{\otimes m}[|m| - 2]$ is formed by the vectors

$$f^{(s)} = v_{m_1} \otimes \dots \otimes v_{s-1} \otimes f v_{m_s} \otimes v_{s+1} \otimes \dots \otimes v_{m_n}, \quad s = 1, \dots, n. \tag{4.1}$$

We have

$$\mathrm{Sing}L^{\otimes m}[|m| - 2] = \Big\{ \sum_{s=1}^{n} c_s f^{(s)} \mid c_s \in \mathbb{F}_p \text{ and } \sum_{s=1}^{n} m_s c_s = 0 \Big\}. \tag{4.2}$$

For $s = 1, \dots, n$, define the vectors $w_s \in \mathrm{Sing}L^{\otimes m}[|m| - 2]$ by the formula

$$w_s = f^{(s)} - \frac{m_s}{|m|} \sum_{l=1}^{n} f^{(l)}. \tag{4.3}$$

We have

$$w_1 + \dots + w_n = 0. \tag{4.4}$$

By [V4, Lemma 4.2], any $n - 1$ of these vectors form a basis of $\mathrm{Sing}L^{\otimes m}[|m| - 2]$.

Let $z^0 = (z^0_1, \dots, z^0_n) \in \mathbb{F}^n_p$ have distinct coordinates. For $i = 1, \dots, n$, the Gaudin Hamiltonian $H_i(z^0)$ acts on $L^{\otimes m}[|m| - 2]$ by the formulas:

$$\begin{aligned} f^{(s)} &\mapsto \sum_{j \neq i} \frac{m_i m_j / 2}{z^0_i - z^0_j} f^{(s)} + \frac{1}{z^0_i - z^0_s} (m_s f^{(i)} - m_i f^{(s)}), \qquad s \neq i, \\ f^{(i)} &\mapsto \sum_{j \neq i} \frac{m_i m_j / 2}{z^0_i - z^0_j} f^{(i)} + \sum_{j \neq i} \frac{1}{z^0_i - z^0_j} (m_i f^{(j)} - m_j f^{(i)}). \end{aligned} \tag{4.5}$$

Hence

$$\begin{aligned} w_s &\mapsto \sum_{j \neq i} \frac{m_i m_j / 2}{z^0_i - z^0_j} w_s + \frac{1}{z^0_i - z^0_s} (m_s w_i - m_i w_s), \qquad s \neq i, \\ w_i &\mapsto \sum_{j \neq i} \frac{m_i m_j / 2}{z^0_i - z^0_j} w_i + \sum_{j \neq i} \frac{1}{z^0_i - z^0_j} (m_i w_j - m_j w_i). \end{aligned} \tag{4.6}$$

Recall that the Bethe algebra of $\mathrm{Sing}L^{\otimes m}[|m| - 2]$ is the subalgebra of $\mathrm{End}(\mathrm{Sing}L^{\otimes m}[|m| - 2])$ generated by the Gaudin Hamiltonians $H_i(z^0)$, $i = 1, \dots, n$, and the identity operator. We denote it by $\mathcal{B}(z^0, m)$.

4.2. **Bethe ansatz equation and algebra $\mathcal{A}(z^0,m)$.** Let $m=(m_1,\dots,m_n)\in\mathbb{Z}^n_{>0}$ and $p>|m|+1$. Let $z^0=(z^0_1,\dots,z^0_n)$ have distinct coordinates. The Bethe ansatz equations of $\mathrm{Sing}L^{\otimes m}[|m|-2]$ is the single equation

$$\frac{m_1}{t-z^0_1}+\dots+\frac{m_n}{t-z^0_n}=0. \tag{4.7}$$

Write

$$\frac{m_1}{t-z^0_1}+\dots+\frac{m_n}{t-z^0_n}=\frac{P(t)}{\prod_{s=1}^n(t-z^0_s)}, \tag{4.8}$$

where

$$P(t)=P(t,z^0,m)=\sum_{s=1}^n m_s\prod_{l\neq s}(t-z^0_l). \tag{4.9}$$

Let $\mathbb{A}_{\mathbb{F}_p}$ be the affine line over $\mathbb{F}_p$ with coordinate t. Denote $U=\mathbb{A}_{\mathbb{F}_p}-\{z^0_1,\dots,z^0_n\}$. Let $\mathcal{O}(U)$ be the ring of rational functions on the affine line $\mathbb{A}_{\mathbb{F}_p}$ regular on U. Introduce the algebra

$$\mathcal{A}(z^0,m)=\mathcal{O}(U)/(P(t)),\qquad \dim_{\mathbb{F}_p}\mathcal{A}(z^0,m)=n-1. \tag{4.10}$$

Here $(P(t))$ is the ideal generated by $P(t)$. Let $u_s\in\mathcal{A}(z^0,m)$, $s=1,\dots,n$, be the image of $\frac{m_s}{t-z^0_s}$ in $\mathcal{A}(z^0,m)$. The elements u_s span $\mathcal{A}(z^0,m)$ as a vector space and

$$u_1+\dots+u_n=0. \tag{4.11}$$

We have

$$\begin{aligned} u_iu_s&=\frac{1}{z^0_i-z^0_s}(m_su_i-m_iu_s),\qquad s\neq i,\\ u_iu_i&=\sum_{j\neq i}\frac{1}{z^0_i-z^0_j}(m_iu_j-m_ju_i). \end{aligned} \tag{4.12}$$

For a function $g(t)\in\mathcal{O}(U)$ denote $[g(u)]$ its image in $\mathcal{A}(z^0,m)$. The elements $[1],[t],\dots,[t^{n-2}]$ form a basis of $\mathcal{A}(z^0,m)$ over $\mathbb{F}_p$. The defining relation in $\mathcal{A}(z^0,m)$ is $P([t])=0$. The following formulas express the elements $[t^i]$ in terms of the elements u_s.

Lemma 4.1. *We have*

$$\begin{aligned} [1]&=\frac{-1}{|m|}(z^0_1u_1+\dots+z^0_nu_n),\\ [t]&=\frac{1}{|m|^2}\left(\sum_{s=1}^n z^0_sm_s\right)\left(\sum_{s=1}^n z^0_su_s\right)+\frac{-1}{|m|}\left(\sum_{s=1}^n(z^0_s)^2u_s\right),\\ [t^i]&=\frac{-1}{|m|}\sum_{j=1}^i\sum_{s=1}^n(z^0_s)^jm_s[t^{i-j}]+\frac{-1}{|m|}\sum_{s=1}^n(z^0_s)^{i+1}u_s,\qquad i\geqslant 0. \end{aligned} \tag{4.13}$$

□

These formulas are related to formulas for the $\widehat{sl}_2$-action on tensor products of modules dual to Verma modules, see [SV2] and in particular to formula (11) in [SV2].

4.3. **Isomorphism of $\mathcal{A}(z^0, m)$ and $\mathcal{B}(z^0, m)$.** Define the isomorphism of vectors spaces

$$\alpha \ : \ \mathcal{A}(z^0, m) \ \to \ \mathrm{Sing}L^{\otimes m}[|m|-2], \qquad u_s \mapsto w_s, \qquad s = 1, \dots, n, \tag{4.14}$$

in particular, we have

$$\langle 1 \rangle := \alpha([1]) = \frac{-1}{|m|}(z_1^0 w_1 + \dots + z_n^0 w_n). \tag{4.15}$$

Theorem 4.2. *The map*

$$[1] \mapsto \mathrm{Id}, \quad u_s \mapsto H_s(z^0) - \sum_{j \neq s} \frac{m_s m_j/2}{z_s^0 - z_j^0}\,\mathrm{Id}, \qquad s = 1, \dots, n, \tag{4.16}$$

extends to an algebra isomorphism

$$\beta : \mathcal{A}(z^0, m) \to \mathcal{B}(z^0, m), \tag{4.17}$$

such that $\alpha(gh) = \beta(g).\alpha(h)$ for any $g, h \in \mathcal{A}(z^0, m)$.

Proof. The proof follows from comparing (4.6) and (4.12). □

Remark. Theorem 4.2 says that the isomorphism α of vector spaces and the isomorphism β of algebras establish an isomorphism between the $\mathcal{B}(z^0, m)$-module $\mathrm{Sing}L^{\otimes m}[|m|-2]$ and the regular representation of the algebra $\mathcal{A}(z^0, m)$.

Example 4.1. Theorem 4.2 in particular says that if $P(t)$ is irreducible then $\mathcal{B}(z^0, m) \cong \mathbb{F}_{p^{n-1}}$, where $\mathbb{F}_{p^{n-1}}$ is the field with p^{n-1} elements.

For example, if $n = 3$, $m = (1, 1, 1)$, then

$$P(t, z^0) = 3t^2 - 2(z_1^0 + z_2^0 + z_3^0)t + z_1^0 z_2^0 + z_1^0 z_3^0 + z_2^0 z_3^0.$$

If $p = 5$, then $P(t, z^0)$ is irreducible in $\mathbb{F}_5[t]$ for all distinct $z_1^0, z_2^0, z_3^0 \in \mathbb{F}_5$ and $\mathcal{B}(z^0, m) \cong \mathbb{F}_{25}$.

Corollary 4.3. *We have*

$$\dim_{\mathbb{F}_p} \mathcal{B}(z^0, m) = n - 1. \tag{4.18}$$

Corollary 4.4. *The operators $\beta([1]) = \mathrm{Id}$, $\beta([t^i])$, $i = 1, \dots, n-2$, form a basis of the vector space $\mathcal{B}(z^0, m)$ over $\mathbb{F}_p$. The operator*

$$\begin{aligned} \{t\} := \beta([t]) \quad &= \quad \frac{1}{|m|^2}\left(\sum_{s=1}^n z_s^0 m_s\right)\left(\sum_{s=1}^n z_s^0 \left(H_s(z^0) - \sum_{j \neq s} \frac{m_s m_j/2}{z_s^0 - z_j^0}\,\mathrm{Id}\right)\right) \\ &+ \quad \frac{-1}{|m|}\left(\sum_{s=1}^n (z_s^0)^2 \left(H_s(z^0) - \sum_{j \neq s} \frac{m_s m_j/2}{z_s^0 - z_j^0}\,\mathrm{Id}\right)\right) \end{aligned} \tag{4.19}$$

generates $\mathcal{B}(z^0, m)$ as an algebra with defining relation $P(\{t\}) = 0$.

Corollary 4.5. *We have*

$$\left(H_s(z^0) - \sum_{j \neq s} \frac{m_s m_j/2}{z_s^0 - z_j^0}\,\mathrm{Id}\right).\langle 1 \rangle = w_s, \qquad s = 1, \dots, n. \tag{4.20}$$

4.4. **Eigenvectors of** $\mathcal{B}(z^0,m)$ **and the polynomial** $P(t)$**.** The elements of the algebra $\mathcal{A}(z^0,m)$ have the form $Q([t])$, where $Q(t)\in\mathbb{F}_p[t]$, $\deg Q(t)<n-1$. An element $Q([t])$ is an eigenvector of all multiplication operators of $\mathcal{A}(z^0,m)$ if and only if $Q([t])$ is an eigenvector of the multiplication by $[t]$. If $t^0\in\mathbb{F}_p$ is the eigenvalue, then $([t]-t^0)Q([t])=0$, that is,

$$(t-t^0)Q(t)=\text{const }P(t),\qquad \text{const}\in\mathbb{F}_p. \tag{4.21}$$

Hence the set of eigenlines of all multiplication operators of $\mathcal{A}(z^0,m)$ is in one-to-one correspondence with the set of distinct roots of the polynomial $P(t)$, namely, a root t^0 with decomposition $(t-t^0)Q(t)=P(t)$ corresponds to the line generated by the element $Q([t])$.

Corollary 4.6. *The set of eigenlines of* $\mathcal{B}(z^0,m)$ *are in one-to-one correspondence with the set of distinct roots of the polynomial* $P(t)$*, namely, a root* t^0 *with decomposition* $(t-t^0)Q(t)=P(t)$ *for some* $Q(t)\in\mathbb{F}_p[t]$ *corresponds to the line generated by the vector*

$$\omega(t^0,z^0):=Q(\{t\}).\langle 1\rangle\quad\in\ \mathrm{Sing}L^{\otimes m}[|m|-2]. \tag{4.22}$$

□

Thus we have two ways to construct the eigenlines of $\mathcal{B}(z^0,m)$ from roots t^0 of the polynomial $P(t)$. The first is given by Theorem 2.1 and the eigenline is generated by the vector

$$W_{1,n,m}(t^0,z^0)=\sum_{s=1}^n\frac{1}{t^0-z_s}f^{(s)}=\sum_{s=1}^n\frac{1}{t^0-z_s}w_s. \tag{4.23}$$

The second is given by Corollary 4.6 and the eigenline is generated by the vector $Q(\{t\}).\langle 1\rangle$.

Theorem 4.7. *The two eigenlines coincide, more precisely, we have*

$$W_{1,n,m}(t^0,z^0)=\text{const}\,Q(\{t\}).\langle 1\rangle,\qquad \text{const}\in\mathbb{F}_p. \tag{4.24}$$

Proof. We need to show that $([t]-t^0)\alpha^{-1}(W_{1,n,m}(t^0,z^0))=0$ in $\mathcal{A}(z^0,m)$. Indeed

$$([t]-t^0)\alpha^{-1}(W_{1,n,m}(t^0,z^0))=\sum_{s=1}^n\frac{m_s}{t^0-z_s}\Big[\frac{t-t^0}{t-z_s}\Big]=\sum_{s=1}^n\frac{m_s}{t^0-z_s}[1]-\sum_{s=1}^n\Big[\frac{m_s}{t-z_s}\Big]=0$$

due to the Bethe ansatz equation (4.7) and formula (4.11).

□

4.5. **Algebra** $\mathcal{C}(T)$**.** In this section, p is a prime number, $p>n+1$. Fix a monic polynomial

$$T(x)=x^n+\sigma_1x^{n-1}+\sigma_2x^{n-2}+\dots+\sigma_n\ \in\ \mathbb{F}_p[x]. \tag{4.25}$$

We consider the two-dimensional subspaces $V\subset\mathbb{F}_p[x]$ consisting of polynomials of degree n and 1 such that $\mathrm{Wr}(g_1(x),g_2(x))=\text{const}\,T(x)$, where $g_1(x),g_2(x)$ is any basis of V and const $\in\mathbb{F}_p$. Such a subspace V has a unique basis of the form

$$g_1(x)=x^n+a_1x^{n-1}+\dots+a_{n-2}x^2+a_0,\qquad g_2=x-t \tag{4.26}$$

with

$$\mathrm{Wr}(g_1(x),g_2(x))=(n-1)T(x). \tag{4.27}$$

Equation (4.27) is equivalent to the system of equations

$$\begin{aligned}&(n-r-1)a_r-(n-r+1)a_{r-1}t-(n-1)\sigma_r=0,\qquad r=1,\dots,n-1,\\&a_n-(n-1)\sigma_n=0,\end{aligned} \tag{4.28}$$

where $a_0 = 1$. Expressing a_1 from the first equation in terms of t, then expressing a_2 from the first and second equations in terms of t and so on, we can reformulate system (4.28) as the system of equations

$$a_r - \frac{n-1}{2}(nt^r + (n-1)\sigma_1 t^{r-1} + \cdots + (n-r)\sigma_r) = 0, \qquad r = 1, \ldots, n-2, \tag{4.29}$$

$$nt^{n-1} + (n-2)\sigma_1 t^{n-2} + \cdots + 2\sigma_2 t + \sigma_1 = 0, \tag{4.30}$$

$$a_n + \sigma_n = 0. \tag{4.31}$$

Notice that equation (4.30) is the equation $\frac{dT}{dt}(t) = 0$, where $T(x)$ is defined in (4.25).

Let $I \subset \mathbb{F}_p[t, a_1, \ldots, a_{n-2}, a_n]$ be the ideal generated by n polynomials staying in the left-hand sides of the equations of the system (4.28). Define the algebra

$$\mathcal{C}(T) = \mathbb{F}_p[t, a_1, \ldots, a_{n-2}, a_n]/I. \tag{4.32}$$

Let $J \subset \mathbb{F}_p[t]$ be the ideal generated by $\frac{dT}{dt}(t)$. Define the algebra

$$\tilde{\mathcal{C}}(T) = \mathbb{F}_p[t]/J. \tag{4.33}$$

Lemma 4.8. *We have an isomorphism of algebras*

$$\tilde{\mathcal{C}}(T) \to \mathcal{C}(T), \quad [t] \mapsto [t]. \tag{4.34}$$

□

Let $m^0 = (1, \ldots, 1) \in \mathbb{Z}^n_{>0}$. Let $z^0 = (z^0_1, \ldots, z^0_n) \in \mathbb{F}_p^n$ be a point with distinct coordinates. The Bethe ansatz for $\mathrm{Sing} L^{\otimes m^0}[|m^0| - 2]$ has the form

$$\frac{1}{t - z^0_1} + \cdots + \frac{1}{t - z^0_n} = \frac{R(t)}{T(t)} = 0, \tag{4.35}$$

where

$$T(t) = \prod_{s=1}^n (t - z^0_s), \qquad R(t) = \frac{dT}{dt}(t). \tag{4.36}$$

Hence for this $T(x)$ we have

$$\tilde{\mathcal{C}}(T) = \mathcal{A}(z^0, m^0). \tag{4.37}$$

Corollary 4.9. *For $T(t)$ and $R(t)$ as in (4.36) we have*

$$\mathcal{A}(z^0, m^0) \cong \mathcal{B}(z^0, m^0) \cong \mathcal{C}(T) \tag{4.38}$$

and the $\mathcal{B}(z^0, m)$-module $\mathrm{Sing} L^{\otimes m^0}[|m^0| - 2]$ *is isomorphic to the regular representation of the algebra $\mathcal{C}(T)$.*

4.6. **Wronski map.** Let X_n be the affine space of all two-dimensional subspaces $V \subset \mathbb{F}_p[x]$, each of which consists of polynomials of degree n and 1. The space X_n is identified with the space of pairs of polynomials given by formula (4.26). Let $\mathbb{F}_p[x]_n \subset \mathbb{F}_p[x]$ be the affine subspace of monic polynomials of degree n. Introduce the *Wronski map*

$$W_n : X_n \to \mathbb{F}_p[x]_n, \quad \langle g_1(x), g_2(x) \rangle \mapsto \frac{1}{n-1} \mathrm{Wr}(g_1(x), g_2(x)), \tag{4.39}$$

cf. [MTV3]. The algebra $\mathcal{C}(T)$ is the algebra of functions on the fiber $W^{-1}(T)$ of the Wronski map.

Example 4.2. Let $n = 3$ and $T(x) = x^3+\sigma_1 x^2+\sigma_2 x+\sigma_3$. Then $W_3^{-1}(T)$ consists of one point if the discriminant $\sigma_1^2 - 3\sigma_2$ of $\frac{dT}{dx}(x)$ equals zero; $W_3^{-1}(T)$ consists of two points if the discriminant is a nonzero square, and is empty otherwise. Thus, p^2 points of X_3 have one preimage, $\frac{p-1}{2}p^2$ points have two preimages, and $\frac{p-1}{2}p^2$ points have none. Cf. Example 4.1.

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