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NONCOMMUTATIVE DEFORMATIONS OF THICK POINTS

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Abstract. Any commutative algebra is of course also an associative algebra, and we may deform it as a non-commutative associative algebra. In particular this is of interest in singularity theory. It turns out that the versal base space of the non-commutative deformation functor of a thick point, in an affine three-dimensional variety, has properties that are rather astonishing. This base space is the main ingredient of a Toy Model for Quantum Theory, published in several books and papers, see [10], [11], [12], [2].

In this paper I shall describe the problems related to the computation of the local moduli suite, see [14], of the singularity consisting of an isolated point with a 3-dimensional tangent space.

1. Introduction

The last time I met with Egbert Brieskorn was, I think, in 2007 at Oberwohlfach. We talked a lot about his work on the legacy of Felix Hausdorff. We were both children during the 2nd World War, and we had both learned topology by reading the Grundzüge der Mengenlehre. I knew very little about Hausdorff's alter ego, Paul Mongré, and I was fascinated about yet another facet of this truly remarkable man, now well-documented by Brieskorn.

The reason why we sat down to talk, at the very end of a conference, was that Brieskorn had some questions. I had talked about my wild idea of modeling quantum theory and cosmology, using non-commutative deformation theory. We separated as good friends, even though I sensed the wise man's doubts about the endeavour.

It is therefore fitting that I, as a tribute to the always curious mathematician Egbert Brieskorn, one of the central workers in singularity theory during my lifetime, shall explain what I think he wanted to understand. It is a purely mathematical element of a Toy Model in physics; the computation of the versal base space of the non-commutative deformation functor of a thick point of imbedding dimension three.

It turns out that this base space is partitioned into a web of subspaces, the moduli suite of the singularity, see [14]. There is a maximal entropy subspace, equal to the Hilbert scheme $Hilb^{(2)}_{\mathbf{A}^3} = \underline{\tilde{H}}/Z_2$, of two points in affine 3-space, and there is a minimal non-trivial room, containing the quaternions $\mathfrak{Q}$

This $\underline{\tilde{H}}$, the blow up of $\mathbf{A}^3 \times \mathbf{A}^3$ in the diagonal $\underline{\Delta}$, turns out to be the base space of a canonical family of associative k -algebras in dimension 4. The study of the corresponding family of derivations leads to a natural way of introducing, on the tangent bundle of $\underline{\tilde{H}}$, an action of a Lie algebra containing the gauge group $\mathfrak{g}$, of the Standard Model. In particular $\mathfrak{g}/Rad = sl(2)$.

The fact that these results fit well with the set-up of the Standard Model, fusing our versions of General Relativity, Yang-Mills and Quantum Field Theory, is part of another story, see [12], [13], and [2].

2. Deformations and Noncommutativity

Non-commutativity comes up in algebraic geometry in many ways. In relation to deformation theory, there are two levels, implying four different mathematical tools.

First, we may ask wether deformations of an algebraic object must necessarily be parametrized by "commuting" parameters, i.e. wether the classifying objects should be restricted to commutative algebras, or should we accept non-commutative solutions?

Second, since any commutative algebra is also an associative algebra, should we restrict the deformations of a commutative algebra only to the commutative ones, or may we accept a non-commutative algebra as a deformation of a commutative one?

These questions comes up, in particular, in singularity theory. And here, in our story, it is related philosophically to some very central questions in physics.

Is our understanding, and the mathematical models we have made of the Universe, suggesting that not only Quantum Theory, but also our theory of gravity and space, the GRT, must be modelled by some sort of non-commutative algebraic geometry? The literature on these questions is huge, see [15] and later work of Majid, for a reasonably mathematical treatment, and references.

We shall first sketch the story of introducing noncommutative parameters. It turns out that this is the first important step in developing a noncommutative algebraic geometry. The very important fact, in the complex commutative case, that any finite-dimensional algebra is the sum of the local rings of its points, has a nice generalization that we have called the Generalised Burnside Theorem. This is a result that we shall use later, and that has been important in the study of the minimal model program in classical algebraic geometry, see [1]. As a tribute to the physics interested readers, we also add a short section on an algebraic geometric version of entropy, to prepare for the main subject of this paper.

Since the goal of this paper is limited to a strictly mathematical result in deformation theory of a thick point singularity, the main focus will be on the second question above. What can we learn by deforming a commutative singularity to associative algebras?

2.1. Non-commutative Deformations of Modules.

In [7], [8] and [9], we introduced non-commutative deformations of families of modules of associative k-algebras, k a field. We shall here recall the definitions, and the main results that will be used in the sequel.

Let $\underline{a}_r$ denote the category of r-pointed not necessarily commutative k-algebras R. The objects are the diagrams of k-algebras

$$k^r \stackrel{\iota}{\to} R \stackrel{\pi}{\to} k^r$$

such that the composition of ι and π is the identity. Any such *r-pointed k-algebra* R is isomorphic to a k-algebra of $r \times r$-matrices $(R_{i,j})$. The radical of R is the bilateral ideal $Rad(R) := ker\pi$, such that $R/Rad(R) \simeq k^r$. The dual k-vector space of $Rad(R)/Rad(R)^2$ is called the tangent space of R.

The usual, category of commutative local Artinian k-algebras with residue field k, commonly denoted by $\underline{l}$, is of course the commutative part of $\underline{a}_1$. Fix a (not necessarily commutative) associative k-algebra A and consider a right A-module M. The ordinary deformation functor,

$$Def_M : \underline{l} \to \underline{Sets}$$

is then defined.

Assuming $Ext^i_A(M,M)$ has finite k-dimension for $i = 1, 2$, it is well known, see [17], or [7], that Def_M has a pro-representing hull H, *the formal moduli of* M. Moreover, the tangent space of H is isomorphic to $Ext^1_A(M,M)$, and H can be computed in terms of $Ext^i_A(M,M),\ i = 1, 2$ and their *matric* Massey products, see [7].

In the general case, consider a finite family $\mathbf{V} = \{V_i\}_{i=1}^r$ of right A-modules, and put $V := \oplus_{i=1}^r V_i$. Assume that $\dim_k Ext_A^1(V_i, V_j) < \infty$, and call any such family of A-modules a *swarm*. We shall define a deformation functor,

$$Def_{\mathbf{V}} : \underline{a}_r \to \underline{Sets},$$

generalising the functor Def_M above. Given an object $\pi : R = (R_{i,j}) \to k^r$ of $\underline{a}_r$, consider the k-vector space and left R-module $(R_{i,j} \otimes_k V_j)$. It is easy to see that

$$End_R((R_{i,j} \otimes_k V_j)) \simeq (R_{i,j} \otimes_k Hom_k(V_i, V_j)).$$

Clearly π defines a k-linear and left R-linear map

$$\pi(R) : (R_{i,j} \otimes_k V_j) \to \oplus_{i=1}^r V_i$$

inducing a homomorphism of R-endomorphism rings,

$$\tilde{\pi}(R) : (R_{i,j} \otimes_k Hom_k(V_i, V_j)) \to \oplus_{i=1}^r End_k(V_i).$$

The right A-module structure on the V_i's is defined by a homomorphism of k-algebras,

$$\eta_0 : A \to \oplus_{i=1}^r End_k(V_i) \subset (Hom_k(V_i, V_j)) =: End_k(V).$$

Notice that this homomorphism also provides each $Hom_k(V_i, V_j)$ with an A- bimodule structure. Let $Def_{\mathbf{V}}(R) \in \underline{Sets}$ be the set of isoclasses of homomorphisms of k-algebras,

$$\eta' : A \to (R_{i,j} \otimes_k Hom_k(V_i, V_j))$$

such that, $\tilde{\pi}(R) \circ \eta' = \eta_0$, where the equivalence relation is defined by inner automorphisms in the R-algebra $(R_{i,j} \otimes_k Hom_k(V_i, V_j))$ inducing the identity on $\oplus_{i=1}^r End_k(V_i)$. One easily proves that $Def_{\mathbf{V}}$ has the same properties as the ordinary deformation functor and we may prove the following, see [7]:

Theorem 2.1. *The functor $Def_{\mathbf{V}}$ has a pro-representable hull, i.e. an object $H := H(\mathbf{V})$ of the category of pro-objects $\hat{\underline{a}}_r$ of $\underline{a}_r$, together with a versal family*

$$\tilde{V} = (H_{i,j} \otimes V_j) \in \varprojlim_{n \geq 1} Def_{\mathbf{V}}(H/\mathfrak{m}^n)$$

where $\mathfrak{m} = Rad(H)$, such that the corresponding morphism of functors on $\underline{a}_r$

$$\kappa : Mor(H, -) \to Def_{\mathbf{V}}$$

defined for $\phi \in Mor(H, R)$ by $\kappa(\phi) = R \otimes_\phi \tilde{V}$, is smooth, and an isomorphism on the tangent level. H is uniquely determined by a set of matric Massey products defined on subspaces

$$D(n) \subseteq Ext^1(V_i, V_{j_1}) \otimes \cdots \otimes Ext^1(V_{j_{n-1}}, V_k)$$

with values in $Ext^2(V_i, V_k)$.

Moreover, the right action of A on $\tilde{V}$ defines a homomorphism of k-algebras,

$$\eta : A \longrightarrow O(\mathbf{V}) := End_H(\tilde{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j)).$$

The k-algebra $O(\mathbf{V})$, called the ring of observables of $\mathbf{V}$, acts on the family of A-modules $\{V_i\}_{i=1}^r$, extending the action of A.

If $dim_k V_i < \infty$, for all $i = 1, \ldots, r$, the operation of associating $(O(\mathbf{V}), \mathbf{V})$ to $(A, \mathbf{V})$ is a closure operation.

There is a very useful result, see [8], [9],[2],

Theorem 2.2 (A Generalised Burnside Theorem)**.** *Let A be a finite-dimensional k-algebra, k an algebraically closed field. Consider the family $\mathbf{V} = \{V_i\}_{i=1}^r$ of all simple A-modules, then*

$$\eta : A \longrightarrow O(\mathbf{V}) = (H_{i,j} \otimes Hom_k(V_i, V_j))$$

is an isomorphism. Moreover the k-algebras A and H are Morita-equivalent.

We also prove that there exists, in the non-commutative deformation theory, an obvious analogy to the notion of pro-representing (modular) substratum H_0 of the formal moduli H, see [5]. The tangent space of H_0 is determined by a family of subspaces

$$Ext^1_0(V_i, V_j) \subseteq Ext^1_A(V_i, V_j), \quad i \neq j$$

the elements of which should be called the almost split extensions (sequences) relative to the family $\mathbf{V}$, and by a subspace, that we denote,

$$T_0(\Delta) \subseteq \prod_i Ext^1_A(V_i, V_i)$$

which is the tangent space of the deformation functor of the full subcategory of the category of A-modules generated by the family $\mathbf{V} = \{V_i\}_{i=1}^r$, see [6]. If $\mathbf{V} = \{V_i\}_{i=1}^r$ is the set of all indecomposable's of some Artinian k-algebra A, we show that the above notion of *almost split sequence* coincides with that of Auslander, see [16].

2.2. **Local Moduli and Entropy.** Consider an algebraic geometric object X, and let $aut(X)$ be the Lie algebra of infinitesimal automorphisms of X. The sub-Lie algebra $aut_0(X)$ that lifts to the formal moduli of X, is a Lie ideal. Put $\mathfrak{a}(X) := aut(X)/aut_0(X)$, then if $X(t)$ is a deformation of some X along a parameter t, we find $dim_k\mathfrak{a}(X(t)) \leq dim_k\mathfrak{a}(X)$. One may phrase this saying that an object X can never gain *information* when deformed. Moreover, deformation is, obviously, not a reversible process, so information can get lost. This measure of information losses, is related, as we shall see, to the notion of gain of entropy (en-ergy and tropos=transform) coined by Clausius (1865) and generalised by Boltzmann and Shannon.

In [14], studying moduli problems of singularities in (classical) algebraic geometry, we were led to consider the notion of *Modular Suite*. This is a canonical partition $\{\mathbf{M}_\alpha\}$, of the versal base space, $\mathbf{M}$, of the deformation functor of an algebraic object, X. The different *rooms*, $\mathbf{M}_\alpha$, correspond to the subsets of equivalence classes of deformations in $\mathbf{M}$, along which the Lie algebra $\mathfrak{a} := aut/aut_0$ deforms as Lie-algebras, and therefore conserves its dimension. Working with Thermodynamics, it occurred to me that the notion of entropy has an interesting parallel in deformation theory. In fact I have proposed the following,

Definition 2.3. *Fix an object X, and let $X(\underline{t})$ corresponds to the point $\underline{t} \in \mathbf{M}_\alpha$, then we shall term Entropy, of the state $\underline{t}$, the integer,*

$$S(\underline{t}) := dim_k(\mathbf{M}_\alpha).$$

In this classical situation, assuming that the field is algebraically closed, and that $\mathbf{M}$ is of finite Krull dimension, the modular suite $\{\mathbf{M}_\alpha\}$ is finite, with an inner room, the *modular substratum* and an *ambiant* (open) *maximal entropy* stratum. But the structure of the modular suite may be very complex, even for simple singularities X, see the example of the quasi homogenous plane curve singularity $x_1^5 + x_2^{11}$, in [14].

It is also clear that for any *algebraic dynamics* in $\mathbf{M}$, the entropy will always stay or grow, see again [14]. To be able to construct situations where the entropy is lowered, or the information goes up, we must leave classical algebraic geometry, and venture into non-commutative algebraic geometry, see [2].

In the general situation, where our algebras of *observables* are associative but not necessarily commutative, the first interesting cases are deformations of associative algebras, see [13].

2.3. **Deformations of Associative Algebras.** Given an associative k-algebra A, The tangent space of the formal moduli of A, as an associative k-algebra is, by deformation theory, see [5], and [14],

$$T_\star := A^1(k, A; A) = Hom_F(ker\rho, A)/Der,$$

where, $\rho : F \to A$ is any surjective homomorphism of a free k-algebra F, onto A, Hom_F means the F-bilinear maps, and Der denotes the subset of the restrictions to $I := ker\rho$ of the k-derivations from F to A.

As an example, let $A = k[x_1, .., x_d]$ be the polynomial algebra, then we find,

$$A^1(k, A; A) = Hom_F(ker\rho, A)$$

where $F = k < x_1, .., x_d >$, and $ker\rho =< [x_i, x_j] >$, and any element in $A^1(k, A; A)$ is a generalised Poisson structure. The technique for this general deformation theory, can be found in loc.cit. [5], see also [4], and we prove the same type of theorems as for modules over an associative algebra,

Now, let us consider the rather innocent singularity,

$$U := k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2,$$

as an associative algebra. U is, geometrically, an isolated point, with a 3-dimensional tangent space. We shall be interested in the versal base space for the deformation-functor of U, as associative algebra.

The tangent space of the formal moduli of the singularity

$$U := k[x_1, x_2, x_3]/(x_1, x_2, x_3)^2,$$

as an associative k-algebra is now,

$$T_\star := A^1(k, U; U) = Hom_F(ker\rho, U)/Der,$$

where, $\rho : F \to U$ is the obvious surjection of the free k-algebra $F = k < x_1, x_2, x_3 >$, with $ker\rho = (\underline{x})^2$, generated as F bi-module by the family $\{x_{i,j} := x_i x_j\}$.

Any F-bilinear morphism $\phi : (\underline{x})^2 \to U$, must be of the form,

$$\phi(x_{i,j}) = a^0_{i,j} + \sum_{l=1}^{3} a^l_{i,j} x_l$$

and the bilinearity is seen to imply that $a^0_{i,j} = 0$. Thus, the dimension of $Hom_F(I, U)$ is 27.

Any derivation $\delta \in Der$, must be given by,

$$\delta(x_i) = b^0_i + \sum_{l=1}^{3} b^l_i x_l$$

and the restriction of this map, to the generators of $I = (\underline{x})^2$, must have the form,

$$\delta(x_{i,j}) = b^0_j x_i + b^0_i x_j,$$

therefore determined by the $b^0_i s$. It follows that the tangent space $T_\star$ is of dimension 27-3=24.

Now, let $o, p \in \mathbf{A}^3$, be two points, $o = (o_1, o_2, o_3), p = (p_1, p_2, p_3)$, with respect to the coordinate system, $\underline{x}$, and put,

$$\phi_{o,p}(x_{i,j}) = p_j x_i + o_i x_j,$$

then it is easy to see that the maps $\{\phi_{o,p}\}$ generate a 6-dimensional sub vector subspace T_0 of $T_\star$. Notice that, if $o = p$ then $\phi_{o,p}$, is a derivation, thus 0 in $T_\star$.

Moreover, the rather unexpected happens. We may integrate the tangent subspace T_0, and obtain a family of flat deformations of U. In fact, it is easy to see that,

$$U(o,p) := k < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - p_j x_i + o_i p_j),$$

is an associative k-algebra of dimension 4, and a deformation of U, in a direction of $\underline{H}$. This defines a family of associative $H := k[o,p]$-algebras,

$$\mathbf{U} := H < x_1, x_2, x_3 > /(x_i x_j - o_i x_j - p_j x_i + o_i p_j).$$

Let us put,

$$\mathbf{x}_{i,j} := (x_i - o_i)(x_j - p_j) = x_i x_j - o_i x_j - p_j x_i + o_i p_j, o := (o_1, o_2, o_3), p = (p_1, p_2, p_3) \in H^3,$$

Notice that if $o = p$ then $U(o,p)$ is isomorphic to U, as it should, and that, $U(o,p) \simeq U(-o,-p)$. Moreover, for any non-zero element $\kappa \in k$, and any 3-vector $c \in \mathbf{A}^3$, we have,

$$U(o,p) \simeq U(\kappa o, \kappa p),\ U(o,p) \simeq U(o-c, p-c).$$

Choosing $c = 1/2(p+o)$, we find $o' := o - c = -(p-c) =: -p'$, and it is easy to see that if $o' \neq 0$ the sub Lie algebra generated by $\{x_1, x_2, x_3\}$ in $U(o', p')$, is isomorphic to the standard 3-dimensional Lie algebra with relations, $[y_1, y_2] = y_2, [y_1, y_3] = y_3, [y_2, y_3] = 0$. Moreover, choosing $c = (p+o)$, we find an isomorphism,

$$U(o,p) \simeq U(-p,-o) \simeq U(p,o),$$

which should be related to the obvious action of Z_2 on $\underline{H}$, which again might be related to the CPT-equivalence in physics, see [12], (4.9).

Let $\epsilon_{i,j,k}$ and $\delta_{i,j}$ be the usual indices, the first one nonzero only for $\{i,j,k\} = \{1,2,3\}$, and the last one the usual delta function. Then the algebra,

$$\mathbf{Q} := k < x_1, x_2, x_3 > /(x_i x_j - \epsilon_{i,j,k} x_k + \delta_{i,j}),$$

is isomorphic to the quaternions, which therefore is another non-trivial deformation of U. Notice that we here have used the ordinary notation for summation, by repeating indexes. Notice, for eventually later use that the discoverer of the Quaternions, Hamilton, wrote about his algebra as the *science of pure time*, see [3].

Consider now the restriction to the subscheme $\underline{H} - \underline{\Delta}$, of the family $\mathbf{U}$, denoted by,

$$\nu' : \mathbf{U}' \to \underline{H} - \underline{\Delta}.$$

Let $\underline{\tilde{H}}$ be the blown up of $\underline{H}$, in $\underline{\Delta}$, and recall that,

$$\mathbf{H} := Hilb^{(2)}_{\mathbf{A}^3} = \underline{\tilde{H}}/Z_2.$$

Since for all non-zero $\kappa \in k$, we have $U(\lambda + \kappa u, -\kappa u + \lambda) \simeq U(\kappa u, -\kappa u) \simeq U(u, -u)$, this family extends uniquely to a family,

$$\nu : \tilde{\mathbf{U}} \to \underline{\tilde{H}},$$

compatible with the action of Z_2.

Let us compute the algebras $U(o,p)$, and their Lie algebras of derivations, $\mathfrak{g}(\underline{t}) := Der_k(U(\underline{t}))$.

First, the 4-dimensional k-algebras $U(o,p)$, with relation,

$$x_{i,j} = (x_i - o_i)(x_j - p_j),$$

with, $o \neq$p, are all isomorphic, since in this case there is an element $\alpha \in Gl_k(3)$ sending (o,p) onto any other pair (o', p'), with $o' \neq p'$. Let us see this, using the generalised Burnside theorem, see [12], (3.2). Obviously $U(o,p)$ has only two simple representations, of dimension 1, call them k_o and k_p. By the O-construction, there is an isomorphisme,

$$\eta : U(o,p) \to \begin{pmatrix} H_{1,1} \otimes End(k_o) & H_{1,2} \otimes Hom_k(k_o, k_p) \\ H_{2,1} \otimes Hom_k(k_p, k_o) & H_{2,2} \otimes End_k(k_p) \end{pmatrix},$$

where, $H_{1,1}$ is a formal algebra with tangent space $Ext^1_{U(o,p)}(k_o, k_o)$, $H_{2,2}$ is a formal algebra with tangent space $Ext^1_{U(o,p)}(k_p, k_p)$, and $H_{1,2}$, respectively $H_{1,2}$, is a bi-module generated by $Ext^1_{U(o,p)}(k_o, k_p)^\star$, respectively by $Ext^1_{U(o,p)}(k_p, k_o)^\star$. There are no problems computing the Ext-groups. Recall that

$$Ext^1_{U(o,p)}(V_1, V_2) = Der_k(U(o,p), Hom_k(V_1, V_2))/Triv,$$

and that $u \in U(o,p)$ operates on $\phi \in Hom_k(V_1, V_2)$, as,

$$(u\phi)(v_1) = u\phi(v_1), (\phi u)(v_1) = \phi(uv_1).$$

In the general case (one may test it in the interesting case, $o = (1,0,0), p = (0,0,0)$ above), we obtain,

$$Ext^1_{U(o,p)}(k_o, k_o) = Ext^1_{U(o,p)}(k_p, k_p) = Ext^1_{U(o,p)}(k_o, k_p) = 0, Ext^1_{U(o,p)}(k_p, k_o) = k^2.$$

Therefore,

$$\eta : U(o,p) \to \begin{pmatrix} k & 0 \\ < \xi_1, \xi_2 > & k \end{pmatrix}$$

is an isomorphism. Here $\xi_i \cdot 1 = \xi_i$. We may pick generators of this algebra,

$$x_1 := \begin{pmatrix} 0 & 0 \\ \xi_1 & 0 \end{pmatrix}, x_2 := \begin{pmatrix} 0 & 0 \\ \xi_2 & 0 \end{pmatrix}, x_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and obtain the relations corresponding to the choice of $o = (0,0,-1), p = (0,0,1)$. We have therefore obtained an algebraic subspace $\underline{\tilde{H}}$, of the miniversal base space $\mathbf{M}$ of the algebra U, corresponding to the algebras $U(o,p)$ that are all isomorphic. This subspace is therefore a *trivialising section* of this miniversal base space.

Remark 2.4. *Deformations of U(o,p)*

Using the same technique as above, computing the deformations of one of these isomorphic algebras, we may show that the tangent space of $Def_{U(o,p)}$ is trivial. In fact, as above, this tangent space is given by,

$$A^1(U(o,p), U(o,p)) = Hom_F(J, U(o,p))/Der,$$

where $J = ker(\pi)$ and $\pi : F \to U(o,p)$ is a surjective homomorphism of the free k-algebra $F = k < x_1, x_2, x_3 >$ onto $U(o,p)$. Obviously $J = ker(\pi)$ is generated by the elements

$$\{x_{i,j} := x_i x_j - o_i x_j - p_j x_i + o_i p_j\},$$

and we have in J the relations,

$$x_{i,j} x_k + o_i x_{j,k} + p_j x_{i,k} = x_i x_{j,k} + o_j x_{j,k} + p_k x_{i,j}.$$

Let $o = (1,0,0), p = (0,1,0)$, then an easy, but quite lengthy computation shows that these relations implies that any bilinear homomorphism, $c \in Hom_F(J, U(o,p))$, is the restriction of a derivation, $\beta \in Der_k(F, U(o,p))$, proving that,

$$A^1(U(o,p), U(o,p)) = 0.$$

Notice that any automorphism of U is reduced to a substitution,

$$y_i := \sum_{k=1}^{3} \alpha_{i,k} x_k, \alpha := (\alpha_{i,k}) \in Gl_k(3).$$

If we change the coordinates, of the point pair (o, p), by the automorphism above, then with obvious indexes,

$$U_x(o,p) \simeq U_y(\alpha(o), \alpha(p)).$$

3. Local Gauge Group

Borrowing notions from quantum physics, we shall call the principal Lie algebra bundle on the space, $\underline{\tilde{H}}$,

$$\mathfrak{g} := Der_H(\mathbf{U})$$

the local gauge group of the H-representation Θ_H.

3.1. **Computation of $\mathfrak{g}$, and its Action.** Any element $\delta \in Der_H(\mathbf{U})$ must be given by its values on the coordinates,

$$\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3,\ \delta_i^j \in H.$$

Now, let us define,

$$\tilde{\Theta}_{\tilde{H}} := \{\kappa \in End_{\tilde{H}}(\mathbf{U}), \kappa(1) = 0\}.$$

Oviously,

$$\mathfrak{g} \subset \tilde{\Theta}.$$

Any $\kappa \in \tilde{\Theta}_{\tilde{H}}$ will correspond to $\kappa_i := \kappa(x_i) \in \mathbf{U}, i = 1,2,3,$ i.e. to a matrix of the type,

$$M := \begin{pmatrix} 0 & 0 & 0 & 0 \\ \kappa_1^0 & \kappa_1^1 & \kappa_1^2 & \kappa_1^3 \\ \kappa_2^0 & \kappa_2^1 & \kappa_2^2 & \kappa_2^3 \\ \kappa_3^0 & \kappa_3^1 & \kappa_3^2 & \kappa_3^3 \end{pmatrix},$$

where, $\kappa_i := (\kappa_i^0, \kappa_i^1, \kappa_i^2, \kappa_i^3), \kappa_i^j \in H$. Moreover, it is clear that $\tilde{\Theta}$ is a Lie algebra, and that $\mathfrak{g}$ is a natural sub-Lie algebra, of this matrix algebra.

Put,

$$\bar{o} = (1, o_1, o_2, o_3), \bar{p} = (1, p_1, p_2, p_3),$$

and consider now the 4-vectors,

$$\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3), i = 1,2,3.$$

Suppose $\delta \in \mathfrak{g}$, then computing in $\mathbf{U}$, we find the formula,

$$\delta(x_i x_j - o_i x_j - p_j x_i + o_i p_j) = (\delta_i \cdot \bar{o}) x_j - (\delta_i \cdot \bar{o}) p_j + x_i(\delta_j \cdot \bar{p}) - o_i(\delta_j \cdot \bar{p})$$

which leads to,

$$\delta \in Der_H(\mathbf{U})$$

if and only if, $\delta(x_i x_j - o_i x_j - p_j x_i + o_i p_j) = 0$, therefore, if and only if,

$$\delta_i \cdot \bar{o} = \delta_i \cdot \bar{p} = 0, i = 1,2,3.$$

Given a point $\underline{t} = (o,p) \in \underline{H}$, let us compute the Lie algebra $\mathfrak{g}(\underline{t}) := Der_k(U(\underline{t}))$. Any element $\delta \in Der_k(U(\underline{t}))$ must have the form,

$$\delta(x_i) = \delta_i^0 + \delta_i^1 x_1 + \delta_i^2 x_2 + \delta_i^3 x_3,\ \delta_i^p \in k.$$

Put, as above, $\bar{o} = (1, o_1, o_2, o_3), \bar{p} = (1, p_1, p_2, p_3)$, and consider the 4-vectors

$$\delta_i = (\delta_i^0, \delta_i^1, \delta_i^2, \delta_i^3), i = 1,2,3.$$

As above, we find that $\delta \in Der_k(U(\underline{t}))$ if and only if $\delta_i \cdot \bar{o} = \delta_i \cdot \bar{p} = 0, i = 1, 2, 3$.

The tangent space $\Theta_{H,\underline{t}}$ of $\underline{H}$, at $\underline{t}$, is represented by the space of all pairs of 3-vectors, (ξ, ν) and we are interested in the action of $\mathfrak{g}(\underline{t})$ on this tangent space. Since all $U(o,p)$ are isomorphic, there must, for any tangent, (ξ, ν), exist an isomorphism of $k[\epsilon]$-algebras,

$$\eta : U(o,p) \otimes k[\epsilon] \to U(o + \xi\epsilon, p + \nu\epsilon),$$

commuting with the projection onto $U(o,p)$. It must be given by formulas,

$$\eta(x_i) = x_i + \kappa(x_i)\epsilon,\ \kappa(x_i) = \kappa_i^0 + \kappa_i^1 x_1 + \kappa_i^2 x_2 + \kappa_i^3 x_3 \in U(o,p), i = 1, 2, 3.$$

Put $\kappa_i := (\kappa_i^0, \kappa_i^1, \kappa_i^2, \kappa_i^3)$, then,

$$\kappa \in \tilde{\Theta}_k.$$

A little computation now shows that we must have,

$$\xi_i = \kappa_i \cdot \overline{o},\ \nu_i = \kappa_i \cdot \overline{p}, i = 1, 2, 3.$$

Therefore, given a point $\underline{t} = (o,p)$, and the corresponding generators $\{x_i, i = 1, 2, 3\}$ of $U(o,p)$, any $\kappa \in \tilde{\Theta}_k$ will correspond to $\kappa_i := \kappa(x_i) \in U(o,p), i = 1, 2, 3$, and therefore to a tangent of $\underline{H}$ at the point $\underline{t} = (o,p)$,

$$(\xi = \overline{\kappa} \cdot \overline{o},\ \nu = \overline{\kappa} \cdot \overline{p}) \in \Theta_{\underline{H},\underline{t}}.$$

We therefore find an exact sequence of bundles on $\underline{\tilde{H}}$,

$$0 \to \mathfrak{g} \to \tilde{\Theta}_{\tilde{H}} \to \Theta_{\tilde{H}} \to 0.$$

The Lie algebra $\mathfrak{g}$, is now seen to operate naturally on $\tilde{\Theta}_{\tilde{H}}$, corresponding to exactly the operation above, drawn from the deformation theory of algebras. Any $\delta \in \mathfrak{g}$, operates on $\kappa \in \tilde{\Theta}_{\tilde{H}}$ as $\delta(\kappa) = \delta \cdot \kappa - \kappa \cdot \delta$. Since $\delta \cdot \overline{o} = \delta \cdot \overline{p} = 0$, we find

$$\delta(\xi, \nu) = (\delta(\xi), \delta(\nu)) := (\delta(\kappa)\overline{o}, \delta(\kappa)\overline{p}).$$

Observe that, since $(\overline{o} - \overline{p}) = (o - p)$, the Lie algebra representation of $\mathfrak{g}$ on the tangent space $\Theta_{\tilde{H},\underline{t}}$, at the point $\underline{t} = (o,p)$, kills the subspace generated by the vectors $\{\xi = (o - p), \nu = (o - p))\}$.

If $o \neq p$, it follows that $\bar{o}$ and $\bar{p}$, are linearly independent, in a 4-dimensional vector space, therefore each vector $\delta_i, i = 1, 2, 3$ is confined to a 2-dimensional vector space. Consequently, $\mathfrak{g}(\underline{t}) := Der_k(U(\underline{t}))$ is of dimension 6. Using the isomorphism, $U(o,p) \simeq U(o - c, p - c)$, mentioned above, we may choose coordinates such that $o = (0,0,0), p = (1,0,0)$.

In fact, we may first put $c = o$, and reduce to the situation where $o = 0$, and p is a non-zero 3-vector. Any $\delta \in Der_k(U(o,p))$ will then be represented by a matrix of the form,

$$M := \begin{pmatrix} \delta_1^1 & \delta_1^2 & \delta_1^3 \\ \delta_2^1 & \delta_2^2 & \delta_2^3 \\ \delta_3^1 & \delta_3^2 & \delta_3^3 \end{pmatrix},$$

where $M(p) = 0$, and we know that the Lie structure is the ordinary matrix Lie-products. Now, clearly we may find a nonsingular matrix N such that $N(p) = (1,0,0)$, and the Lie algebra of matrices M, will be isomorphic to the Lie-algebra of the matrices, NMN^{-1}, which are those corresponding to $p = e_1 := (1,0,0)$, and we are working with $U(0, e_1)$. Notice that in this picture, the fundamental vector $\overline{op} = (1,0,0)$. With this we find that, $\delta \in \mathfrak{g}(\underline{t})$ imply,

$$\delta_i^0 = \delta_i^1 = 0, i = 1, 2, 3.$$

The following result is now easily seen.

Theorem 3.1. *The Lie algebra $\mathfrak{g}(\underline{t})$ is isomorphic to the Lie algebra of matrices of the form,*

$$\begin{pmatrix} 0 & \delta_1^2 & \delta_1^3 \\ 0 & \delta_2^2 & \delta_2^3 \\ 0 & \delta_3^2 & \delta_3^3 \end{pmatrix}.$$

The radical $\mathfrak{r}$, is generated by 3 elements, $\{u, r_1, r_2\}$, with

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $u \notin [\mathfrak{g}, \mathfrak{g}]$, $[u, r_i] = -r_i, [r_1, r_2] = 0$, and the quotient,

$$\mathfrak{g}(\underline{t})/\mathfrak{r} = \mathfrak{sl}(2).$$

with the usual generators h, e, f,

$$h = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

In particular, we find that $\mathfrak{sl}(2) \subset \mathfrak{g}(\underline{t})$.

Remark 3.2. *The tangent space at a point $\underline{t} \in \underline{H}$, decomposes into,*

$$\Theta_{H,\underline{t}} = \tilde{\Delta} \oplus \tilde{c}$$

where, $\tilde{\Delta} = \{(\xi, \xi)\}, \tilde{c} = \{(\nu, -\nu)\}, \xi, \nu \in k^3$. We know that the action of $\mathfrak{g}$ kills the tangent vectors of the type $((o-p), (p-o))$, or $((o-p), (o-p))$, and the operator h, generating the Cartan subalgebra of $\mathfrak{g}$, picks out two eigenvectors, together forming a unique homogenous coordinate system for $\Theta_{H,\underline{t}}$,

$$\{d_1, d_2, d_3\} \subset \tilde{\Delta}, \{c_1, c_2, c_3\} \subset \tilde{c},$$

where, $d_3 = (o-p, o-p), c_3 = (o-p, (p-o))$, d_1 and c_1 positive eigenvectors for h, and d_2 and c_2 negative eigenvectors for h.

With this done, we may write up the action of $\mathfrak{g}$ on Θ_H.

3.2. **Action of the Local Gauge Group in Canonical Coordinates.** If

$$o = (0, 0, 0), p = (1, 0, 0),$$

then we have seen that the Lie algebra $\mathfrak{g}(\underline{t})$ comes out isomorphic to the Lie algebra of matrices of the form,

$$\begin{pmatrix} 0 & \delta_1^2 & \delta_1^3 \\ 0 & \delta_2^2 & \delta_2^3 \\ 0 & \delta_3^2 & \delta_3^3 \end{pmatrix}.$$

The radical $\mathfrak{r}$, is generated by 3 elements, $\{u, r_1, r_2\}$, with

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, r_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $u \notin [\mathfrak{g}, \mathfrak{g}]$, and

$$[u, r_i] = -r_i, [r_1, r_2] = 0 \quad \text{and the quotient} \quad \mathfrak{g}(\underline{t})/\mathfrak{r} = \mathfrak{sl}(2).$$

with the usual generators h, e, f,

$$h = u_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, e = u_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, f = u_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

In particular, we find that $\mathfrak{sl}(2) \subset \mathfrak{g}(\underline{t})$.

Notice also that, in this case, the unique 0-tangent line at the point

$$\underline{t}_0 = (o, p),\ o = (0,0,0),\ p = (1,0,0),$$

killed by $\mathfrak{g}$, is represented by the pair $d_3 := ((1,0,0),(1,0,0))$, and the unique light-velocity line is represented by $c_3 := ((1,0,0),(-1,0,0))$.

Let $d_1 := ((0,1,0),(0,1,0))$, $d_2 := ((0,0,1),(0,0,1))$ and let $c_1 := ((0,1,0),(0,-1,0))$, $c_2 := ((0,0,1),(0,0,-1))$. Then $\{c_1, c_2, c_3, d_1, d_2, d_3\}$ is a basis for the tangent space $\Theta_{\underline{t}_0}$, and $\{d_1, d_2, d_3\}$ is a basis for $\tilde{\Delta}_{\underline{t}_0}$.

We observe that the generator h of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acts in this basis as,

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which makes the choice of basis above canonical, i.e. determines $\{c_1, c_2, d_1, d_2\}$ as (± 1) eigenvectors of h, in $\tilde{c}$, resp. in $\tilde{\Delta}$. The actions of the gauge fields $\mathfrak{g}$ can then be given canonically: The generators, $h, e, f \in \mathfrak{sl}(2) \subset \mathfrak{g}$ act, in the above basis, like,

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$e = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The generators, $u, r_1, r_2 \in rad(\mathfrak{g})$ act, in the above basis, like,

$$u = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$r_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$r_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

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