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KULIKOV SINGULARITIES

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Dedicated to the memory of Egbert Brieskorn.

Abstract. In the study of normal surface singularities the relation between analytical and topological properties and invariants of the singularity is a very rich problem. This relation is particularly close for surface singularities constructed from families of curves. We use these Kulikov singularities to reexamine results of Némethi-Okuma and Tomaru.

Introduction

The first time I met Brieskorn was when I started my Ph.D. studies in Leiden and he was spending some months there. Horst Knörrer was then also working there. Through his students Brieskorn has influenced my career and work very much. And of course through his work, in the first place through his book with Knörrer on plane algebraic curves [2]. This is a most remarkable book, not only because of its value for money (Brieskorn negotiated a price below DM 50) and its white cover, but mainly because its style and contents. Ever since curve singularities and algebraic curves have been central in my work.

Trying to describe singularities one may ask the question:

Which discrete data are needed to know a singularity?

One interpretation of 'knowing a singularity' is that we can write down equations. As we only have discrete data, such equations necessarily describe an equisingular family of singularities.

For plane curve singularities there are very satisfactory answers to the question, which can be found in Brieskorn's book [2]. There is the link of the singularity, which gives the embedded topology (without the embedding one has only the number of components); another invariant is the resolution graph. Since Brieskorn's work on the exotic spheres as links of singularities it is realised that in high dimension the abstract link contains not enough information. In the surface case the situation is different. The topology of the link, encoded in the resolution graph, is a strong invariant. For rational and minimally elliptic singularities it determines the equisingularity class. For higher geometric genus this is no longer the case and the study of the relation between analytical and topological properties and invariants of singularities is a very rich problem.

To have a strong relation we have to look at special classes of singularities. In the work of Neumann and Wahl (for an overview see [18]) and of Némethi two kind of restrictions are imposed, an analytical one, that the singularity is $\mathbb{Q}$-Gorenstein, and a topological one, that the link is a rational homology sphere. Neumann and Wahl came even up with a way to write down equations from the resolution graph, provided certain special numerical conditions are satisfied. The so called splice type equations describe a complete intersection singularity in a particular simple form, however not for a singularity with the original graph, but for its universal abelian cover (which is a finite cover due to the rational homology sphere condition). In a recent paper Némethi and Okuma [12] study which analytic structures can occur for a specific resolution graph, giving details for an example already mentioned by Némethi [11]. One of the occurring structures is that of a Kodaira or Kulikov singularity.

Kodaira singularities were introduced by Karras [4], using a construction similar to the one earlier described by Kulikov [6]. In my thesis [16] I introduced the term *Kulikov singularities*. The construction starts from a (degenerating) 1-parameter family $\pi\colon W \to D$ of curves of genus g. Let $\sigma\colon \widetilde{W} \to W$ be the blow up of W in r points of the special fibre W_0, each point a smooth point on a component occurring with multiplicity 1. Then the strict transform of the special fibre can be blown down to a singular point $p \in \overline{W}$. By definition $(\overline{W}, p)$ is a Kulikov singularity. The study of properties of such singularities reduces in two ways to the study of curves. The morphism π descends to a function on the singularity, which defines a general hyperplane section. This curve singularity is more accessible and invariants like its multiplicity and embedding dimension determine the corresponding invariants of the surface singularity. The other occurrence of curves is by construction: the properties of the central fibre, considered as curve of arithmetic genus g, are essential.

Kulikov introduced his construction to give a uniform construction of the unimodal and bimodal singularities. These are the simplest types of minimally elliptic singularities. For higher genus Kulikov singularities should also be considered as simplest types. The generalisation of Laufer's minimally elliptic cycle [7] is the characteristic cycle, introduced by Karras for Kodaira singularities [5] and in [16] in general. Tomaru studied for which Brieskorn singularities the characteristic cycle is equal to the fundamental cycle [17] .

Karras' work on Kodaira singularities of higher genus [5] and my work on Kulikov singularities [16] was never published. When referred to, these singularities are mainly seen as singularities where there is a function defining the fundamental cycle Z, which is moreover reduced at components E_i with $E_i \cdot Z < 0$. In this paper I actually take this as definition (see Definition 2.1), being the shortest, but it is the construction using a family of curves which gives a good understanding of the singularity. As illustration I treat the results of Némethi and Okuma [12] and of Tomaru [17] from this point of view.

1. Invariants of surface singularities

The topological type of a normal complex surface singularity is determined by and determines the resolution graph of the minimal good resolution. But a resolution graph can be defined for any resolution, not necessarily good.

Definition 1.1. Let $\pi\colon (M, E) \to (V, p)$ be a resolution of a surface singularity with exceptional divisor $E = \bigcup_{i=1}^r E_i$. The *resolution graph* Γ is a weighted graph with vertices corresponding to the irreducible components E_i. Each vertex has two weights, the self-intersection $-b_i = E_i^2$, and the arithmetic genus $p_a(E_i)$, the second traditionally written in square brackets and omitted if zero. There is an edge between distinct vertices if the corresponding components E_i and E_j intersect, weighted with the intersection number $E_i \cdot E_j$ (only written out if larger than one).

Other definitions, which record more information, are possible: one variant is to have an edge for each intersection point $P \in E_i \cap E_j$, with weight the local intersection number $(E_i \cdot E_j)_P$. This is the more common definition in the case that the intersections are transverse.

The classes of the curves E_i form a preferred basis of $H := H_2(M, \mathbb{Z})$. Following algebro-geometric tradition the elements of H are called *cycles*. They are written as linear combinations of the E_i. The intersection form on M gives a negative definite quadratic form on H. Let $K \in H^2(M, \mathbb{Z})$ be the canonical class. It can be written as rational cycle in $H_{\mathbb{Q}} = H \otimes \mathbb{Q}$ by solving the adjunction equations $E_i \cdot (E_i + K) = 2p_a(E_i) - 2$. The function $-\chi(A) = \frac{1}{2} A \cdot (A + K)$, $A \in H$, makes H into a quadratic *quadratic lattice*, in the sense of [8, 1.4]. We prefer to work with the genus $p_a(A) = 1-\chi(A)$. Note that the genus function determines the intersection form, as

$$p_a(A+B) = p_a(A) + p_a(B) + A \cdot B - 1 \; .$$

The data (H, p_a) is equivalent to $(H, \{E_i \cdot E_j\}, \{p_a(E_i)\})$, encoded in the resolution graph Γ.

There are some important cycles on E, some of which only depend on the quadratic lattice, while others depend on the analytic structure.

Definition 1.2. The *fundamental cycle* Z is the is the smallest positive cycle such that $E_i \cdot Z \le 0$ for all i. The *maximal ideal cycle* $Z_{\mathfrak{m}}$ is the smallest cycle occurring as compact part of the divisor of a function $f \in \mathfrak{m}_{(V,p)}$. The *canonical cycle* Z_K is the rational cycle on E, which is numerically equivalent to the anticanonical class of the resolution M.

We recall that the *geometric genus* $p_g(V,p)$ is the dimension of $R^1\pi_*\mathcal{O}_M$. This is equal to the maximal value of $h^1(\mathcal{O}_D)$ over all positive cycles. In fact, there is a unique minimal *cohomological cycle* with this maximal value (see [15, 4.8]). A topological lower bound for p_g is the arithmetic genus $p_a(V,p)$, which is the maximal value of $p_a(D)$ over all positive cycles. The genus $p_a(Z)$ of the fundamental cycle is also a topological invariant of the singularity, which is called the *fundamental genus* $p_f(V,p)$ [17].

Obviously $p_f \le p_a \le p_g$, and all inequalities can be strict; the easiest example with $p_a > p_f$ is the case of an irreducible exceptional curve E of genus $g > 1$ and self-intersection -1.

Definition 1.3. The *characteristic cycle* C of a nonrational singularity is the smallest cycle which realises the fundamental genus: it is the cycle $C \le Z$ with $p_a(C) = p_a(Z)$ and $p_a(D) < p_a(C)$ for all cycles $0 < D < C$.

This cycle is a generalisation of Laufer's minimally elliptc cycle and its existence is proved in the same way. It was first introduced by Karras for Kodaira singularities [5]. The general definition is in [16]; Tomaru also introduced it under the name *minimal cycle* [17].

2. Kulikov singularities

In this section we introduce the Kulikov construction, give some properties and discuss when the resulting singularity is Gorenstein.

Definition 2.1. Let (V,p) be a normal surface singularity with fundamental cycle Z on the minimal resolution. It is called a *Kulikov singularity* if there exists a function $f\colon (V,p) \to (\mathbb{C},0)$ with $(X,p) = (f^{-1}(0),p)$ a reduced curve singularity with divisor on the minimal resolution of the form $Z + \widetilde{X}$, such that the strict (or proper) transform $\widetilde{X}$ of X intersects the exceptional set E transversally in smooth points on components having multiplicity one in the fundamental cycle Z.

Such singularities are the result of a construction first given by Kulikov [Kulikov], to describe the unimodal and bimodal singularities. He starts from a (degenerating) family $\pi\colon W \to D$ of curves of genus g. This is a proper morphism of a non-singular surface to a small disc. The special fibre $W_0 = \pi^{-1}(0)$ over $0 \in D$ can be written as $W_0 = n_1C_1 + \dots n_kC_k$, where the C_i are the irreducible components of this fibre. The intersection matrix $(C_i \cdot C_j)$ is negative semi-definite. Let $\sigma\colon \widetilde{W} \to W$ be the blow up of W in r points $q_1, \dots, q_r$, each a smooth point of a component C_i which has multiplicity $n_i = 1$ in W_0. We denote the strict transform of a component C_i by E_i. Then the special fibre $\widetilde{W_0}$ of $\tilde\pi = \pi \circ \sigma$ can be written as

$$\widetilde{W_0} = n_1E_1 + \dots n_kE_k + \widetilde{X}_1 + \dots + \widetilde{X}_r \; ,$$

where the $\widetilde{X}_j$ are (-1)-curves. Now the intersection matrix $(E_i \cdot E_j)$ is negative definite and $E = \bigcup E_i$ can be blown down to a singular point $p \in \overline{W}$.

Lemma 2.2. *Kulikov's construction results in a Kulikov singularity. Conversely, every Kulikov singularity can be obtained by this construction.*

Proof. The construction yields the minimal resolution if there are no (-1)-curves in the family $\pi\colon W \to D$ except possibly curves containing a point q_j. If there are other (-1)-curves we blow them down without changing the resulting singularity. So we may assume that $\widetilde{W} \to \overline{W}$ is the minimal resolution of the singularity $p \in \overline{W}$. We write $\widetilde{W}_0 = Y + \widetilde{X}$ and have to show that Y is the fundamental cycle of the singularity $(\overline{W}, p)$. We put $Y = Z + D$ with D an effective cycle supported on E. Then D does not intersect $\widetilde{X}$, as each $\widetilde{X}_i$ intersects Y in a component with multiplicity one. Now $0 = D \cdot \widetilde{W}_0 = D \cdot (Z + D + \widetilde{X}) = D \cdot Z + D \cdot D \leq 0$, so $D \cdot D = 0$ and therefore $D = 0$.

Conversely, given a function $f\colon (V,p) \to (\mathbb{C},0)$ with divisor $Z + \widetilde{X}$ we compactify to a family of curves, following Karras [1980,Thm 2.9]: in each point $q \in E \cap \widetilde{X}$ there are local coordinates such that f is given by $xy = 0$, and $\widetilde{X}$ by $y = 0$. We glue the blow-up of the origin to it: with coordinates (u,y) we have two charts, given by $(u,y) = (u, u\eta) = (xy, y)$. The glueing is by identifying the (x,y) coordinates. Then $u = xy$ extends the function f. □

Kulikov singularities are a special case of Kodaira singularities, defined by Karras [4, 5]. In his construction it is allowed that points to be blown up coincide: one blows up consecutively, and it is allowed to blow up the strict transform of the fibre in a point of intersection with a previously blown up curve. Then the curve $(X,p) = (f^{-1}(0),p)$ is not necessarily a reduced curve.

The advantage of the more strict definition of Kulikov singularities is that the curve (X,p) is a general hyperplane section. The function $f\colon (V,p) \to (\mathbb{C},0)$ defines a smoothing of this curve with Milnor number $\mu = 2g + r - 1$. The structure of the hyperplane section is often much easier to describe than that of the singularity itself. It allows conclusion about the multiplicity and the embedding dimension of the singularity.

An alternative description of the construction starts from a *minimal family* $\pi\colon W \to D$, meaning that W does not contain (-1)-curves. One then blows up points consecutively, each time blowing up a point with multiplicity one in the special fibre. In each stage a (-1)-curve intersects only one other curve, so in the final surface the (-1)-curves are ends, and their complement is connected. Write as before $\widetilde{W}_0 = Y + \widetilde{X}$ with $\widetilde{X}$ the union of the (-1)-curves. Then the support of Y can be blown down.

We have the following properties.

Proposition 2.3.

(1) *For a Kulikov singularity the maximal ideal cycle $Z_{\mathfrak{m}}$ is equal to the fundamental cycle Z.*
(2) *The fundamental genus is equal to the genus of the curves in the family used in the construction: $p_f(V,p) = g$.*
(3) *A rational singularity is Kulikov if and only if the fundamental cycle is reduced.*
(4) *The characteristic cycle of a nonrational Kulikov singularity is the strict transform of the special fibre of the minimal family resulting in the singularity.*

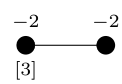
Proof. Only the last property needs a proof. It suffices to consider the case that the strict transform is the whole fundamental cycle. Suppose that $C < Z$ and choose a computation sequence $Z_j = Z_{j-1} + E_{i_j}$ from $Z_0 = C$ to $Z_k = Z$. As $p_a(Z_j) = p_a(Z)$ for all j, each E_{i_j} is a smooth rational curve with $E_{i_j} \cdot Z_{j-1} = 1$. This holds in particular for the last one and therefore $E_{i_k} \cdot Z = 1 + E_{i_k}^2 < 0$. This implies that E_{i_k} has multiplicity one in the fundamental cycle and

$E_{i_k} \cdot = -E_{i_k} \cdot \widetilde{X}$. After blowing down $\widetilde{X}$ the strict transform of E_{i_k} has self-intersection (-1), contradicting that the family we started from was a minimal family. □

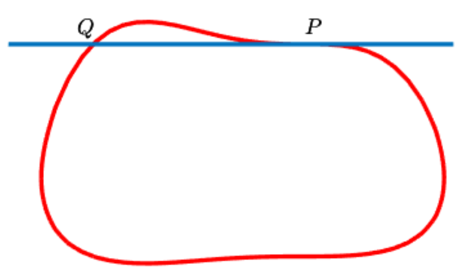
To obtain a Gorenstein Kulikov singularity we have to perform the construction in special points. Let $\pi\colon W \to D$ be a minimal family of curves of genus g. The relative dualising sheaf $\omega_{W/D}$ is isomorphic to Ω_W. Let (ω) be the divisor of a global section. It consists of an horizontal, non-compact part N and a divisor supported on the special fibre, determined up to a multiple of this fibre. Suppose that each component of N intersects the special fibre only transversally in components of multiplicity one. Now we perform the Kulikov construction starting from the minimal family, blowing up at least these intersection points, in such a way that in the final family $\tilde\pi\colon \widetilde{W} \to D$ the pull back of ω has the same multiplicity m along all (-1)-curves X_i, and that the horizontal part of its divisor intersects the special fibre only in $\widetilde{X}$. Let $f = \tilde\pi^*(t)$, with t a coordinate function on D. Then the meromorphic two-form $f^{-m}\omega$ is holomorphic and nowhere zero on $U \setminus E$, U a neighbourhood of E. Therefore the Kulikov singularity is Gorenstein.

Example 2.4. We give an example of a 1-parameter family of weighted homogeneous Gorenstein singularities V_a such that V_0 is not Kulikov but V_a is Kulikov for $a \neq 0$. It is the simplest of the series of examples of Briançon and Speder of a family which is μ-constant, but not μ^*-constant [1].

Consider
$$f_a(x, z, t) = z^3 + azx^3 + tx^4 + t^9 \ .$$
The resolution graph is



The exceptional divisor on the minimal resolution is $E = E_1 + E_2$ with E_1 a curve of genus 3 with self-intersection -2, and E_2 a rational (-2)-curve. The canonical model of E_1 is the plane quartic $\eta\zeta^3 + a\zeta\xi^3 + \xi^4 + \eta^4$; this curve has a flex in $P = (0 : 0 : 1)$, and the tangent $\eta = 0$ intersects the curve in $Q = (-a : 0 : 1)$, so for $a = 0$ there is a hyperflex. The normal bundle of E_1 has $P + Q$ as divisor, and E_2 intersects E_1 in Q. The general hyperplane section has two branches for $a \neq 0$; the strict transform of one branch passes through P, and the other intersects E_2 in a smooth point of E. For $a = 0$ the curve is irreducible, its strict transform passes through $P = Q = E_1 \cap E_2$.



To construct this singularity we start from the trivial family $W = E_1 \times D$. A canonical divisor is $3P \times D + Q \times D$. After blowing up in $P \times \{0\}$ the multiplicity along the newly introduced exceptional divisor is 4. Blowing up in $Q \times \{0\}$ gives multiplicity 2. We blow up again in intersection point of special fibre and strict transform of section $Q \times D$, resulting in multiplicity 4. By dividing with t^4 we see that the singularity is Gorenstein with $K = -4E_1 - 2E_2$. The functions t, $x = t^2\xi/\eta$ and $z = t^3\zeta/\eta$ are holomorphic on neighbourhood of E, giving $(t^3\zeta/\eta)^3 + a(t^3\zeta/\eta)(t^2\xi/\eta)^3 + t(t^2\xi/\eta)^4 + t^9 = 0$; this formula works also for $a = 0$. The blowing up can be done in family over a base $D \times A$, with a a coordinate on A. We first blow up in $P \times 0 \times A$, then in Q as lying on the strict transform of $C \times 0 \times A$ and then once again in the intersection point with the strict transform of the appropriate section. For $a = 0$ this means that we blow up in a double point of the special fibre, which is not allowed in the Kulikov construction.

3. The characteristic cycle of Brieskorn-Pham singularities

The simplest type of quasi-homogeneous hypersurface singularities has an equation, which is a sum of perfect powers, and is usually called a Brieskorn-Pham polynomial. We write in the surface case

$$x^a + y^b + t^c \tag{3.1}$$

with $a \le b \le c$. It is well known how to get the resolution of this surface singularity from the exponents a, b and c [14]. The precise form is not important for us now.

Lemma 3.1. *If $c \ge \operatorname{lcm}(a,b)$, the Brieskorn-Pham singularity (3.1) is a Kulikov singularity of genus $g = (\mu - r + 1)/2$, where $\mu = (a-1)(b-1)$ is the Milnor number of the curve singularity $x^a + y^b$ and $r = \gcd(a,b)$ is the number of branches.*

Proof. We construct the singularity with the Kulikov construction. We start with an affine family of curves, whose equation is in fact given by a Brieskorn-Pham polynomial, but with lower exponent c. Put $d = \operatorname{lcm}(a,b)$. Let $r = \gcd(a,b)$, then $d = \frac{ab}{r}$. Consider the family $\xi^a + \eta^b + t^{c-d} = 0$ as family of affine plane curves, parametrised by t, and complete it in the weighted projective space with weights $(\frac{d}{a}, \frac{d}{b}, 1)$. The homogeneous equation is then $\xi^a + \eta^b + t^{c-d} w^d = 0$. We resolve the singularity at the origin. We look at the chart $\xi = 1$. There the equation is $1 + \bar\eta^b + t^{c-d}\bar w^d = 0$, modulo the action $\frac{a}{d}(\frac{d}{b}, 1)$. For $t = 0$ we have $1 + \bar\eta^b = 0$, so there are indeed $\frac{b}{a/d} = r$ points on the compactification of the special fibre. The coordinate transformation from $(\xi, \eta, 1)$ coordinates to $(1, \bar\eta, \bar w)$ is $\xi = \bar w^{-\frac{d}{a}}$, $\eta = \bar\eta \bar w^{-\frac{d}{b}}$. We blow up in the r points at infinity on the special fibre. The functions $x := \xi t^{\frac{d}{a}}$, $y := \eta t^{\frac{d}{a}}$ and t are holomorphic in a neighbourhood of the strict transform of the special fibre, and generate the local ring of the Kulikov singularity. They satisfy $x^a + y^b + t^c = 0$. $\square$

It follows that the family of curves obtained by resolving the singularity of $\xi^a + \eta^b + t^{c-d} w^d = 0$ is not minimal if $c - d \ge d = \operatorname{lcm}(a,b)$. Furthermore the resolution graph of $x^a + y^b + t^{c-d}$ is a subgraph of the resolution graph of $x^a + y^b + t^c$.

Proposition 3.2. *Write $c = c_0 + c_1 d$ with $0 \le c_0 < d$. The characteristic cycle of the Brieskorn-Pham singularity (3.1) has support on the subgraph corresponding to the singularity $x^a + y^b + t^{c_0+d}$ and is the fundamental cycle of that singularity. In particular, the characteristic cycle is equal to the fundamental cycle if and only if $d \le c < 2d$.*

Proof. If the family used in the construction above is not minimal, one can blow down each component of the strict transform of the affine curve $\xi^a + \eta^b = 0$ and still have a family of the same type. So the family is minimal if and only $c - d < d$. The result now follows from Proposition 2.3.(4). $\square$

The Proposition was proved by Tomaru [17] using an explicit description of the resolution of the singularity. As to this resolution, we note that there are r chains of $c_1 - 1$ (-2)-curves from the characteristic cycle to the components of $\widetilde{X}$.

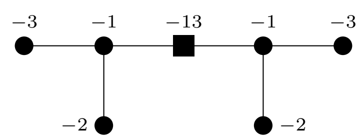
Remark 3.3. The above result extends with the same proof to the case of Brieskorn complete intersections. A proof in the style of [17] was given by Meng, Yuan and Wang [9].

4. Singularities with a specific resolution graph

A recent paper Némethi and Okuma [12] concerns the problem of determining upper and lower bounds for the geometric genus in terms of the resolution graph. The Authors study which analytic structures can occur for a specific resolution graph, giving details for an example already mentioned by Némethi [11]. Here we rederive their results from our point of view.

The main feature of the example is that the topological upper bound for p_g is not realised. The maximal p_g occurs for a non Gorenstein Kulikov singularity and for a Gorenstein splice type singularity.

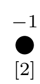
The singularity considered has an integral homology sphere link. The resolution graph for the minimal good resolution is:



This graph satisfies the semigroup condition of Neumann and Wahl [13] so there exist singularities of splice type with this graph, with $p_g = 3$. The defining equations of this complete intersection singularity have 'leading' forms

$$z_1^2 z_2 + z_3^2 + z_4^3, \qquad z_1^3 + z_2^2 + z_4^2 z_3\ . \tag{4.1}$$

On the minimal resolution the exceptional curve is an irreducible two-cuspidal rational curve, of self-intersection -1. Therefore the resolution graph for the minimal resolution is simply:

(4.2) 

with a possibly singular central curve. This is the same graph as when the exceptional divisor is a smooth curve of genus two. We note that there exists a Gorenstein Kulikov singularity with this graph, namely the hypersurface $z^2 = y^5 + x^{10}$; it has the maximal geometric genus: $p_g = 4$.

We first analyse the Gorenstein condition. On the minimal resolution M adjunction gives for the exceptional curve that $\omega_E = \omega_M \otimes \mathcal{O}_E(E)$. The singularity is Gorenstein if and only if $\omega_M = \mathcal{O}_M(-3E)$. This happens if and only if $\omega_E = \mathcal{O}_E(-2E)$, that is, if the conormal bundle of E is a theta characteristic.

Lemma 4.1. *A singularity with resolution graph (4.2) satisfies $2 \le p_g \le 4$. If $p_g = 4$ then it is a Gorenstein Kulikov singularity. If $p_g = 3$ it is either non Gorenstein Kulikov of multiplicity 3 or a non Kulikov complete intersection.*

Proof. To analyse the possible values for p_g we look at a computation sequence. Here one compares the different $\mathcal{O}(-kE)$ via the short exact sequences

$$0 \longrightarrow \mathcal{O}(-(k+1)E) \longrightarrow \mathcal{O}(-kE) \longrightarrow \mathcal{O}_E(-kE) \longrightarrow 0$$

As $H^1(\widetilde{X}, \mathcal{O}(-3E)) = 0$ one gets the exact sequences

$$0 \longrightarrow H^1(\widetilde{X}, \mathcal{O}(-E)) \longrightarrow H^1(\widetilde{X}, \mathcal{O}) \longrightarrow H^1(E, \mathcal{O}_E) \longrightarrow 0$$

$$H^0(E,\mathcal{O}_E(-E)) \longrightarrow H^1(\widetilde{X},\mathcal{O}(-2E)) \longrightarrow H^1(\widetilde{X},\mathcal{O}(-E)) \longrightarrow H^1(E,\mathcal{O}_E(-E)) \longrightarrow 0$$

and the isomorphism $H^1(\widetilde{X},\mathcal{O}(-2E)) \cong H^1(E,\mathcal{O}_E(-2E))$.

This gives $2 \le p_g \le 4$. If $p_g = 4$ then $\mathcal{O}_E(-2E) = \omega_E$, so the singularity is Gorenstein. Moreover, the theta characteristic is odd. Indeed, on a smooth genus two curve the divisor of a Weierstrass point is an odd theta characteristic. The Kulikov construction starting from a trivial family and blowing just one Weierstrass point lying on the central fibre, yields the example $z^2 = y^5 + x^{10}$.

A two-cuspidal rational curve has only one theta characteristic, which is even [3]. This can also be seen from the description of the pencil with this special fibre in the list of Namikawa and Ueno [10]: their example is $y^2 = (x^3+t)((x-1)^3+t)$, and one sees that three Weierstrass points come together in cusp. This shows that there cannot be a singularity with this exceptional divisor with $p_g = 4$. But any computation with the quadratic lattice H cannot distinguish between such a curve and a smooth curve.

A non Gorenstein Kulikov singularity is obtained by blowing up one smooth point of the special fibre; for a smooth curve this point should not be a Weierstrass point. By construction the general hyperplane section is a curve with Milnor fibre of genus two, so $\delta = 2$. The only irreducible non Gorenstein curve singularity is the monomial curve (t^3, t^4, t^5). Therefore the surface singularity has multiplicity 3 and embedding dimension 4. In this case $H^0(E,\mathcal{O}_E(-E)) = \mathbb{C}$, so $p_g = 3$.

If the singularity is Gorenstein, but not Kulikov, then $p_g = 3$ and the curve E has an even theta characteristic. For a smooth E there exists a quasi-homogeneous singularity. Let $y^2 = f_6(x,\bar{x})$ be a hyperelliptic curve E, and write $f_6 = PQ$ with P, Q of degree 3. Consider the divisor $(P) = 2D$, with D a divisor of degree 3 on E, consisting of three Weierstrass points. Then $\mathcal{O}_E(D - K_E)$ is an even theta characteristic. The graded ring $\bigoplus H^0(E,\mathcal{O}_E(k(D-K_E)))$ is generated by $z = xP$, $\bar{z} = \bar{x}P$, $w = yP$ and $v = P^2$. The equations are then

$$w^2 = Q(z,\bar{z}), \qquad v^2 = P(z,\bar{z})\ .$$

The singularity with two-cuspidal curve as exceptional curve is a superisolated complete intersection singularity. The graded tangent cone is obtained in the same way as above, by taking $P = x^3$, $Q = \bar{x}^3$. We have to add terms of lowest degree to make the singularity isolated, resulting in splice diagram equations of the form (4.1):

$$w^2 = \bar{z}^3 + vz^2, \qquad v^2 = z^3 + w\bar{z}^2\ .$$

□

Finally a quasi-homogeneous singularity with $p_g = 2$ is obtained from a divisor $D - K_E$ with$D$ a general effective divisor of degree 3 on a smooth curve E. The graded ring $\bigoplus H^0(E,\mathcal{O}_E(k(D-K_E)))$ has 7 generators. The same ring for the two-cuspidal rational curve gives a weighted tangent cone of a singularity in $\mathbb{C}^7$.

References

[1] Joël Briançon et Jean-Paul Speder, *La trivialité topologique n'implique pas les conditions de Whitney*. C. R. Acad. Sci. Paris Sér. A-B **280** (1975), A365–A367.

[2] Egbert Brieskorn and Horst Knörrer, Ebene algebraische Kurven. Basel, Boston, Stuttgart: Birkhäuser (1981).

[3] Joe Harris, *Theta-characteristics on algebraic curves*. Trans. Amer. Math. Soc. **271** (1982), 611–638. DOI: 10.1090/S0002-9947-1982-0654853-6

[4] Ulrich Karras, *On pencils of curves and deformations of minimally elliptic singularities*. Math. Ann. **247** (1980), 43–65. DOI: 10.1007/BF01359866

[5] U. Karras Methoden zur Berechnung von Algebraischen Invarianten und zur Konstruktion von Deformationen normaler Flächensingularitäten, Habilitationsschrift, Dortmund, 1981.

[6] V. S. Kulikov, *Degenerate elliptic curves and resolution of uni- and bimodal singularities*. Funktsional. Anal. i Prilozhen., **9** (1975), 72–73.

[7] Henry B. Laufer, *On minimally elliptic singularities*. Amer. J. Math. **99** (1977), 1257–1295. DOI: 10.2307/2374025

[8] Eduard Looijenga and Jonathan Wahl, *Quadratic functions and smoothing surface singularities*. Topology **25** (1986), 261–291.

[9] Fanning Meng, Wenjun Yuan and Zhigang Wang *The Minimal Cycles over Brieskorn Complete Intersection Surface Singularities* Taiwanese J. Math. **20** (2016), 277–286. DOI: 10.11650/tjm.20.2016.6434

[10] Yukihiko Namikawa and Kenji Ueno, *The complete classification of fibres in pencils of curves of genus two*. Manuscripta Math. **9** (1973), 143–186. DOI: 10.1007/BF01297652

[11] András Némethi, *Lattice cohomology of normal surface singularities*. Publ. Res. Inst. Math. Sci. **44** (2008), 507–543. DOI: 10.2977/prims/1210167336

[12] András Némethi and Tohomiro Okuma, *Analytic singularities supported by a specific integral homology sphere link*. Methods Appl. Anal. **24** (2017), 303-320. DOI: 10.4310/MAA.2017.v24.n2.a7

[13] Walter D. Neumann and Jonathan Wahl, *Complex surface singularities with integral homology sphere links*. Geom. Topol. **9** (2005), 757–811. DOI: 10.2140/gt.2005.9.757

[14] Orlik, Peter; Wagreich, Philip *Isolated singularities of algebraic surfaces with $\boldsymbol{C}^*$-action*. Ann. of Math. **93** (1971), 205–228. DOI: 10.2307/1970772

[15] Miles Reid, *Chapters on algebraic surfaces*. In: Complex algebraic geometry (Park City, UT, 1993), IAS/Park City Math. Ser., **3**, Amer. Math. Soc., Providence, RI, 1997, pp. 3–159.

[16] Jan Stevens, Kulikov singularities, a study of a class of complex surface singularities with their hyperplane sections. Thesis, Leiden 1985

[17] Tadashi Tomaru, *On Gorenstein surface singularities with fundamental genus $p_f \geq 2$ which satisfy some minimality conditions*. Pacific J. Math. **170** (1995), 271–295. DOI: 10.2140/pjm.1995.170.271

[18] Jonathan Wahl, *Topology, geometry, and equations of normal surface singularities*. In: Singularities and Computer Algebra, London Math. Soc. Lect. Note Ser., **324** (Cambridge University Press, 2006), pp. 351–371. DOI: 10.1017/CBO9780511526374.018

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