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PICARD GROUPS FOR LINE BUNDLES WITH CONNECTIONS

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To the memory of Egbert Brieskorn

Abstract. We study analogues of the usual Picard group for complex manifolds or non-singular complex algebraic varieties but instead of line bundles we study line bundles with connections. We choose an approach which works for both cases. We identify obstructions for the existence of a connection, or a connection which is even integrable or regular (integrable), and point out where one should be careful when passing from the analytic to the algebraic case.

Introduction

It was Egbert Brieskorn who brought the authors together already in 1970 when he was professor at Göttingen. As a result of the first meeting a cooperation started which lasted over decades up to now, the main subject being theorems of Lefschetz type, we are therefore very grateful to him! In this context it was natural for us to turn to the Picard group. In the present paper we consider Picard groups of line bundles with a connection.

In order to be more precise, let X be a reduced complex analytic space. It is known that the isomorphism classes of line bundles on X define a group, called the analytic Picard group $Pic^{an}(X)$ of X.

Remember that one can pass from a line bundle to the invertible sheaf of its sections, after all we may work with invertible sheaves instead of line bundles because we have an equivalence of categories.

If X is a complex manifold, it is natural to consider line bundles on the space X with a connection or with an integrable connection. The isomorphism classes of these line bundles define groups that we shall denote by $Pic^{an}_c(X)$ for line bundles with a connection and $Pic^{an}_{ci}(X)$ for line bundles with an integrable connection.

We are going to compare these groups with the original Picard group $Pic^{an}(X)$ using certain exact sequences. In particular, these give obstructions for the existence of a connection resp. an integrable connection. As we will see these results are not really new (in the analytic case) but the important point is that we use an elementary approach which also goes over to the algebraic case without problems. It avoids hypercohomology (which is basic for Deligne cohomology) or the curvature of differentiable connections. But in order to make the results plausible we relate our approach to one which uses the well-known relation to Deligne cohomology.

An important special case is the one of compact Kähler manifolds. Here we show that we can avoid to go back to (p,q)-forms explicitly but we can argue with the abstract framework of Hodge theory. This has the advantage that we can easily pass afterwards to smooth complete algebraic varieties which might not be projective. We prove that in the compact Kähler (or complete algebraic) case every connection on a line bundle is automatically integrable - a fact which may be surprising before seeing the proof (which is easy).

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The essential point for us is to pass to the algebraic case. As already said our approach goes over easily. To work with hypercohomology, similar to Deligne cohomology, requires some care but we discuss how to argue then. Also, we deal with regular (integrable) connections and study different ways to describe the obstructions for their existence. After all we show that an algebraic line bundle admits a regular integrable connection if and only if its complex first Chern class vanishes - a result which does not follow from the Riemann-Hilbert correspondence!!

By the way, the theory of D-modules will not be considered here because it is only related to the integrable case

At the end we discuss some illustrative examples.

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1. Analytic Comparisons

1.1. $Pic^{an}(X)$ and $Pic_c^{an}(X)$

In this section let X be a complex manifold which is paracompact (e.g. Stein or compactifiable; the condition is not automatically fulfilled, see [4]). A connection on an invertible $\mathcal{O}_X$-module $\mathcal{L}$ is a $\mathbb{C}$-linear morphism $\nabla : \mathcal{L} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}$ such that $\nabla(fs) = f\nabla(s) + df \otimes s$, see [5] I Déf. 2.4, p. 7.

If $\mathcal{L} = \mathcal{O}_X$, a connection is defined by a form $\omega \in H^0(X, \Omega^1_X)$: $\nabla(1) = \omega$, so $\nabla(f) = df + f\omega$. If $\mathcal{L}$ is trivial, s a nowhere vanishing section of $\mathcal{L}$ and $\omega \in H^0(X, \Omega^1_X)$, there is a uniquely defined connection ∇ on $\mathcal{L}$ such that $\nabla(s) = \omega \otimes s$: we say that it is defined by ω with respect to s. Two line bundles $(\mathcal{L}, \nabla)$, $(\mathcal{L}', \nabla')$ are called isomorphic if there is an isomorphism $\phi : \mathcal{L} \to \mathcal{L}'$ such that the diagram

$$\begin{array}{ccc} \mathcal{L} & \stackrel{\nabla}{\to} & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L} \\ \phi\downarrow & & id \otimes \phi \downarrow \\ \mathcal{L}' & \stackrel{\nabla'}{\to} & \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{L}' \end{array}$$

is commutative.

The isomorphism classes of invertible $\mathcal{O}_X$-modules with connection form a group $Pic_c^{an}(X)$.

We have an exact sequence of sheaves:

$$0 \to \mathbb{C}^*_X \to \mathcal{O}^*_X \to d\mathcal{O}_X \to 0$$

where $\mathbb{C}^*_X \to \mathcal{O}^*_X$ is given by the inclusion and $\mathcal{O}^*_X \to d\mathcal{O}_X$ is defined by $f \mapsto df/f$.

This latter morphism is surjective, because, if $\omega \in d\mathcal{O}_{X,x}$, there is $f \in \mathcal{O}_{X,x}$ such that $\omega = df$. Then $e^f \in \mathcal{O}^*_{X,x}$ has its image equal to ω. The rest of the sequence is exact because of Poincaré Lemma.

This exact sequence of sheaves gives an exact sequence of cohomology:

$$\ldots \to H^0(X, d\mathcal{O}_X) \to H^1(X, \mathbb{C}^*_X) \to H^1(X, \mathcal{O}^*_X) \to H^1(X, d\mathcal{O}_X) \to \ldots$$

Here we only use the mapping $H^1(X, \mathcal{O}^*_X) \to H^1(X, d\mathcal{O}_X)$ from the exact sequence (see also the proof of Theorem 2.2.22 of [3]).

Now we can prove the following exact sequence in an elementary way. We will see that it can also be obtained easily using hypercohomology (Deligne cohomology).

Theorem 1.1. *We have an exact sequence:*

$$H^0(X, \mathcal{O}^*_X) \to H^0(X, \Omega^1_X) \to Pic_c^{an}(X) \to Pic^{an}(X) \to H^1(X, \Omega^1_X)$$

Proof. The map $H^0(X, \mathcal{O}_X^*) \to H^0(X, \Omega_X^1)$ is defined by $g \mapsto \frac{dg}{g}$, $H^0(X, \Omega_X^1) \to Pic_c^{an}(X)$ by $\omega \mapsto (\mathcal{O}_X, \nabla(f) = df + f\omega)$.

The map $Pic^{an}(X) \to H^1(X, \Omega_X^1)$ is the composition of $H^1(X, \mathcal{O}_X^*) \to H^1(X, d\mathcal{O}_X)$ (see above) and the natural map from $H^1(X, d\mathcal{O}_X)$ to $H^1(X, \Omega_X^1)$, since $Pic^{an}(X) \simeq H^1(X, \mathcal{O}_X^*)$.

Now, notice that we have a group structure on $Pic_c^{an}(X)$. According to Deligne in [5] p. 8, consider the invertible sheaves (i.e. invertible $\mathcal{O}_X$-modules) $\mathcal{L}$ and $\mathcal{L}'$ defined by the (s_i) and (s_i') on an open covering $\mathcal{U}$, with the connections ∇ and ∇' defined by (α_i) and (α_i') on the open covering $\mathcal{U}$, then $\mathcal{L} \otimes \mathcal{L}'$ is invertible and defined by $(s_i \otimes s_i')$, and the connection ∇_0 on this invertible sheaf is defined by $(\alpha_i + \alpha_i')$.

(i) Now let us prove the exactness. First, the function $g \in H^0(X, \mathcal{O}_X^*)$ is mapped onto $\frac{dg}{g} \in H^0(X, \Omega_X^1)$, and this in turn to the element of $Pic_c^{an}(X)$ represented by $(\mathcal{O}_X, \nabla)$, where $\nabla(f) := df + f\frac{dg}{g}$. This is the inverse image of $(\mathcal{O}_X, d)$ under the isomorphism $\cdot g : \mathcal{O}_X \to \mathcal{O}_X$, so its class in $Pic_c^{an}(X)$ is trivial: we have a commutative diagram

$$\begin{array}{ccc} H^0(X, \mathcal{O}_X) & \overset{\nabla}{\to} & H^0(X, \Omega_X^1) = H^0(X, \Omega_X^1 \otimes \mathcal{O}_X) \\ \cdot g \downarrow & & \cdot g \downarrow \\ H^0(X, \mathcal{O}_X) & \overset{d}{\to} & H^0(X, \Omega_X^1) = H^0(X, \Omega_X^1 \otimes \mathcal{O}_X) \end{array}$$

Suppose now that $\omega \in H^0(X, \Omega_X^1)$ is mapped onto the trivial element of $Pic_c^{an}(X)$, which means that $(\mathcal{O}_X, d)$ is isomorphic to $(\mathcal{O}_X, f \mapsto df + \omega f)$. The isomorphism gives a mapping from $\mathcal{O}_X$ onto itself, which is of the form $\cdot g$ for some $g \in H^0(X, \mathcal{O}_X^*)$. Then, the image of $1 \in H^0(X, \mathcal{O}_X)$ is $\omega \in H^0(X, \Omega_X^1)$ and by the multiplication by g, it is dg. Therefore $\omega = \frac{dg}{g}$.

(ii) It is obvious that the composition of the two middle arrows gives the trivial mapping.

The kernel of the map $Pic_c^{an}(X) \to Pic^{an}(X)$ defined by $(\mathcal{L}, \nabla) \mapsto \mathcal{L}$ is given by the pairs $(\mathcal{O}_X, \nabla)$, so it coincides with the image of the morphism $H^0(X, \Omega_X^1) \to Pic_c^{an}(X)$ defined by $\omega \mapsto (\mathcal{O}_X, \nabla(f) = df + f\omega)$. So, the middle part of the sequence is exact.

(iii) Now let $\mathcal{L}$ be an invertible sheaf which is in the kernel of $Pic^{an}(X) \to H^1(X, \Omega_X^1)$. Let $\mathcal{U} = (U_i)_{i \in I}$ be a covering of X, such that $\mathcal{L}|U_i$ is isomorphic to $\mathcal{O}_X|U_i$ by a map $\mathcal{O}_X|U_i \to \mathcal{L}|U_i$ which corresponds to $1 \mapsto s_i$. Let g_{ij} be the complex analytic transition map defined on $U_{ij} = U_i \cap U_j$ from $\mathcal{L}|U_i$ to $\mathcal{L}|U_j$. We have $s_j = g_{ij}s_i$ on $U_i \cap U_j$.

Since $s_j = g_{ij}s_i = g_{ij}g_{ki}s_k = g_{kj}s_k$ on $U_i \cap U_j \cap U_k$, we have $g_{kj} = g_{ij}g_{ki}$ on $U_i \cap U_j \cap U_k$. The family (g_{ij}) defines a 2-cocycle of $H^1(X, \mathcal{O}_X^*)$, a fact which is well-known. Since

$$H^1(\mathcal{U}, \Omega_X^1) \subset H^1(X, \Omega_X^1)$$

cf. [9] Hilfssatz 12.4, p. 91, the image of $\mathcal{L}$ in $H^1(X, \Omega_X^1)$ being trivial, the 2-cocycle (dg_{ij}/g_{ij}) is trivial, i.e. a coboundary. Therefore there are differential forms ω_i and ω_j defined respectively on U_i and U_j, such that:

$$\frac{dg_{ij}}{g_{ij}} = \omega_j - \omega_i$$

on $U_i \cap U_j$.

Consider for each i the connection $\tilde{\nabla}_i$ on $\mathcal{O}_X|U_i$ defined by:

$$\tilde{\nabla}_i(f) = df + f\omega_i$$

This defines on $\mathcal{L}|U_i$ a connection:

$$\nabla_i(fs_i) = df \otimes s_i + f\omega_i \otimes s_i,$$

which gives for $f = 1$:

$$\nabla_i(s_i) = \omega_i \otimes s_i.$$

On $U_i \cap U_j$, we have $g_{ij}s_i = s_j$. Therefore, on $U_i \cap U_j$:

$$\nabla_i(fg_{ij}s_i) = d(fg_{ij}) \otimes s_i + fg_{ij}\omega_i \otimes s_i = g_{ij}df \otimes s_i + fdg_{ij} \otimes s_i + fg_{ij}\omega_i \otimes s_i,$$

which implies, with $f = 1$, on $U_i \cap U_j$:

$$\nabla_i(g_{ij}s_i) = dg_{ij} \otimes s_i + g_{ij}\omega_i \otimes s_i.$$

Therefore:

$$\nabla_i(s_j) = g_{ij}(\frac{dg_{ij}}{g_{ij}} + \omega_i) \otimes s_i = (\frac{dg_{ij}}{g_{ij}} + \omega_i) \otimes g_{ij}s_i = (\omega_j - \omega_i + \omega_i) \otimes s_j$$

which yields:

$$\nabla_i(s_j) = \nabla_j(s_j)$$

on $U_i \cap U_j$.

Therefore the $(\nabla_i)_{i\in I}$ define on $\mathcal{L}$ a connection ∇ and the class of the element $\mathcal{L}$ which lies in the kernel of the map $Pic^{an}(X) \to H^1(X, \Omega^1_X)$ is the image of the class of $(\mathcal{L}, \nabla)$.

It remains to prove that the image of $(\mathcal{L}, \nabla)$ in $H^1(X, \Omega^1_X)$ in the above sequence vanishes.

Let $(U_i)_{i\in I}$ be an open covering of X such that $\mathcal{L}|U_i$ is isomorphic to $\mathcal{O}_X|U_i$ by a map $s_i \mapsto 1$. We write $\nabla s_i = \omega_i \otimes s_i$. Let (g_{ij}) be the cocycle of transition functions such that $s_j = g_{ij}s_i$. Then (dg_{ij}/g_{ij}) is a cocycle which represents an element of $H^1(X, \Omega^1_X)$. Since:

$$\nabla(s_j) = \nabla(g_{ij} \otimes s_i) = dg_{ij} \otimes s_i + g_{ij}\omega_i \otimes s_i = \omega_j \otimes s_j = g_{ij}\omega_j \otimes s_i,$$

we obtain:

$$\frac{dg_{ij}}{g_{ij}} = \omega_j - \omega_i.$$

Therefore the class of the element given by the elements (dg_{ij}/g_{ij}) vanishes in $H^1(X, \Omega^1_X)$.

This shows that the above sequence is exact.

We shall give an interpretation of this exact sequence below.

Implicitly we have used:

Lemma 1.2. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is represented by a cocycle (g_{ij}) in $C^1(\mathcal{U}, \mathcal{O}^*_X)$. Then, a connection ∇ on $\mathcal{L}$ is represented by an element (ω_i) in $C^0(\mathcal{U}, \Omega^1_X)$ which is mapped by $\delta : C^0(\mathcal{U}, \Omega^1_X) \to C^1(\mathcal{U}, \Omega^1_X)$ onto $(\frac{dg_{ij}}{g_{ij}}) \in C^1(\mathcal{U}, \Omega^1_X)$.*

Note that $(d\omega_i) \in \mathcal{C}^0(\mathcal{U}, \Omega^2_X)$ is a cocycle, i.e. defines an element of $H^0(X, \Omega^2_X)$, which is the curvature of ∇, see below.

Particularly easy is the case of Stein manifolds. Then $H^1(X, \Omega^1_X) = 0$, because of Cartan's Theorem B, so from Theorem 1.1 we obtain:

Lemma 1.3. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on a Stein manifold X. Then there is a complex analytic connection on $\mathcal{L}$.*

In the following subsection we shall show how our reasoning above is related to the literature ("Atiyah obstruction").

1.2. **Atiyah obstruction.**

Atiyah ([1] §2) has studied complex analytic connections on a holomorphic principal fibre bundle P. Whereas differentiable connections always exist there is an obstruction to the existence of a complex analytic one. In particular, there is an obstruction $b(E)$ to the existence of a complex analytic connection on the principal fibre bundle which corresponds to a holomorphic vector bundle E (see [1] p. 194). We call it the Atiyah obstruction. In the case of a line bundle L we have that $b(L) \in H^1(X, \Omega_X^1)$.

Here we use again invertible sheaves $\mathcal{L}$ instead of line bundles L. Then a complex analytic connection on L corresponds to a connection on the sheaf $\mathcal{L}$ of holomorphic sections of L.

Let us recall the definition of $b(\mathcal{L})$, see [1] p. 193. Let $D(\mathcal{L})$ be the locally free $\mathcal{O}_X$-module defined as follows:
as a $\mathbb{C}_X$-module, $D(\mathcal{L}) := \mathcal{L} \oplus (\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{L})$, and the $\mathcal{O}_X$-module structure is given by:

$$f \cdot (s, \beta) := (fs, f\beta + df \otimes s),$$

if f is a section of $\mathcal{O}_X$, s a section of $\mathcal{L}$ and β is a section of $\Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{L}$.
Then we get an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{L} \to D(\mathcal{L}) \to \mathcal{L} \to 0$$

where the second arrow is given by $\beta \mapsto (0, \beta)$ and the third one by $(s, \beta) \mapsto s$.
Applying $Hom(\mathcal{L}, \cdots)$ we obtain a long exact cohomology sequence

$$\ldots \to H^0(X, Hom(\mathcal{L}, D(\mathcal{L})) \to H^0(X, Hom(\mathcal{L}, \mathcal{L})) \to H^1(X, Hom(\mathcal{L}, \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{L})) \to \ldots$$

Now $b(\mathcal{L})$ is defined as the image of $1 \in H^0(X, \mathcal{O}_X)$ in $H^1(X, \Omega_X^1)$ under the mapping:

$$H^0(X, \mathcal{O}_X) \overset{\simeq}{\to} H^0(X, Hom(\mathcal{L}, \mathcal{L})) \to H^1(X, Hom(\mathcal{L}, \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{L})) \overset{\simeq}{\to} H^1(X, \Omega_X^1)$$

(so the mapping depends on $\mathcal{L}$!).

Lemma 1.4. $b(\mathcal{L}) = 0$ *if and only if* $\mathcal{L}$ *admits a connection.*

Proof: A splitting of the first exact sequence above is given by an $\mathcal{O}_X$-linear mapping of the form $s \mapsto (s, \nabla(s))$, such that ∇ is a connection on $\mathcal{L}$, and vice versa.

Look at the second exact sequence. The second arrow maps 1 onto $b(\mathcal{L})$, by definition of $b(\mathcal{L})$, with the identifications made in the definition. The inverse images of 1 with respect to the first arrow correspond to the splittings of the first exact sequence, i.e. to the connections on $\mathcal{L}$. This implies our statement.

Lemma 1.5. $b(\mathcal{L})$ *is the image of* $-[\mathcal{L}] \in H^1(X, \mathcal{O}_X^*)$ *in* $H^1(X, \Omega_X^1)$*, i.e.* $b(\mathcal{L})$ *is represented by the cocycle* $-(\frac{dg_{ij}}{g_{ij}})$.

Proof: Let $\mathcal{U} = (U_i)$ be an open Stein covering of X such that $\mathcal{L}|U_i$ is trivial. Let s_i be a nowhere vanishing section of $\mathcal{L}|U_i$. Then, $s_j = g_{ij}s_i$, where g_{ij} are the corresponding transition functions. Let ∇_i be the connection on $\mathcal{L}|U_i$ such that $\nabla_i(s_i) = 0$. Now, let us describe $H^0(\mathcal{U}, Hom(\mathcal{L}, \mathcal{L})) \to H^1(\mathcal{U}, Hom(\mathcal{L}, \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{L}))$ using the exact sequence of complexes:

$$0 \to C^{\cdot}(\mathcal{U}, Hom(\mathcal{L}, \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{L})) \to C^{\cdot}(\mathcal{U}, Hom(\mathcal{L}, D(\mathcal{L})) \to C^{\cdot}(\mathcal{U}, Hom(\mathcal{L}, \mathcal{L})) \to 0.$$

Consider $(\sigma_i) \in C^0(\mathcal{U}, Hom(\mathcal{L}, D(\mathcal{L}))$, where σ_i is the homomorphism $\mathcal{L}|U_i \to D(\mathcal{L})|U_i$ which maps s_i to $(s_i, 0)$ (note that $\nabla_i(s_i) = 0$), i.e. $s_j = g_{ij}s_i$ to $(s_j, \frac{dg_{ij}}{g_{ij}} \otimes s_j)$. Then (σ_i) is mapped to $(\tau_i) \in C^0(\mathcal{U}, Hom(\mathcal{L}, \mathcal{L}))$ with $\tau_i = id : \mathcal{L}|U_i \to \mathcal{L}|U_i$.

The coboundary of (σ_i) is given by $\sigma_j - \sigma_i : \mathcal{L}|U_i \cap U_j \to D(\mathcal{L})|U_i \cap U_j$:

$$(\sigma_j - \sigma_i)(s_j) = (0, -\frac{dg_{ij}}{g_{ij}} \otimes s_j),$$

so $\sigma_j - \sigma_i$ can be identified with $-\frac{dg_{ij}}{g_{ij}} \in H^0(U_i \cap U_j, \Omega^1_X)$.

Note that the relation established in the preceding lemma is taken up to sign as definition of the Atiyah class in [18] Def. 4.2.18.

Corollary 1.6. *An invertible sheaf $\mathcal{L}$ admits a connection if and only if its image in $H^1(X, \Omega^1_X)$ is* 0.

This corollary is consequence of Lemmas 1.4 and 1.5. This coincides with our result from Theorem 1.1.

1.3. $Pic^{an}_c(X)$ and $Pic^{an}_{ci}(X)$

Recall that a connection ∇ is integrable if its curvature vanishes.

When $\mathcal{L} = \mathcal{O}_X$ and $\nabla(f) = df + f\omega$, the value of the curvature R_∇ of the connection ∇ on $\mathcal{L}$ is $d\omega$ (see I 3.2.2 of [5], p. 23).

More generally, recall that a connection is given by a $\mathbb{C}$-linear morphism:

$$\nabla^1 : \mathcal{L} \to \Omega^1_X \otimes \mathcal{L} = \Omega^1_X(\mathcal{L})$$

It defines a $\mathbb{C}$-linear morphism:

$$\nabla^2 : \Omega^1_X(\mathcal{L}) \to \Omega^2_X(\mathcal{L})$$

by the formula: $\nabla^2(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s)$ (see I (2.4) and (2.9) of [5]).

Definition 1.7. *The connection $\nabla = \nabla^1$ is said to be integrable if $\nabla^2 \circ \nabla^1 = 0$.*

In particular, if s is a global nowhere vanishing section of $\mathcal{L}$ and if ∇ is defined by ω with respect to s we have $R_\nabla(s') = d\omega \otimes s'$ for every section of $\mathcal{L}$. So ∇ is integrable if and only if $d\omega = 0$.

Obviously we have, similarly to Lemma 1.2:

Lemma 1.8. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module which is represented by a cocycle (g_{ij}) in $C^1(\mathcal{U}, \mathcal{O}^*_X)$. Then, an integrable connection ∇ on $\mathcal{L}$ is represented by an element (ω_i) in $C^0(\mathcal{U}, \Omega^1_X)$, ω_j closed, which is mapped by $\delta : C^0(\mathcal{U}, \Omega^1_X) \to C^1(\mathcal{U}, \Omega^1_X)$ onto $(\frac{dg_{ij}}{g_{ij}}) \in C^1(\mathcal{U}, \Omega^1_X)$.*

In particular the trivial connection d on $\mathcal{O}_X$ is integrable. As we did for the group $Pic^{an}_c(X)$, the isomorphism classes of analytic invertible sheaves with integrable connection form a group $Pic^{an}_{ci}(X)$ in which the neutral element is the class of $(\mathcal{O}_X, d)$ and the product of the classes of $(\mathcal{L}_1, \nabla_1)$ and of $(\mathcal{L}_2, \nabla_2)$ is the class of $(\mathcal{L}_1 \otimes \mathcal{L}_2, \nabla)$, where:

$$\nabla(s_1 \otimes s_2) = \nabla_1(s_1) \otimes s_2 + s_1 \otimes \nabla_2(s_2).$$

One can prove (see [5] using Théorème 2.17 Chap. I p. 12) that, if $(\mathcal{L}_1, \nabla_1)$ and $(\mathcal{L}_2, \nabla_2)$ are integrable connections, the connection:

$$(\mathcal{L}_1 \otimes \mathcal{L}_2, \nabla)$$

is also integrable. One can see this directly, too, using that the sum of closed forms is closed.

The curvature of a connection $(\mathcal{L}, \nabla)$ defines an $\mathcal{O}_X$-homomorphism:

$$\mathcal{L} \to \Omega^2_X \otimes \mathcal{L}$$

Now $\mathrm{Hom}(\mathcal{L}, \Omega_X^2 \otimes \mathcal{L}) \simeq H^0(X, Hom(\mathcal{L}, \Omega_X^2 \otimes \mathcal{L})) \simeq H^0(X, \Omega_X^2)$, so it is given by an element ω of $H^0(X, \Omega_X^2)$. If this cohomology group vanishes, we have $Pic_{ci}^{an}(X) \simeq Pic_c^{an}(X)$.

One can prove the following proposition also by Deligne cohomology, see below, but it is much easier to proceed directly.

Proposition 1.9. *Let X be a complex manifold. We have an exact sequence*

$$0 \to Pic_{ci}^{an}(X) \to Pic_c^{an}(X) \to H^0(X, \Omega_X^2).$$

Proof. Let $(\mathcal{L}, \nabla)$ be an integrable connection.

Assume this connection is isomorphic to the trivial connection $(\mathcal{O}_X, d)$, the class of the connection $(\mathcal{L}, \nabla)$ is therefore the class of the trivial connection. This means that the map $Pic_{ci}^{an}(X) \to Pic_c^{an}(X)$ is an injection.

The mapping $Pic_c^{an}(X) \to H^0(X, \Omega_X^2)$ associates the curvature of ∇ with the isomorphism class of $(\mathcal{L}, \nabla)$. It is well-defined: if $(\mathcal{L}, \nabla)$ and $(\mathcal{L}', \nabla')$ are isomorphic and if we take local sections of $\mathcal{L}$ and $\mathcal{L}'$ which correspond each other with respect to the isomorphism, the two connections are defined by the same differential forms with respect to these sections. The exactness at $Pic_c^{an}(X)$ is obvious.

In fact the following proposition shows that $Pic_{ci}^{an}(X)$ is of topological nature:

Proposition 1.10. *We have the isomorphism:*

$$Pic_{ci}^{an}(X) \simeq H^1(X, \mathbb{C}^*).$$

Proof. According to Théorème 2.17 in chapter I of [5] there is an equivalence of categories between the category of local systems of one-dimensional complex vector spaces on X with the category of line bundles with an integrable connection.

The resulting bijection is compatible with the group structure given by the tensor product.

We can observe that the group $H^1(X, \mathbb{C}^*)$ classifies the local systems of one dimensional complex vector spaces on X (see Theorem 3.3 of [23]), up to isomorphism, because the local transition functions are locally constant. The same is true for $Pic_{ci}^{an}(X)$ as mentioned at the beginning of this paragraph.

Corollary 1.11. *Let $f : X \to Y$ be a holomorphic map between two complex manifolds such that it induces an isomorphism $H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$, then:*

$$Pic_{ci}^{an}(X) \simeq Pic_{ci}^{an}(Y).$$

Proof: Note that

$$\mathrm{Ext}^1(H_0(X, \mathbb{Z}), \mathbb{C}^*) = 0,$$

because the abelian group $H_0(X, \mathbb{Z})$ is free, and the Universal coefficient formula implies

$$H^1(X, \mathbb{C}^*) \simeq Hom(H_1(X, \mathbb{Z}), \mathbb{C}^*).$$

So we get isomorphisms

$$Hom(H_1(X, \mathbb{Z}), \mathbb{C}^*) \simeq H^1(X, \mathbb{C}^*) \simeq Pic_{ci}^{an}(X)$$

1.4. **Relation to Deligne cohomology.** The preceding subsection is closely related to special cases of Deligne cohomology. We start by recalling the notion of Čech hypercohomology.

Let $\mathcal{S}^\cdot$ be a non-negative complex of sheaves of abelian groups on a topological space X. If $\mathcal{U}$ is an open covering of X we can define $\mathbb{H}^k(\mathcal{U}, \mathcal{S}^\cdot) := H^k(C^\cdot(\mathcal{U}, \mathcal{S}^\cdot)_{tot})$ where $(C^\cdot(\mathcal{U}, \mathcal{S}^\cdot))_{tot}$ is the total (or the simple) complex associated to the bi-graded complex $C^\cdot(\mathcal{U}, \mathcal{S}^\cdot)$ (see e.g. [3] p. 14, p. 28). Taking the direct limit with respect to open coverings $\mathcal{U}$, we get $\check{\mathbb{H}}^k(X, \mathcal{S}^\cdot) := \lim_{\rightarrow} \mathbb{H}^k(\mathcal{U}, \mathcal{S}^\cdot)$, see [3] p. 32. We can proceed in a slightly different way, similarly to [11] II 5.8

p.223 in the case of sheaves : we consider only open coverings $\mathcal{U} = (U_x)_{x \in X}$ with $x \in U_x$, put $\check{C}^{\cdot}(X, \mathcal{S}^{\cdot}) := \lim\limits_{\to} C^{\cdot}(\mathcal{U}, \mathcal{S}^{\cdot})$, then $\check{\mathbb{H}}^k(X, \mathcal{S}^{\cdot}) = H^k((\check{C}^{\cdot}(X, \mathcal{S}^{\cdot}))_{tot})$.

Now let X be as before and let $\mathcal{U} = (U_i)$ be an open covering of X. We assume that the U_i are Stein, which can be achieved by refinement. Let $Pic^{an}\,\mathcal{U}$ be the group of isomorphism classes of invertible $\mathcal{O}_X$-modules which are trivial on the U_i, and let $Pic_c^{an}\,\mathcal{U}$ be the group of isomorphism classes of such sheaves with connection. First, $Pic^{an}\,\mathcal{U} \simeq H^1(\mathcal{U}, \mathcal{O}_X^*)$.

Let $\mathcal{S}^{\cdot}$ be the non-negative complex:

$$\mathcal{O}_X^* \xrightarrow{g \mapsto \frac{dg}{g}} \Omega_X^1 \to 0 \to \dots.$$

Then we have a description of $Pic_c^{an}\, X$ as a (Čech) hypercohomology group:

Lemma 1.12. *a) $Pic_c^{an}\,\mathcal{U} \simeq \mathbb{H}^1(\mathcal{U}, \mathcal{S}^{\cdot})$.*
b) $Pic_c^{an}\, X \simeq \check{\mathbb{H}}^1(X, \mathcal{S}^{\cdot}) \simeq \mathbb{H}^1(X, \mathcal{S}^{\cdot})$ (cf. [3] Theorem 2.2.20, p. 80).

Proof: a) Argue as in the proof of Lemma 1.2 (See 2.2).
b) Take the direct limit with respect to open Stein coverings $\mathcal{U}$. The second isomorphism holds because X is paracompact (see [3] Theorem 1.3.13, p. 32).

As a consequence, we obtain the exact sequence of Theorem 1.1 again:
We have an exact sequence of complexes:

$$0 \to C^{\cdot+1}(\mathcal{U}, \Omega_X^1) \to (C^{\cdot}(\mathcal{U}, \mathcal{S}^{\cdot}))_{tot} \to C^{\cdot}(\mathcal{U}, \mathcal{O}_X^*) \to 0$$

Note that $H^1(V, \Omega_X^1) = 0$ for $V = U_{i_0} \cap \ldots \cap U_{i_q}$ because V is Stein: recall that the intersection of two open Stein subets is Stein, see [19] Prop. 51.7, p. 225. So we have exactness on the right.

This exact sequence induces a long exact cohomology sequence

$$\ldots \to H^k(\mathcal{U}, \mathcal{O}_X^*) \to H^k(\mathcal{U}, \Omega_X^1) \to \mathbb{H}^{k+1}(\mathcal{U}, \mathcal{S}^{\cdot}) \to H^{k+1}(\mathcal{U}, \mathcal{O}_X^*) \to \ldots$$

After this take the direct limit and replace Čech (hyper)cohomology by the usual one.

In fact, using Proposition 2.2 below we have an easier proof.

Now let us turn to Deligne cohomology. Let us recall its definition (see [8] p. 45). Put $\mathbb{Z}(p) := (2\pi i)^p \mathbb{Z} \subset \mathbb{C}$. Let $\mathbb{Z}(p)_{\mathcal{D}}$ be the following non-negative complex:

$$\mathbb{Z}(p)_X \to \Omega_X^0 \to \ldots \to \Omega_X^{p-1} \to 0 \to \ldots$$

where the first arrow is the inclusion. Then the Deligne cohomology $H_{\mathcal{D}}^*(X, \mathbb{Z}(p))$ is defined as the hypercohomology $\mathbb{H}^*(X, \mathbb{Z}(p)_{\mathcal{D}})$.

Looking at the commutative diagram

$$\begin{array}{ccccccc}
\mathbb{Z}(p)_X & \to & \mathcal{O}_X & \to & \Omega_X^1 & \to \ldots \to & \Omega_X^{p-1} \\
\downarrow & & \downarrow & & \downarrow \cdot(2\pi i)^{-p+1} & & \downarrow \cdot(2\pi i)^{-p+1} \\
0 & \to & \mathcal{O}_X^* & \stackrel{f \mapsto \frac{df}{f}}{\to} & \Omega_X^1 & \to \ldots \to & \Omega_X^{p-1}
\end{array}$$

where the second verical arrow is given by $f \mapsto exp((2\pi i)^{-p+1} f)$ we see that the complex above is quasi-isomorphic to

$$0 \to \mathcal{O}_X^* \stackrel{f \mapsto \frac{df}{f}}{\to} \Omega_X^1 \to \ldots \to \Omega_X^{p-1} \to 0 \to \ldots$$

For $p = 1$, we obtain that $\mathbb{Z}(1)_{\mathcal{D}}$ is quasi-isomorphic to $\mathcal{O}_X^*(-1)$, cf. [2] p. 2038, so $H_{\mathcal{D}}^1(X, \mathbb{Z}(1)) \simeq H^0(X, \mathcal{O}_X^*)$ and $H_{\mathcal{D}}^2(X, \mathbb{Z}(1)) \simeq Pic^{an}(X)$.

For $p = 2$, we get that $\mathbb{Z}(2)_{\mathcal{D}}$ is quasi-isomorphic to $\mathcal{S}^{\cdot}(-1)$, cf. [8] p. 46, so $Pic_c^{an}(X) \simeq H_{\mathcal{D}}^2(X, \mathbb{Z}(2))$ because of Lemma 1.12 (see the remark of Deligne quoted in [2] at the bottom of p. 2039).

For $p \geq \dim X + 1$ the complex is quasi-isomorphic to $0 \to \mathcal{O}_X^* \to d\mathcal{O}_X \to 0 \to \ldots$, see beginning of subsection 1.1; by Poincaré Lemma, it is also quasi-isomorphic to

$$0 \to \mathbb{C}_X^* \to 0 \to \ldots.$$

So $H^2_{\mathcal{D}}(X, \mathbb{Z}(p)) \simeq H^1(X, \mathbb{C}_X^*) \simeq Pic^{an}_{ci}(X)$, using Proposition 1.10.

For $p > 2$, $H^2_{\mathcal{D}}(X, \mathbb{Z}(p))$ does not depend on p:

Let $\pi : \mathbb{Z}(p+1)_{\mathcal{D}} \to \mathbb{Z}(p)_{\mathcal{D}}$ be the projection, then $\mathbb{H}^k(X, ker\,\pi) \simeq H^{k-p-1}(X, \Omega^p_X) = 0$, $k \leq 3$.

We obtain altogether, cf. [10] p. 156:

Lemma 1.13. *a)* $H^2_{\mathcal{D}}(X, \mathbb{Z}(1)) \simeq Pic^{an}(X)$.
b) $H^2_{\mathcal{D}}(X, \mathbb{Z}(2)) \simeq Pic^{an}_c(X)$.
c) $H^2_{\mathcal{D}}(X, \mathbb{Z}(p)) \simeq Pic^{an}_{ci}(X)$ *for* $p > 2$.

1.5. $Pic^{an}(X)$ and $Pic^{an}_{ci}(X)$

The first exact sequence of §1.1 gives a long exact sequence which fits into a commutative diagram:

Theorem 1.14. *We have a commutative diagram with exact rows:*

$$\begin{array}{ccccccccccccc}
0 & \to & H^0(X,\mathbb{C}_X^*) & \to & H^0(X,\mathcal{O}_X^*) & \to & H^0(X,d\mathcal{O}_X) & \to & Pic^{an}_{ci}(X) & \to & Pic^{an}(X) & \to & H^1(X,d\mathcal{O}_X) \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & H^0(X,\mathbb{C}_X^*) & \to & H^0(X,\mathcal{O}_X^*) & \to & H^0(X,\Omega^1_X) & \to & Pic^{an}_{c}(X) & \to & Pic^{an}(X) & \to & H^1(X,\Omega^1_X)
\end{array}$$

Proof. The exactness of the upper line is consequence of Proposition 1.10 and the exactness of the sequence $0 \to \mathbb{C}_X^* \to \mathcal{O}_X^* \to d\mathcal{O}_X \to 0$.

Since the vertical map $H^0(X, d\mathcal{O}_X) \to H^0(X, \Omega^1_X)$ is injective we conclude that

$$0 \to H^0(X, \mathbb{C}_X^*) \to H^0(X, \mathcal{O}_X^*) \to H^0(X, \Omega^1_X)$$

is exact. Because of Theorem 1.1 we conclude that the lower line is exact, too.

Remark: We may also argue using hypercohomology:

In the upper row compare $\mathcal{O}_X^* \to d\mathcal{O}_X \to 0$ with $\mathcal{O}_X^* \to 0$, in the lower row $\mathcal{O}_X^* \to \Omega^1_X \to 0$ with $\mathcal{O}_X^* \to 0$.

In particular, we observe that:

Lemma 1.15. *If the complex manifold X is compact with an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X and if ∇_1 and ∇_2 are two connections on $\mathcal{L}$ such that $(\mathcal{L}, \nabla_1) \simeq (\mathcal{L}, \nabla_2)$, we must have $\nabla_1 = \nabla_2$.*

Proof. We have $(\nabla_1 - \nabla_2)(s) = \omega \otimes s$ where $\omega \in H^0(X, \Omega^1_X)$ is mapped to $0 \in Pic^{an}_c(X)$. So there is $g \in H^0(X, \mathcal{O}_X^*)$ such that $\omega = \frac{dg}{g}$. Since $H^0(X, \mathbb{C}^*) = H^0(X, \mathcal{O}_X^*)$ because global functions on X are locally constant on a compact space, we have that $\omega = 0$.

Now let us drop the compactness condition again.

Lemma 1.16. *a) An element $x \in H^2(X, \mathbb{Z})$ is sent onto 0 in $H^2(X, \mathbb{C})$ if and only if it is the first Chern class of an invertible $\mathcal{O}_X$-module which can be endowed with an integrable connection.*
b) If X is Stein, an invertible sheaf $\mathcal{L}$ admits an integrable complex analytic connection on X if and only if the complex first Chern class vanishes.

Proof. a) We have a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z} & \to & \mathbb{C} & \to & \mathbb{C}^* & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \mathbb{Z} & \to & \mathcal{O}_X & \to & \mathcal{O}_X^* & \to & 0
\end{array}$$

with exact rows. This leads to a commutative diagram:

$$\begin{array}{ccc} H^1(X,\mathbb{C}^*_X) & \to & H^2(X,\mathbb{Z}) \\ \downarrow & & \downarrow \\ H^1(X,\mathcal{O}^*_X) & \to & H^2(X,\mathbb{Z}) \end{array}$$

The lower arrow associates to each invertible sheaf its first Chern class, therefore the upper arrow associates to each invertible sheaf with an integrable connection the first Chern class of the invertible sheaf. Now consider the upper row of the first diagram. It leads to an exact sequence:

$$H^1(X,\mathbb{C}^*) \to H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{C}),$$

which gives our result.

b) Note that we have $H^1(X,\mathcal{O}^*_X) \simeq H^2(X,\mathbb{Z})$, too, because $H^k(X,\mathcal{O}_X) = 0$, $k = 1, 2$.

Remark: We can make Proposition 1.9 more precise: There is an exact sequence

$$0 \to Pic^{an}_{ci}(X) \to Pic^{an}_c(X) \to H^0(X, d\Omega^1_X) \to H^2(X,\mathbb{C}^*_X)$$

Compare the non-negative complexes $\mathcal{O}^*_X \to d\mathcal{O}_X \to 0$ and $\mathcal{O}^*_X \to \Omega^1_X \to 0$, see subsection 1.4. The cokernel is quasi-isomorphic to $0 \to \Omega^1_X/d\mathcal{O}_X \to 0$, i.e. to $0 \to d\Omega^1_X \to 0$.

1.6. **Compact Kähler manifolds.** In the case X is a compact Kähler manifold, we can apply Hodge Theory.

We prefer an approach which can be transferred later on to the case of smooth complete complex algebraic varieties which might not be Kähler:

We have $H^k(X;\mathbb{C}) \simeq \mathbb{H}^k(X,\Omega^\cdot_X)$, by Poincaré lemma.

Let us look at the Hodge filtration F on $\Omega^\cdot$: let $F^p\Omega^\cdot$ be the subcomplex

$$0 \to \ldots \to 0 \to \Omega^p_X \to \Omega^{p+1}_X \to \ldots$$

The corresponding spectral sequence degenerates at E_1, cf. [6] p. 28, so $\mathbb{H}^{k-p}(X,\Omega^{\geq p}) = \mathbb{H}^k(X,F^p\Omega^\cdot)$ can be considered as a subspace $F^pH^k(X;\mathbb{C})$ of $H^k(X;\mathbb{C})$.

Let $\overline{F^p}H^k(X;\mathbb{C})$ be the image of $F^pH^k(X;\mathbb{C})$ under conjugation in $H^k(X;\mathbb{C})$. Assume $p+q = k$. Then $H^{p,q}(X) := F^pH^k(X;\mathbb{C}) \cap \overline{F^q}H^k(X;\mathbb{C}) \simeq F^pH^k(X;\mathbb{C})/F^{p+1}H^k(X;\mathbb{C}) \simeq H^q(X,\Omega^p_X)$.

In particular, $H^{1,1}(X)$ is a subspace of $H^2(X;\mathbb{C})$ which is isomorphic to $H^1(X,\Omega^1_X)$.

Then the first part of the following Lemma is well-known:

Lemma 1.17. *Let X be a compact Kähler manifold, $\mathcal{L}$ an invertible sheaf on X.*

a) (see [12] Ch. 3.3, p. 417) *The complex first Chern class $c_1(\mathcal{L})_{\mathbb{C}}$ of $\mathcal{L}$ is in $H^{1,1}(X)$.*

b) (see [1] Prop. 12, p. 196) *With the identifications above, $b(\mathcal{L}) = -2\pi i c_1(\mathcal{L})_{\mathbb{C}}$.*

Proof. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Z}_X & \to & \mathcal{O}_X & \xrightarrow{f\mapsto e^{2\pi i f}} & \mathcal{O}^*_X & \to & 0 \\ & & \downarrow \cdot 2\pi i & & \downarrow \cdot 2\pi i & & \downarrow & & \\ 0 & \to & \mathbb{C}_X & \to & \mathcal{O}_X & \to & d\mathcal{O}_X & \to & 0 \end{array}$$

We get a commutative diagram

$$\begin{array}{ccc} H^1(X,\mathcal{O}^*_X) & \to & H^2(X;\mathbb{Z}) \\ \downarrow & & \downarrow \cdot 2\pi i \\ H^1(X,d\mathcal{O}_X) & \to & H^2(X;\mathbb{C}) \\ \downarrow & & \\ H^1(X,\Omega^1_X) & & \end{array}$$

Note that $d\mathcal{O}_X$ is quasiisomorphic to $\Omega_X^{\geq 1}$, hence we may replace $H^1(X, d\mathcal{O}_X)$ by $F^1H^2(X;\mathbb{C})$. In particular, the middle horizontal arrow is injective.

a) Look at the images of (g_{ij}).

By [17] Theorem 4.3.1, p. 62, we have that the image in $H^2(X;\mathbb{C})$ is $2\pi i c_1(\mathcal{L})_{\mathbb{C}}$. The second commutative diagram shows that $2\pi i c_1(\mathcal{L})_{\mathbb{C}} \in F^1H^2(X;\mathbb{C})$. Since the first Chern class is real it is invariant under conjugation, so we obtain our statement.

b) By Lemma 1.5, the image of (g_{ij}) in $H^1(X, \Omega_X^1)$ is $-b(\mathcal{L})$. If we identify $H^{1,1}$ with $H^1(X, \Omega_X^1)$ we obtain our statement because of a).

Note that the proof of b) in [1] loc. cit. works only if dim $X = 1$ because it uses an exact sequence of the form

$$0 \to \mathbb{C}_X \to \mathcal{O}_X \to \Omega_X^1 \to 0$$

Now in the Kähler case we have a stronger result than Lemma 1.16:

Lemma 1.18. *Let X be a compact Kähler manifold, $\mathcal{L}$ an invertible sheaf on X. Then the following conditions are equivalent:*

a) $\mathcal{L}$ admits an integrable connection,

b) $\mathcal{L}$ admits a connection,

c) the first Chern class of $\mathcal{L}$ is a torsion element.

For b) $\Rightarrow$ c) cf. [3] Cor. 2.2.25.

Proof. That the first Chern class is a torsion element means that the complex first Chern class vanishes, because it is known that the cohomology group $H^2(X,\mathbb{Z})$ is finitely generated when X is compact, hence triangulable.

a) $\Leftrightarrow$ c): $\mathcal{L}$ admits an integrable connection if and only if the image of $\mathcal{L}$ in $H^1(X, d\mathcal{O}_X)$ vanishes, by Theorem 1.14.

The composition $Pic^{an}\, X \to H^1(X, d\mathcal{O}_X) \to H^2(X;\mathbb{C})$ is given by $[\mathcal{L}] \mapsto 2\pi i c_1(\mathcal{L})_{\mathbb{C}}$, see the proof of the preceding lemma.

To prove a) $\Leftrightarrow$ c) it is therefore sufficient to show that the mapping $H^1(X, d\mathcal{O}_X) \to H^2(X;\mathbb{C})$ is injective, which has been done in the preceding proof.

Now b) $\Leftrightarrow$ c), because we know that b) holds if and only if $b(\mathcal{L}) = 0$ by Lemma 1.4. The rest follows from the preceding lemma 1.17.

In the preceding Lemma 1.18, i.e. in the case of compact Kähler manifolds, we can sharpen the fact that a) $\Leftrightarrow$ b):

Theorem 1.19. *If X is a compact Kähler manifold, a connection on an invertible sheaf is integrable.*

Proof. We have another connection ∇' which is integrable, by Lemma 1.18. Then the difference of the connections is the multiplication by a form $\omega \in H^0(X, \Omega_X^1)$. By Hodge theory, the Hodge spectral sequence degenerates at E_1, so $d\omega = 0$. Hence the two connections have the same curvature, so the original connection must be integrable, too.

2. Algebraic case

2.1. Suppose now that X is a smooth complex algebraic variety. The underlying analytic space X^{an} is a paracompact complex manifold. One has an analogue of Theorem 1.14 but one has to be careful with the upper row because one has no longer a Poincaré lemma. In fact we have to replace the sheaf $d\mathcal{O}_X$ by the sheaf

$$^c\Omega^1_X := ker(d : \Omega^1_X \to \Omega^2_X)$$

of closed Pfaffian forms on X.

We will always use Zariski topology (even in the case of $H^0(X, \mathbb{C}^*_X)$ below) if we do not write X^{an}. However, $c_1(X) := c_1(X^{an})$.

We will see that the following theorem can be proved using an algebraic analogue of Deligne cohomology, too, i.e. using hypercohomology, but we can proceed in an elementary way:

Theorem 2.1. *Let X be a smooth complex algebraic variety. Then we have a commutative diagram with exact rows*

$$\begin{array}{ccccccccccc} 0 & \to & H^0(X, \mathbb{C}^*_X) & \to & H^0(X, \mathcal{O}^*_X) & \to & H^0(X, {}^c\Omega^1_X) & \to & Pic_{ci}(X) & \to & Pic(X) & \to & H^1(X, {}^c\Omega^1_X) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^0(X, \mathbb{C}^*_X) & \to & H^0(X, \mathcal{O}^*_X) & \to & H^0(X, \Omega^1_X) & \to & Pic_c(X) & \to & Pic(X) & \to & H^1(X, \Omega^1_X) \end{array}$$

Proof. We can no longer use the exact sequence of the beginning of section 1.1. Therefore we must proceed in a different way.

Let us check first that the lower row is exact.

Note that the sequence of sheaves: $0 \to \mathbb{C}^*_X \to \mathcal{O}^*_X \to \Omega^1_X$ is exact, where $\mathcal{O}^*_X \to \Omega^1_X$ is defined by $h \mapsto \frac{dh}{h}$. In fact:

Suppose that $h \in \mathcal{O}^*_{X,x}$, where x is a closed point of X, $\frac{dh}{h} = 0$: Then $h^{an} \in \mathcal{O}^*_{X^{an},x}$ is mapped to $0 \in \Omega^1_{X^{an},x}$, so h^{an} is constant, which implies that h is constant.

Therefore the sequence:

$$0 \to H^0(X, \mathbb{C}^*_X) \to H^0(X, \mathcal{O}^*_X) \to H^0(X, \Omega^1_X)$$

is exact.

The rest goes as in the proof of Theorem 1.1.

The upper row is treated in an analogous way. Note that the connection ∇ on $\mathcal{O}_X$:

$$\nabla(f) = df + f\omega$$

is integrable if and only if ω is closed, because the curvature of ∇ is $d\omega$.

Note that $0 \to \mathbb{C}^*_X \to \mathcal{O}^*_X \to {}^c\Omega^1_X \to 0$ is in general not exact, in contrast to the analytic case: take $X = \mathbb{C}^*$, $\omega := \frac{dz}{z} \in {}^c\Omega^1_X$.

Proposition 1.9 has an algebraic counterpart:

Proposition 2.2. *Let X be a non-singular complex algebraic variety. We have an exact sequence*

$$0 \to Pic_{ci}(X) \to Pic_c(X) \to H^0(X, \Omega^2_X).$$

The proof is similar to the one of Proposition 1.9.

2.2. **Use of Čech hypercohomology.** Similarly as in the analytic case (see §1.4) we can observe that $Pic_c(X)$ is isomorphic to the first Čech hypercohomology group $\check{\mathbb{H}}^1(X, \mathcal{S}^\cdot)$ of the complex $\mathcal{S}^\cdot$:

$$\mathcal{O}^*_X \to \Omega^1_X \to 0 \to \dots$$

on X (but not, up to a shift, of the complex $\mathbb{Z}_X \to \mathcal{O}_X \to {}^c\Omega^1_X \to 0 \to \dots$).

For Čech hypercohomology, we refer to subsection (1.4).

More precisely:

Lemma 2.3. *If X is a non-singular complex variety, we have:*

$$Pic_c(X) \simeq \check{\mathbb{H}}^1(X, \mathcal{S}^{\cdot}) \simeq \mathbb{H}^1(X, \mathcal{S}^{\cdot}).$$

Proof: Let $\mathcal{U} = (U_i)$ be a covering of X by open Zariski subsets of X. An element of $\mathbb{H}^1(\mathcal{U}, \mathcal{S}^{\cdot})$ is given by an element $((\omega_i), (g_{ij})) \in C^0(\mathcal{U}, \Omega^1_X) \oplus C^1(\mathcal{U}, \mathcal{O}^*_X)$ such that (g_{ij}) is a cocycle, i.e. $g_{ij} = g_{ik}g_{kj}$ on $U_i \cap U_j \cap U_k$, and $\omega_j - \omega_i = \frac{dg_{ij}}{g_{ij}}$ on $U_i \cap U_j$.

Assume now that $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module on X which is endowed with a connection ∇. There is a Zariski open covering $\mathcal{U}$ of X such that for each U_i we have a trivialization of $\mathcal{L}|U_i$. Then $\mathcal{L}$ is represented by some cocycle (g_{ij}) in $C^1(\mathcal{U}, \mathcal{O}^*_X)$, and $\nabla|U_i$ corresponds to a connection $g \mapsto dg + g\omega_i$ on $\mathcal{O}_{U_i}$. Then $\omega_j - \omega_i = \frac{dg_{ij}}{g_{ij}}$ on $U_i \cap U_j$, so we obtain an element of $\mathbb{H}^1(\mathcal{U}, \mathcal{S}^{\cdot})$, hence of $\check{\mathbb{H}}^1(X, \mathcal{S}^{\cdot})$.

On the other hand, an element of $\check{\mathbb{H}}^1(X, \mathcal{S}^{\cdot})$ comes from an element of $\mathbb{H}^1(\mathcal{U}, \mathcal{S}^{\cdot})$ which is represented by a cocycle (g_{ij}) and (ω_i) for a suitable open Zariski covering $\mathcal{U}$ of X. Then (g_{ij}) defines an invertible $\mathcal{O}_X$-module $\mathcal{L}$, and (ω_i) defines a connection on $\mathcal{L}$.

One verifies that one obtains well-defined mappings between $Pic_c(X)$ and $\check{\mathbb{H}}^1(X, \mathcal{S}^{\cdot})$. We obtain $Pic_c(X) \simeq \check{\mathbb{H}}^1(X, \mathcal{S}^{\cdot})$.

Now in the case of sheaves we have isomorphisms $\check{H}^k \to H^k$ for $k = 0, 1$, see [11] II 5.9 Corollaire, p. 227 (note that X is not paracompact and that we are not only dealing with coherent algebraic sheaves!). This result still holds in the case of hypercohomology, as shown in the following proposition. So our lemma is proved.

Proposition 2.4. *Let X be a topological space and $\mathcal{S}^{\cdot}$ a non-negative complex of sheaves of abelian groups on X. Then the homomorphism $\check{\mathbb{H}}^k(X, \mathcal{S}^{\cdot}) \to \mathbb{H}^k(X, \mathcal{S}^{\cdot})$ is bijective for $k \leq 1$ and injective for $k = 2$.*

Proof: (i) First we may reduce to the case that $\mathcal{S}^{\cdot}$ is a bounded complex:
Choose $p > 0$. Let $\pi : \mathcal{S}^{\cdot} \to \mathcal{S}^{\leq p-1}$ be the canonical projection. Then the exact sequence $0 \to ker\, \pi \to \mathcal{S}^{\cdot} \to \mathcal{S}^{\leq p-1} \to 0$ of *presheaf*(!) complexes yields a short exact sequence of double complexes:

$$\check{C}^{\cdot}(X, ker\pi) \to \check{C}^{\cdot}(X), \mathcal{S}^{\cdot}) \to \check{C}^{\cdot}(X, \mathcal{S}^{\leq p-1}) \to 0$$

cf. [11] II Th. 5.8.1, p. 204, hence a long exact sequence

$$\check{\mathbb{H}}^{q-1}(X, \mathcal{S}^{\leq p-1}) \to \check{\mathbb{H}}^{q-p}(X, \mathcal{S}^{\geq p}) \to \check{\mathbb{H}}^q(X, \mathcal{S}^{\cdot}) \to \check{\mathbb{H}}^q(X, \mathcal{S}^{\leq p-1}) \to \check{\mathbb{H}}^{q-p+1}(X, \mathcal{S}^{\geq p})$$

Now put $p := 4$. Since $\check{\mathbb{H}}^q(X, \mathcal{S}^{\geq p}) = 0$ for $q < 0$ we obtain $\check{\mathbb{H}}^q(X, \mathcal{S}^{\cdot}) \simeq \check{\mathbb{H}}^q(X, \mathcal{S}^{\leq p-1})$ for $q \leq 2$. The same holds for $\mathbb{H}$ instead of $\check{\mathbb{H}}$.

(ii) So we may assume that $\mathcal{S}^{\cdot}$ is a bounded complex. Then we proceed by induction on the length of the complex, the case where the length is 0 being trivial.

Induction step: We may assume that $\mathcal{S}^0 \neq 0$. Putting $p = 1$ we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\check{H}^{q-1}(X, \mathcal{S}^0) & \to & \check{\mathbb{H}}^{q-1}(X, \mathcal{S}^{\geq 1}) & \to & \check{\mathbb{H}}^q(X, \mathcal{S}^{\cdot}) & \to & \check{H}^q(X, \mathcal{S}^0) & \to & \check{\mathbb{H}}^q(X, \mathcal{S}^{\geq 1}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^{q-1}(X, \mathcal{S}^0) & \to & \mathbb{H}^{q-1}(X, \mathcal{S}^{\geq 1}) & \to & \mathbb{H}^q(X, \mathcal{S}^{\cdot}) & \to & H^q(X, \mathcal{S}^0) & \to & \mathbb{H}^q(X, \mathcal{S}^{\geq 1})
\end{array}$$

Using the fact that the case of a sheaf is established by [11] p. 227, see above, and the Five Lemma we obtain the induction step.

Remark: The proof of the preceding Theorem gives the following exact sequence:

$$\check{H}^0(X, \mathcal{O}^*_X) \to \check{H}^0(X, \Omega^1_X) \to \check{\mathbb{H}}^1(X, \mathcal{S}^{\cdot}) \to \check{H}^1(X, \mathcal{O}^*_X) \to \check{H}^1(X, \Omega^1_X)$$

This exact sequence can also be obtained as follows:

Look at the exact sequence of *presheaf* (!) complexes:

$$0 \to \Omega^1_X\{1\} \to \mathcal{S}^{\cdot} \to \mathcal{O}^*_X\{0\} \to 0$$

where, for any presheaf $\mathcal{T}$, the complex $\mathcal{T}\{k\}$ denotes the complex $\mathcal{T}^{\cdot}$ with $\mathcal{T}^l = \mathcal{T}$ for $l = k$ and $= 0$ otherwise.

This gives the long exact Čech cohomology sequence in question.

We can proceed in the same way to prove the exactness of the upper line of the diagram of Theorem 2.1 by replacing Ω^1_X by ${}^c\Omega^1_X$. See Remark after Theorem 1.14.

We have special cases:

Lemma 2.5. *Let X be complete, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module on X.*

a) $Pic(X) \simeq Pic^{an}(X^{an})$, similarly for Pic_c, Pic_{ci}.

b) $\mathcal{L}$ admits an integrable connection if and only if $c_1(\mathcal{L})$ is a torsion element.

c) Every connection on $\mathcal{L}$ is integrable, so $Pic_c(X) \simeq Pic_{ci}(X)$.

Proof: a) This follows from GAGA (see [22] and also [20] p. 152/153) if X is projective. In general, use [13] Théorème 4.4 instead of [22].

Instead of [20] we can also compare 2.1 and 2.2 with the corresponding analytic statements. b), c): If X is projective we know that X^{an} is compact Kähler, so the results follows by GAGA and Lemma 1.18, Theorem 1.19.

In general we know by [7] §5 that we can still apply Hodge theory to X, so Lemma 1.18 and Theorem 1.19 still hold. In fact, the Hodge filtration is still defined via $\Omega^{\cdot}_X$.

For part b) of the lemma it will turn out that the hypothesis that X is complete is unnecessary, see Corollary 2.11 below. For c) we must in general restrict to regular connections, see below (Theorem 2.13).

Remember that compact Kähler manifolds are not automatically algebraic, cf. the case of complex tori, see [21] Cor. p. 35.

Lemma 2.6. *Let X be affine. Then every invertible $\mathcal{O}_X$-module on X admits a connection.*

Proof: Obvious from Theorem 2.1, because $H^1(X, \Omega^1_X) = 0$.

2.3. **Regularity.** It is useful to take the notion of regularity into account.

The regularity has been introduced by P.Deligne in [5] Chap II §4. For the sake of convenience we define here the regularity of integrable connections on an invertible sheaf:

Definition 2.7. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and ∇ an integrable connection on $\mathcal{L}$. Then ∇ is called regular if there exists a smooth compactification $\bar{X}$ of X such that $D := \bar{X} \setminus X$ is a divisor with normal crossings and that, for all $x \in D$, there exists an open Zariski neighbourhood V of x and there exists $s \in H^0(V, j_*\mathcal{L})$, s nowhere vanishing on $V' := V \setminus D$, such that $\nabla(s|V') = (\alpha|V') \otimes (s|V')$ with $\alpha \in H^0(V, \Omega^1_{\bar{X}}(log\, D))$. Here $j : X \to \bar{X}$ is the inclusion.*

Note that we can replace:

"there exists $s \in H^0(V, j_*\mathcal{L}), s$ nowhere vanishing on $V' = V \setminus D$, such that $\nabla(s|V') = (\alpha|V') \otimes s|V'$"

by

"for any $s \in H^0(V, j_*\mathcal{L}), s$ nowhere vanishing on $V' = V \setminus D$, we have $\nabla(s|V') = (\alpha|V') \otimes s|V'$". Here it is important that we deal with invertible sheaves!

In fact, let $s, s' \in H^0(V, j_*\mathcal{L})$, s, s' nowhere vanishing on $V' = V \setminus D$. Then $s' = hs$, where h is a rational function on V which has neither zeroes nor poles inside V'. If $\nabla(s|V') = (\alpha|V') \otimes (s|V')$ with $\alpha \in H^0(V, \Omega^1_{\bar{X}}(\log D))$ we get $\nabla(s'|V') = (\alpha'|V') \otimes (s'|V')$ with $\alpha' = \frac{dh}{h} + \alpha \in H^0(V, \Omega^1_{\bar{X}}(\log D))$.

As P. Deligne noticed, the notion of regularity does not depend on the compactification of X such that the divisor at ∞ is a normal crossing divisor (see [5] p. 90).

We can define the Picard group $Pic_{cir}X$ of regular integrable connections in an obvious way. Now let us fix a compactification $\bar{X}$ of X as in the preceding definition.

Lemma 2.8. *There is an exact sequence:*

$$H^0(X, \mathcal{O}^*_X) \to H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(\log D)) \to Pic_{cir}(X) \to Pic(X) \to H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(\log D))$$

Proof: The proof is analogous to the proof of Theorem 1.1.

But first observe that, for any invertible $\mathcal{O}_X$-module $\mathcal{L}$, there is a Zariski open covering $\mathcal{U} = (\bar{U}_i)$ of $\bar{X}$ such that the restriction of $\mathcal{L}$ to $U_i = \bar{U}_i \setminus D$ is trivial.

For this, we may assume that X is connected, hence irreducible. One considers a non-empty and therefore dense Zariski open subspace U of X on which $\mathcal{L}$ is trivial. On U, the restriction $\mathcal{L}|U$ has a nowhere vanishing section s. This section extends to $\bar{X}$ as a rational section s_1 of $\mathcal{L}$. Let D_1 be the divisor of this section - this makes sense because $\mathcal{L}$ is locally trivial. Now D_1 extends to a divisor $\bar{D}_1$ on $\bar{X}$. For any $x \in \bar{X}$ there is an open affine neighbourhood $\bar{V}$ such that $\bar{D}_1|\bar{V}$ is a principal divisor, i.e. divisor of some rational function ϕ_x. Then $\phi_x^{-1}s_1$ is a nowhere vanishing section of $\mathcal{L}|V$ with $V := \bar{V} \setminus D$; it gives a trivialization of $\mathcal{L}|V$.

The first arrow is induced by the homomorphism $j_*\mathcal{O}^*_X \to {}^c\Omega^1_{\bar{X}}(\log D)$ which is defined as follows. Locally, a section g of $j_*\mathcal{O}^*_X$ is of the form $h^{-1}\tilde{g}$, where $h, \tilde{g}$ are regular functions which do not vanish inside X. Then the image is defined to be $\frac{dg}{g} = \frac{d\tilde{g}}{\tilde{g}} - \frac{dh}{h}$ which is indeed a closed logarithmic form.

Assume now that $g \in H^0(X, \mathcal{O}^*_X)$ is given. Then the image in $Pic_{cir}(X)$ is given by $\mathcal{O}_X$, together with the connection $f \mapsto df + f\frac{dg}{g}$. This is isomorphic to $\mathcal{O}_X$, together with the connection $f \mapsto df$, so we have the trivial element of $Pic_{cir}(X)$.

On the other hand, suppose that $\omega \in H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(\log D))$ is mapped onto the trivial element of $Pic_{cir}(X)$. Then there is a $g \in H^0(X, \mathcal{O}^*_X)$ such that $\omega = \frac{dg}{g}$.

This shows the exactness at $H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(\log D))$.

Then, an element of $Pic\, X$ is represented by a cocycle (g_{ij}) on a covering $\mathcal{U}$ as defined before. This covering comes from an affine covering $\bar{\mathcal{U}}$ of $\bar{X}$, where each g_{ij} extends as a rational function with poles inside D which is a regular and non-vanishing function on $\bar{U}_i \cap \bar{U}_j \setminus D$. Then $\frac{dg_{ij}}{g_{ij}}$ is a closed logarithmic form on $\bar{U}_i \cap \bar{U}_j$: After refining $\mathcal{U}$ if necessary we may assume that we can write $g_{ij} = h_{ij}^{-1}\tilde{g}_{ij}$ where h_{ij} and $\tilde{g}_{ij}$ are regular on $\bar{U}_i \cap \bar{U}_j$ and without zeroes in $U_i \cap U_j$. Then:

$$\frac{dg_{ij}}{g_{ij}} = \frac{d\tilde{g}_{ij}}{\tilde{g}_{ij}} - \frac{dh_{ij}}{h_{ij}}$$

is a closed logarithmic form. This defines the map:

$$Pic\, X \to H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(\log D)).$$

On the other hand, a regular integrable connection on $\mathcal{O}_X$ is of the form $g \mapsto dg + g\omega$ with $\omega \in H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(\log D))$, and the map from $H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(\log D))$ into $Pic_{cir}(X)$ is given by:

$$\omega \mapsto (\mathcal{O}_X, \nabla)$$

where $\nabla(g) = dg + g\omega$. Then, the composition:

$$H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D)) \to Pic_{cir}(X) \to Pic\, X$$

is zero. Let $(\mathcal{L}, \nabla)$ a regular integrable connection on the invertible $\mathcal{O}_X$-module $\mathcal{L}$ where $\mathcal{L}$ is isomorphic to $\mathcal{O}_X$. The pair $(\mathcal{L}, \nabla)$ is isomorphic to $(\mathcal{O}_X, \nabla_0)$ for some connection ∇_0, and there is a closed logarithmic form $\omega \in H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$, such that $\nabla_0(g) = dg + g\omega$. This proves the exactness of the sequence at $Pic_{cir}(X)$.

Now fix an element of $Pic\, X$ whose image in $H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$ is trivial. Such an element is given by an affine covering $\mathcal{U}$ and a cocycle

$$(\frac{dg_{ij}}{g_{ij}})$$

such that:

$$\frac{dg_{ij}}{g_{ij}} = \omega_j - \omega_i$$

where ω_i is a closed form in ${}^c\Omega^1_{\bar{X}}(log\, D)$ over the Zariski open subset U_i of X.

As we did in the proof of Theorem 1.1, the element (ω_i) defines a regular integrable connection ∇ on an invertible $\mathcal{O}_X$-module $\mathcal{L}$ and the image of the isomorphism class of $\mathcal{L}$ is the element of $H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$ given by the cocycle $(\frac{dg_{ij}}{g_{ij}})$.

It remains to prove that the composition:

$$Pic_{cir}(X) \to Pic\, X \to H^1(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D))$$

is zero. As in the proof of Theorem 1.1, an element of $Pic_{cir}(X)$ is given by $(\mathcal{L}|U_i, \nabla|U_i)_i$ such that $(\mathcal{L}|U_i, \nabla|U_i)$ is isomorphic over the Zariski open subspace U_i to $(\mathcal{O}_{U_i}, \tilde{\nabla}_i)$ where:

$$\tilde{\nabla}_i(f) = df + \omega_i f$$

for some $\omega_i \in H^0(U_i, {}^c\Omega^1_{\bar{X}}(log\, D))$, and, if the element (g_{ij}) is the cocycle which defines $\mathcal{L}$, we have:

$$\frac{dg_{ij}}{g_{ij}} = \omega_j - \omega_i.$$

Since the forms ω_i are closed, reasoning as in the proof of Theorem 1.1, we obtain our assertion.

Remarks. 1. In fact, at the beginning we have shown that $j_*\mathcal{L}$ is an invertible $j_*\mathcal{O}_X$-module, $j : X \to \bar{X}$ being the inclusion.
2. Again we can prove the lemma by showing that $Pic_{cir}X \simeq \check{\mathbb{H}}^1(\bar{X}, \mathcal{T}^\cdot) \simeq \mathbb{H}^1(\bar{X}, \mathcal{T}^\cdot)$, where $\mathcal{T}^\cdot$ is the non-negative complex

$$j_*\mathcal{O}^*_X \overset{g \mapsto \frac{dg}{g}}{\longrightarrow} {}^c\Omega^1_{\bar{X}}(log\, D) \longrightarrow 0 \longrightarrow \dots$$

with $j : X \to \bar{X}$ being the inclusion.

In this context it is useful to have:

Lemma 2.9. *$Pic\, X \simeq H^1(\bar{X}, j_*\mathcal{O}^*_X)$.*

Proof: It is sufficient to show that $R^1 j_*\mathcal{O}^*_X = 0$. An element of $(R^1 j_*\mathcal{O}^*_X)_x$ is represented by an element of $H^1(U \cap X, \mathcal{O}^*_X)$, where U is an open neighbourhood of x, so by a line bundle $\mathcal{L}$ on $U \cap X$. After shrinking U if necessary we know that $\mathcal{L}$ is trivial, by the proof of Lemma 2.8. This implies our assertion.

Theorem 2.10. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X. Then $\mathcal{L}$ admits a regular integrable connection if and only if its first Chern class is a torsion element.*

Proof: Since the integral cohomology of X is an abelian group of finite type, the implication $\Rightarrow$ is proved by Lemma 1.16.

Now, consider the implication $\Leftarrow$.

Suppose first that $c_1(\mathcal{L}) = 0$.

Let $\bar{X}$ be a smooth compactification of X such that $D := \bar{X} \setminus X$ is a normal crossing divisor. Suppose that D has r irreducible components. Then $\mathcal{L}$ extends to an algebraic invertible sheaf $\mathcal{L}'$ on $\bar{X}$ with first Chern class $c_1(\mathcal{L}') = 0$.

To prove this, we consider the diagram with exact rows:

$$\begin{array}{ccccccc} \mathbb{Z}^r & \to & Pic\,\bar{X} & \to & Pic\,X & \to & 0 \\ \downarrow\simeq & & \downarrow c_1 & & \downarrow c_1 & & \\ H^2(\bar{X}^{an}, X^{an}; \mathbb{Z}) & \to & H^2(\bar{X}^{an}; \mathbb{Z}) & \to & H^2(X^{an}; \mathbb{Z}) & & \end{array}$$

Let $[\mathcal{L}]$ be the class of $\mathcal{L}$. We have assumed that its first Chern class is $c_1(\mathcal{L}) = 0$. Let $\mathcal{L}_1$ be a invertible $\mathcal{O}_{\bar{X}}$-module whose class has its image equal to $[\mathcal{L}]$. The first Chern class of $\mathcal{L}_1$ comes from an element of $H^2(\bar{X}^{an}, X^{an}; \mathbb{Z})$ which corresponds to an element of $\mathbb{Z}^r$ whose image in $Pic\,\bar{X}$ is $\mathcal{L}_2$ which has the same first Chern class as $\mathcal{L}_1$. The invertible sheaf $\mathcal{L}' := \mathcal{L}_1 \otimes \mathcal{L}_2^{-1}$ has a first Chern class equal to 0 and it extends $\mathcal{L}$.

On the complete non-singular variety $\bar{X}$ we have obtained an invertible sheaf $\mathcal{L}'$ which extends $\mathcal{L}$ and has first Chern class $c_1(\mathcal{L}') = 0$. By Lemma 2.5 the invertible sheaf $\mathcal{L}'$ is endowed with a integral connection ∇'. The restriction of ∇' to $\mathcal{L}$ is a regular integral connection.

If $c_1(\mathcal{L}) = c$, c being a torsion element, by Lemma 1.16 there is an analytic invertible sheaf $\mathcal{L}'$ with integrable connection on X^{an} having c as first Chern class. By Deligne's existence theorem (Théorème 5.9 Chap. II of [5] p. 97) we can find an invertible sheaf $\mathcal{L}_1$ on X with an integrable connection such that $\mathcal{L}_1^{an} = \mathcal{L}'$, so $c_1(\mathcal{L}_1) = c$. Now $c_1(\mathcal{L} \otimes (\mathcal{L}_1)^{-1}) = 0$, so by the preceding result there is a regular integrable connection on $\mathcal{L} \otimes (\mathcal{L}_1)^{-1}$. So we get a regular integrable connection on $\mathcal{L} = \mathcal{L}_1 \otimes (\mathcal{L}' \otimes (\mathcal{L}_1)^{-1})$, too.

Therefore if the first Chern class of $\mathcal{L}$ is a torsion element, the invertible sheaf $\mathcal{L}$ admits a regular integrable connection.

Corollary 2.11. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. Then the following conditions are equivalent:*

1. *$\mathcal{L}$ admits a regular integrable connection;*
2. *$\mathcal{L}$ admits an integrable connection;*
3. *$\mathcal{L}^{an}$ admits an analytic integrable connection;*
4. *the first Chern class $c_1(\mathcal{L})$ of $\mathcal{L}$ is a torsion element.*

2.4. **Remark on integrability and regularity.** One may define a notion of regularity for connections which does not suppose that the connection is integrable - at least in the case of invertible sheaves.

This may seem to be superfluous because we will see that such a connection is automatically integrable. The situation changes, however, if we generalize the notions of regularity by asking regularity with respect to a partial compactification only.

Definition 2.12. *Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module and ∇ a connection on $\mathcal{L}$. Then ∇ is called regular if there exists a smooth compactification $\bar{X}$ of X such that $D := \bar{X} \setminus X$ is a divisor with normal crossings and that, for all $x \in D$ there exists an affine neighbourhood V of x and there exists $s \in H^0(V, j_*\mathcal{L})$ which does not vanish on D, such that $\nabla(s|V) = (\alpha|V) \otimes s|V$ with $\alpha \in H^0(V, \Omega^1_{\bar{X}}(log\,D))$. Here $j : X \to \bar{X}$ is the inclusion.*

As in the definition of a regular integrable connection we may again replace "there exists s... such that..." by "for all s... we have...".

The independence of the compactification will follow from the next theorem.

We can define the group $Pic_{cr}X$ of isomorphism classes of invertible $\mathcal{O}_X$-modules with a regular connection in an obvious way.

In fact, such a regular connection is automatically integrable, because we have:

Theorem 2.13. *If $\mathcal{L}$ is an invertible $\mathcal{O}_X$-module, every regular connection on $\mathcal{L}$ is integrable.*

Proof: We proceed as in the proof of Theorem 1.19.

First we show that the mapping:

$$Pic_{cir}X \to Pic_{cr}X$$

is surjective. In fact, we have the following Lemma:

Lemma 2.14. *There is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
H^0(X,\mathcal{O}_X^*) & \to & H^0(\bar{X},{}^c\Omega^1_{\bar{X}}(log\,D)) & \to & Pic_{cir}(X) & \to & Pic(X) & \to & H^1(\bar{X},{}^c\Omega^1_{\bar{X}}(log\,D)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^0(X,\mathcal{O}_X^*) & \to & H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D)) & \to & Pic_{cr}(X) & \to & Pic(X) & \to & H^1(\bar{X},\Omega^1_{\bar{X}}(log\,D))
\end{array}$$

Proof: As in Lemma 2.8, the proof is analogous to the proof of Theorem 1.1.

The upper line is exact, as we saw in Lemma 2.8. Concerning the lower row, we define the map $Pic\,X \to H^1(\bar{X},\Omega^1_{\bar{X}}(log\,D))$ as the composition $Pic\,X \to H^1(\bar{X},{}^c\Omega^1_{\bar{X}}(log\,D)) \to H^1(\bar{X},\Omega^1_{\bar{X}}(log\,D))$.

The map $H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D)) \to Pic_{cr}(X)$ is given by:

$$\omega \mapsto (\mathcal{O}_X,\nabla)$$

where the connection ∇ is defined by $\nabla(f) = df + f\omega$. This defines a connection on $\mathcal{O}_X$ which is regular since $\omega \in H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D))$. Therefore, the composition:

$$H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D)) \to Pic_{cr}(X) \to Pic\,X$$

is zero.

Let $(\mathcal{L},\nabla)$ be an invertible sheaf with a regular connection whose image is zero in $Pic(X)$. Then $\mathcal{L}$ is isomorphic to the trivial invertible sheaf $\mathcal{O}_X$ and there is a connection ∇_0 on $\mathcal{O}_X$ such that $(\mathcal{L},\nabla)$ is isomorphic to $(\mathcal{O}_X,\nabla_0)$. So ∇_0 is a regular connection. On the other hand there is a global form ω on X, such that $\nabla_0(f) = df + f\omega$. If ∇_0 is regular, one can choose the form ω as a global rational form on $\bar{X}$ in $H^0(\bar{X},\Omega^1_{\bar{X}}(log\,D))$. Then the lower row is exact at $Pic_{cr}(X)$.

Now, let us check the exactness at $Pic(X)$. Let $\bar{\mathcal{U}} = \bar{U}_i$ be an affine covering of $\bar{X}$ as in the proof of Lemma 2.8, such that (U_i) is a covering of X and $(\mathcal{L}|U_i,\nabla|U_i = \nabla_i)$ is isomorphic to $(\mathcal{O}_X|U_i,\tilde{\nabla}_i)$, where:

$$\tilde{\nabla}_i(f) = df + f\omega_i$$

with a rational differential form ω_i defined on $\bar{U}_i$ with poles contained in D. On this covering (U_i) of X, the invertible sheaf $\mathcal{L}$ defines the cocycle (g_{ij}) and its image in $H^1(\bar{X},\Omega^1_{\bar{X}}(log\,D))$ is the cocycle $\frac{d\hat{g}_{ij}}{\hat{g}_{ij}}$ defined by the rational functions on the covering $(\bar{U}_i)$ which extend (g_{ij}) and, again:

$$\frac{d\hat{g}_{ij}}{\hat{g}_{ij}} = \omega_j - \omega_i.$$

If the image of the class of $\mathcal{L}$ in $H^1(\bar{X}, \Omega^1_{\bar{X}}(log\, D))$ is trivial, we have:

$$\frac{d\hat{g}_{ij}}{\hat{g}_{ij}} = \omega_j - \omega_i.$$

where $\hat{g}_{ij}$ is a rational function which extends g_{ij} to $\bar{X}$ and ω_i is a logarithmic differential form along D on $\bar{U}_i$. The invertible sheaf $\mathcal{L}$ is endowed with a regular connection ∇ locally defined on U_i by:

$$\tilde{\nabla}_i(f) = df + f\omega_i.$$

This ends the proof of Lemma 2.14.

Then, we have:

Lemma 2.15. $H^0(\bar{X}, {}^c\Omega^1_{\bar{X}}(log\, D)) = H^0(\bar{X}, \Omega^1_{\bar{X}}(log\, D))$

Proof. We know that the spectral sequence $E_1^{pq} = H^q(\bar{X}, \Omega^p_{\bar{X}}(log\, D)) \to H^{p+q}(X^{an}; \mathbb{C})$ degenerates at E_1 (see [6] Corollaire 3.2.13 page 38), so the mapping:

$$H^0(\bar{X}, \Omega^1_{\bar{X}}(log\, D)) \xrightarrow{d} H^0(\bar{X}, \Omega^2_{\bar{X}}(log\, D))$$

is the zero map which precisely means that the forms in $H^0(\bar{X}, \Omega^1_{\bar{X}}(log\, D))$ are closed as stated in the lemma.

This proves the Lemma.

Proof of Theorem 2.13:

The two preceding lemmas show that $Pic_{cir}(X) \to Pic_{cr}(X)$ is surjective.

Now let ∇ be a regular connection on $\mathcal{L}$. Because of the surjectivity just mentioned there is a line bundle $\mathcal{L}'$ on X and an integrable regular connection ∇' on $\mathcal{L}'$ such that $\mathcal{L}' \simeq \mathcal{L}$; we may assume moreover that $\mathcal{L}' = \mathcal{L}$. Then $\nabla(s) = \nabla'(s) + \omega \otimes s$ with $\omega \in H^0(\bar{X}, \Omega^1_{\bar{X}}(log\, D))$. Because of the last lemma: $d\omega = 0$, hence ∇ is integrable, too.

Remark: Since $Pic_{cir}\, X = Pic_{cr}\, X$ we have $Pic_{cir}\, X \simeq \mathbb{H}^1(X, \tilde{\mathcal{T}}^{\cdot})$, where $\tilde{\mathcal{T}}^{\cdot}$ is the complex $j_*\mathcal{O}^*_X \to \Omega^1_{\bar{X}}(log\, D) \to 0 \to \ldots$.

So it may seem that discussing regular connections without the hypothesis of integrability was useless.

What is useful, however, is the lower exact sequence of Lemma 2.14.

Furthermore let us look at the following situation: $X \subset \bar{X}$, $\bar{X}$ being a smooth complex algebraic variety which is not assumed to be complete, $D := \bar{X} \setminus X$ divisor with normal crossings. Let ∇ be connection on an invertible sheaf on X. Then we may define when ∇ is regular resp. regular integrable with respect to D in an obvious way. In the case $\bar{X} = X$ this means that no regularity condition is imposed at all, so we can no longer expect coincidence of the two notions.

3. Some examples

In the following examples we only consider complex algebraic varieties.

3.1. For the complex projective line, the invertible sheaf $\mathcal{O}(k)$ has no connection whenever $k \neq 0$. Consider $X = \mathbb{P}^1$. One knows that $Pic(X) = \mathbb{Z}$. We shall see that $Pic_{ci}(X) \simeq Pic_c(X) = \{0\}$.

In fact, as we have proved in the section 2, for any compact connected complex Kähler manifold X (in particular any complex projective variety without singularities) we have:

$$Pic^{an}_{ci}(X) \simeq Pic^{an}_c(X) \simeq H^1(X, \mathbb{C}^*).$$

For the complex line $\mathbb{P}^1$ the cohomology $H^1(X^{an}, \mathbb{C}^*) = 0$. By GAGA (see [22], [20]) we have $Pic_{ci}(X) \simeq Pic_{ci}^{an}(X^{an})$ and $Pic_c(X) \simeq Pic_c^{an}(X^{an})$ which yields our result.

3.2. We give an example of an invertible $\mathcal{O}_X$-module which has a connection but no integrable connection.

Let $\bar{X} := \{z_0 z_1 - z_2 z_3 = 0\} \subset \mathbb{P}^3$. Notice that $\bar{X}$ is a complex surface isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. Let $D := \bar{X} \cap \{z_0 + z_1 + z_2 - z_3 = 0\}$. Let $X := \bar{X} \setminus D$.

One verifies that D is a smooth hypersurface of $\bar{X}$. Using the Lefschetz Theorem on hyperplane sections, one shows that D is connected. In fact, D is a non-singular projective plane curve of degree 2. So $D \simeq \mathbb{P}^1$. Then $H^1(D^{an}; \mathbb{Z}) = 0$.

By [14] (p. 75) we have a commutative diagram whose lines are exact:

$$\begin{array}{ccccccc}
\mathbb{Z} & \to & Pic\,\bar{X} & \to & Pic\,X & \to & 0 \\
\downarrow\simeq & & \downarrow & & \downarrow & & \\
H^2(\bar{X}^{an}, X^{an}; \mathbb{Z}) & \to & H^2(\bar{X}^{an}; \mathbb{Z}) & \to & Im\,\phi & \to & 0
\end{array}$$

where $\phi : H^2(\bar{X}^{an}; \mathbb{Z}) \to H^2(X^{an}; \mathbb{Z})$.

We have (see [16] Chap. III Exercise 12.6, p. 292):

$$Pic\,\bar{X} \simeq Pic\,\mathbb{P}^1 \times Pic\,\mathbb{P}^1 \simeq \mathbb{Z} \times \mathbb{Z}.$$

According to Künneth formula, we have:

$$H^2(\bar{X}^{an}; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

One verifies that the middle vertical arrow in the diagram above given by the first Chern class is an isomorphism: one has to compute $c_1(p_i^*(\mathcal{O}_{\mathbb{P}^1}(n)), i = 1, 2$, where p_1 and p_2 are the projections of $\bar{X}$ onto $\mathbb{P}^1$.

By the Five Lemma, the last vertical arrow is an isomorphism.

Moreover the lower line of the diagram gives an exact sequence:

$$\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z} \to Im\,\phi$$

because $H^2(\bar{X}^{an}, X^{an}; \mathbb{Z}) \simeq H_2(D^{an}; \mathbb{Z})$ by Lefschetz duality and:

$$H_2(D^{an}; \mathbb{Z}) \simeq \mathbb{Z}$$

because $D \simeq \mathbb{P}^1$.

Therefore, there is an element $c \in Im\,\phi$ which is not a torsion element.

The surjectivity of the third vertical arrow gives that there is an invertible sheaf $\mathcal{L}$ on X such that $c_1(\mathcal{L}) = c$.

Since X is affine, we have:

$$H^1(X, \Omega_X^1) = 0.$$

According to Lemma 2.6 there is a connection on the sheaf $\mathcal{L}$. But according to Lemma 1.16, there is no integrable connection on $\mathcal{L}$.

3.3. Notice that it is easier to find an example where there are connections which are not integrable or regular. One may consider $X = \mathbb{C}^2$. In this case both $Pic^{an}(X^{an})$ and $Pic(X)$ are trivial.

A connection on $\mathcal{O}_X$ (resp. $\mathcal{O}_{X^{an}}$) is given by a global algebraic (resp. analytic) differential form ω:

$$\nabla(f) = df + f\omega.$$

If one considers $\omega = dz_1$, the corresponding connection is integrable but not regular.

If $\omega = z_1 dz_2$, the corresponding connection is not integrable because the form is not closed.

We can compute $Pic_c(X)$ and $Pic_{ci}(X)$ by using the diagram of Theorem 2.1. Then:

$$Pic_c(X) \simeq H^0(X, \Omega^1_X)$$

because for $X = \mathbb{C}^2$, the map $H^0(X, \mathbb{C}^*_X) \to H^0(X, \mathcal{O}^*_X)$ is an isomorphism.

Similarly, we have:

$$Pic_{ci}(X) \simeq H^0(X, {}^c\Omega^1_X).$$

In the analytic case, we know that:

$$Pic^{an}_{ci}(X^{an}) \simeq H^1(X^{an}, \mathbb{C}^*),$$

so it is trivial.

For $Pic^{an}_c(X^{an})$ the exact sequence of 1.14 gives that $Pic^{an}_c(X^{an})$ is isomorphic to the group $H^0(X^{an}, d\Omega^1_{X^{an}})$. The elements of $Pic^{an}_c(X^{an})$ are given by their curvature. Note that $H^2(X^{an}; \mathbb{C}^*) = 0$.

3.4. Let X be a non-singular algebraic variety. It may happen that all invertible sheaves on X admit an integrable connection whereas this is not true for X^{an}, as shown by the following example:

Consider the algebraic variety $X = \mathbb{C}^* \times \mathbb{C}^*$.

Notice that for this variety $Pic(X) = 0$, because $X = \mathbb{C}^2 \setminus Z$ where Z is the closed algebraic subspace given by the union of the lines $\mathbb{C} \times \{0\}$ and $\{0\} \times \mathbb{C}$, then using the Proposition 6.5 in Chapter II of [16] p. 133, we have a surjection:

$$Pic(\mathbb{C}^2) \to Pic(X).$$

Then, $Pic(X) = 0$.

On the other hand $Pic^{an}(X^{an}) \simeq H^2(X^{an}; \mathbb{Z}) = \mathbb{Z}$ because X^{an} is a Stein space; use the exact exponential sequence.

Therefore, there are invertible $\mathcal{O}_{X^{an}}$-modules for which the complex first Chern class is $\neq 0$. According to Lemma 1.16 these sheaves do not have integrable connections. However, by Theorem 1.1 they have a connection because $H^1(X^{an}, \Omega^1_{X^{an}}) = 0$. But these do not come from an algebraic invertible sheaf, because the latter ones are trivial.

3.5. Let X be a non-singular complex algebraic variety and $\mathcal{L}$ an invertible sheaf on X. By Corollary 2.11, there is a connection on $\mathcal{L}$ (and even an integrable one) as soon as $c^1_{\mathbb{C}}(\mathcal{L}) = 0$. This is no longer true if we pass to the analytic situation as shown by the following example:

Put $X := \mathbb{C}^2 \setminus \{0\}$. Note that X^{an} is simply connected.

On the other hand, $H^1(X^{an}, \mathcal{O}_{X^{an}}) \neq 0$: Let $\mathcal{U}$ be the open Stein covering by $U_1 = \mathbb{C} \times \mathbb{C}^*$, $U_2 = \mathbb{C}^* \times \mathbb{C}$. Then $H^1(X^{an}, \mathcal{O}_{X^{an}})$ is the cokernel of :

$$H^0(U_1, \mathcal{O}_{U_1}) \oplus H^0(U_2, \mathcal{O}_{U_1}) \to H^0(U_1 \cap U_2, \mathcal{O}_{U_1 \cap U_2})$$

$$(a, b) \mapsto r_1(a) - r_2(b)$$

where r_1, r_2 are restrictions, so

$$H^1(X^{an}, \mathcal{O}_{X^{an}}) \simeq V$$

where V is the vector space of all globally convergent Laurent series in two variables with negative exponents, so $V \neq 0$.

As usual, let $Pic_0(X^{an})$ be the group of isomorphism of line bundles on X^{an} with trivial first Chern class. The exact sequence:

$$0 = H^1(X; \mathbb{Z}) \to H^1(X^{an}, \mathcal{O}_{X^{an}}) \to Pic_0(X^{an}) \to 0$$

shows that $Pic_0(X^{an}) \neq 0$. On the other hand, $Pic(X) = Pic(\mathbb{C}^2) = 0$. So there are invertible $\mathcal{O}_{X^{an}}$-modules with first Chern class 0 which are not algebraizable. These cannot admit a connection: The composition $H^1(X^{an}, \mathcal{O}_{X^{an}}) \overset{\cong}{\to} Pic(X^{an}) \to H^1(X^{an}, \Omega^1_{X^{an}})$ is given by $(f_{ij}) \mapsto (2\pi i df_{ij})$, so $b(\mathcal{L}) \neq 0$ if (f_{ij}) does not represent the trivial element: note that the mapping $H^1(X^{an}, \mathcal{O}_{X^{an}}) \to H^1(X^{an}, \Omega^1_{X^{an}})$ corresponds to the mapping $V \to V^2$: $h \mapsto (\frac{\partial h}{\partial z_1}, \frac{\partial h}{\partial z_2})$ which is injective.

This shows that there are invertible sheaves on X^{an} whose first Chern class vanishes and which do not admit a holomorphic connection. In particular, we cannot improve Lemma 1.16 in general. On the other hand, cf. Lemma 1.18.

3.6. Let X be a non-singular complex algebraic variety, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module, ∇ a connection on $\mathcal{L}$.

Then we have:

$$\nabla \text{ regular integrable } \Rightarrow \nabla \text{ integrable}$$

This implication is not invertible, as shown by the example $X = \mathbb{C}^2, \mathcal{L} = \mathcal{O}_X$ (see above 3.3).

Note that ∇ is integrable if and only if ∇^{an} is integrable.

In fact, we can consider the existence of connections on $\mathcal{L}$ (resp. $\mathcal{L}^{an}$):

$$\begin{array}{ccccc} \exists\,\nabla \text{ regular integrable} & \Leftrightarrow & \exists\,\nabla \text{ integrable} & \Rightarrow & \exists\,\nabla \\ & & \Updownarrow & & \Downarrow \\ & & \exists\,\nabla \text{ analytic integrable} & \Rightarrow & \exists\,\nabla \text{ analytic} \end{array}$$

For the left upper and the middle vertical equivalence see Corollary 2.11.

Note that there may be no connection at all on $\mathcal{L}$ or $\mathcal{L}^{an}$, as shown by the example $X = \mathbb{P}_1, \mathcal{L} = \mathcal{O}(k), k \neq 0$.

The right horizontal arrows are not invertible, as shown by the complicated example 3.2.

The right vertical arrow is not invertible if the answer to the following question is positive:

Let X be the Serre example of a non-singular algebraic surface which is not affine but the corresponding complex analytic manifold is Stein (see [15] p. 232 Example 3.2). Is there an invertible $\mathcal{O}_X$-module $\mathcal{L}$ on X such that its image in $H^1(X, \Omega^1_X)$ does not vanish? (Note that X is not affine, so it is possible that $H^1(X, \Omega^1_X) \neq 0$). Then, $\mathcal{L}$ does not admit a connection.

On the other hand, X^{an} is Stein, so $H^1(X^{an}, \Omega^1_{X^{an}}) = 0$, which implies that there is a connection on $\mathcal{L}^{an}$.

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