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SCHUBERT DECOMPOSITION FOR MILNOR FIBERS OF THE VARIETIES OF SINGULAR MATRICES

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Abstract. We consider the varieties of singular $m \times m$ complex matrices which may be either general, symmetric or skew-symmetric (with m even). For these varieties we have shown in another paper that they had compact "model submanifolds" for the homotopy types of the Milnor fibers which are classical symmetric spaces in the sense of Cartan. In this paper we use these models, combined with results due to a number of authors concerning the Schubert decomposition of Lie groups and symmetric spaces via the Cartan model, together with Iwasawa decomposition, to give cell decompositions of the global Milnor fibers.

The Schubert decomposition is in terms of "unique ordered factorizations" of matrices in the Milnor fibers as products of "pseudo-rotations". In the case of symmetric or skew-symmetric matrices, this factorization has the form of iterated "Cartan conjugacies" by pseudo-rotations. The decomposition respects the towers of Milnor fibers and symmetric spaces ordered by inclusions. Furthermore, the "Schubert cycles", which are the closures of the Schubert cells, are images of products of suspensions of projective spaces (complex, real, or quaternionic as appropriate). In the cases of general or skew-symmetric matrices the Schubert cycles have fundamental classes, and for symmetric matrices mod 2 classes, which give a basis for the homology. They are also shown to correspond to the cohomology generators for the symmetric spaces. For general matrices the duals of the Schubert cycles are represented as explicit monomials in the generators of the cohomology exterior algebra; and for symmetric matrices they are related to Stiefel-Whitney classes of an associated real vector bundle.

Furthermore, for a matrix singularity of any of these types. the pull-backs of these cohomology classes generate a characteristic subalgebra of the cohomology of its Milnor fiber.

We also indicate how these results extend to exceptional orbit hypersurfaces, complements and links, including a characteristic subalgebra of the cohomology of the complement of a matrix singularity.

Preamble: Motivation from the Work of Brieskorn

After Milnor developed the basic theory of the Milnor fibration and the properties of Milnor fibers and links for isolated hypersurface singularities, Brieskorn was involved in fundamental ways in developing a more complete theory of isolated hypersurface singularities. Furthermore through the work of his many students the theory was extended to isolated complete intersection singularities.

For isolated hypersurface singularities Brieskorn developed the importance of the intersection pairing on the Milnor fiber [Br]. This includes the computation of the intersection index for Pham-Brieskorn singularities, leading to the discovery that for a number of these singularities the link is an exotic topological sphere. He also demonstrated in a variety of ways that group theory in various forms plays an essential role in understanding the structure of singularities. This

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includes the relation between the monodromy and the Milnor fiber cohomology by the Gauss-Manin connection, and including the intersection pairing [Br2]. This includes the relation with Lie groups, especially for the ADE classification for simple hypersurface singularities, where he identified the intersection pairing with the Dynkin diagrams for the corresponding Lie groups. He also gave the structure of the discriminant for the versal unfoldings using the Weyl quotient map on the subregular elements of the Lie group [Br3]. In combined work with Arnold [Br4], he further showed that for the simple ADE singularities the complement of the discriminant is a $K(\pi, 1)$. He continued on beyond the simple singularities to understand the corresponding structures for unimodal singularities [Br5], setting the stage for further work in multiple directions.

The approaches which he initiated provide models for approaching questions for highly non-isolated hypersurface singularities which are used in this paper. For matrix singularities, the high-dimensional singular set means that the Milnor fiber, complement and link have low connectivity and hence can have (co)homology in many degrees [KMs]. To handle this complexity for matrix singularities of the various types, Lie group methods are employed to answer these questions. Partial answers were already given in [D3], including determining the (co)homology of the Milnor fibers using representations as symmetric spaces. This continues here by obtaining geometric models for the homology classes, understanding the analogue of the intersection pairing on the Milnor fiber via a Schubert decomposition, determining the structure of the link and complement, and their relations with the cohomology structure. We see that there is the analogue of the ADE classification which is given for the matrix singularities by the ABCD classification for the infinite families of simple Lie groups. We also indicate how these geometric methods extend to complements and links, including more general exceptional orbit hypersurfaces for prehomogeneous spaces.

Introduction

In this paper we derive the Schubert cell decomposition of the Milnor fibers of the varieties of singular matrices for $m \times m$ complex matrices which may be either general, symmetric, or skew-symmetric (with m even). We show that there is a homology basis obtained from "Schubert cycles", which are the closures of these cells. We further identify these homology classes with the cohomology. For general matrices we identify the correspondence with monomials of the generators for the exterior cohomology algebra and for symmetric matrices we identify the Schubert classes with monomials in the Stiefel-Whitney classes of an associated vector bundle. We also indicate how these results extend to more general exceptional orbit varieties and for the complements and links for all of these cases. Furthermore, for general matrix singularities defined from these matrix types, we define characteristic subalgebras of the cohomology of the Milnor fibers and complements representing them as modules over these subalgebras.

In [D3] we computed the topology of the exceptional orbit hypersurfaces for classes of prehomogeneous spaces which include these varieties of singular matrices. This included the topology of the Milnor fiber, link, and complement. This used the representation of the complements and the global Milnor fibers as homogeneous spaces which are homotopy equivalent to compact models which are classical symmetric spaces studied by Cartan. These symmetric spaces have representations as "Cartan models", which can be identified as compact submanifolds of the global Milnor fibers.

We use the Schubert decomposition for these symmetric spaces developed by Kadzisa-Mimura [KM] building on the earlier results for Lie groups and Stiefel manifolds by J. H. C. Whitehead [W], C.E. Miller, [Mi], I. Yokota [Y]. This allows us to give a Schubert decomposition for the compact models of the Milnor fibers, which together with Iwasawa decomposition provides a cell decomposition for the global Milnor fibers in terms of the Schubert decomposition for these symmetric spaces.

The Schubert decompositions are in terms of cells defined by the unique "ordered factorizations" of matrices in the Milnor fibers into "pseudo-rotations" of types depending on the matrix type, and their relation to a flag of subspaces. For symmetric or skew-symmetric matrices, this factorization has the form of iterated "Cartan conjugacies" by the pseudo-rotations. These are given by a modified form of conjugacy which acts on the Cartan models.

The Schubert decomposition is then further related to the co(homology) of the global Milnor fibers. We do so by showing the Schubert cycles for the symmetric spaces are images of products of suspensions of projective spaces of various types (complex, real, and quaternionic as appropriate). This allows us to relate the duals of the fundamental classes of the Schubert cycles (mod 2 classes for symmetric matrices) to the cohomology classes given for Milnor fibers in [D1]. These are given for the different matrix types and various coefficients as exterior algebras. In the symmetric matrix case the cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients is given as an exterior algebra on the Stiefel-Whitney classes of an associated real vector bundle. For coefficient fields of characteristic zero the generators are classes which transgress to characteristic classes of appropriate types.

We further indicate how these methods also apply to exceptional orbit hypersurfaces in [D3] and how they further extend to the complements of the varieties and their links.

Lastly, we show that for matrix singularities of these matrix types, we can pull-back the cohomology algebras of the global Milnor fibers to identify characteristic subalgebras of the Milnor fibers for these matrix singularities. This represents the cohomology of the Milnor fiber of a matrix singularity of any of these types as a module over the corresponding characteristic subalgebra. We also indicate how this also holds for the cohomology of the complement.

1. Cell Decomposition for Global Milnor Fibers in Terms of their Compact Models

We consider the varieties of singular $m \times m$ complex matrices which may be either general, symmetric, or skew-symmetric (with m even). In [D1] we investigated the topology of these singularities, including the topology of the Milnor fiber, link and complement. This was done by viewing them as the exceptional orbit varieties obtained by the representation of a complex linear algebraic group G on a complex vector space V with open orbit. For example this includes the cases where $V = M$ is one of the spaces of complex matrices $M = Sym_m$ or $M = Sk_m$ (for $m = 2k$) acted on by $\mathrm{GL}_m(\mathbb{C})$ by $B \cdot A = BAB^T$, or , $M = M_{m,m}$ and $\mathrm{GL}_m(\mathbb{C})$ acts by left multiplication. Each of these representations have open orbits and the resulting prehomogeneous space has an exceptional orbit variety $\mathcal{E}$ which is a hypersurface of singular matrices.

Definition 1.1. The *determinantal hypersurface* for the space of $m \times m$ symmetric or general matrices, denoted by $M = Sym_m$ or $M = M_{m,m}$ is the hypersurface of singular matrices defined by $\det : M \to \mathbb{C}$ and denoted by $\mathcal{D}_m^{(sy)}$ for $M = Sym_m$, or $\mathcal{D}_m$ for $M = M_{m,m}$. For the space of $m \times m$ skew-symmetric matrices $M = Sk_m$ (for $m = 2k$) the determinantal hypersurface of singular matrices is defined by the Pfaffian $\mathrm{Pf} : Sk_m \to \mathbb{C}$, and is denoted by $\mathcal{D}_m^{(sk)}$. In the following we uniformly denote any of these functions as f.

Then, we showed in [D3] that the Milnor fibers for each of these singularities at 0 are diffeomorphic to their global Milnor fibers $f^{-1}(1)$ which are denoted by: F_m for general case, $F_m^{(sy)}$ for the symmetric case, and $F_m^{(sk)}$ for the skew-symmetric case. Then, we show in Theorem 3.1 in [D3, §3] that each global Milnor fiber is acted on transitively by a linear algebraic group and so is a homogeneous space. In particular, $F_m = SL_m(\mathbb{C})$, $F_m^{(sy)} \simeq SL_m(\mathbb{C})/SO_m(\mathbb{C})$, and

$F_{2m}^{(sk)} \simeq SL_{2m}(\mathbb{C})/Sp_m(\mathbb{C})$. Moreover, these spaces have as deformation retracts spaces which are symmetric spaces of classical type studied by Cartan: $SL_m(\mathbb{C})$ has as deformation retract SU_m; $SL_m(\mathbb{C})/SO_m(\mathbb{C})$ has as deformation retract SU_m/SO_m; and $SL_{2m}(\mathbb{C})/Sp_m(\mathbb{C})$ has as deformation retract SU_{2m}/Sp_m. These are compact models for the Milnor fibers and we denote them as F_m^c, $F_m^{(sy)\,c}$, and $F_{2m}^{(sk)\,c}$ respectively.

This allowed us to obtain the rational (co)homology (and integer cohomology for the general and skew-symmetric cases and the $\mathbb{Z}/2\mathbb{Z}$ cohomology for the symmetric cases), as well as using the Bott periodicity theorem to compute the homotopy groups in the stable range.

We will now further use the cell decompositions of the symmetric spaces together with Iwasawa decomposition to give the cell decompositions for the global Milnor fibers. We recall the Iwasawa decomposition for $SL_m(\mathbb{C})$ has the form KAN where $K = SU_m$, A_m consists of diagonal matrices with real positive entries of $\det = 1$, and N_m is the nilpotent group of upper triangular complex matrices with 1's on the diagonal. In particular, this means that the map

$$SU_m \times A_m \times N_m \to SL_m(\mathbb{C})$$

sending $(U, B, C) \mapsto U \cdot B \cdot C$ is a real algebraic diffeomorphism. Alternatively $A_m \cdot N_m$ consists of the upper triangular matrices of $\det = 1$ with complex entries except having real positive entries on the diagonal. As a manifold it is diffeomorphic to a Euclidean space of real dimension $2\binom{m}{2} + m - 1$. We denote this subgroup of $SL_m(\mathbb{C})$ as Sol_m, which is a real solvable subgroup of $SL_m(\mathbb{C})$.

For any of the preceding cases, let F denote the Minor fiber and Y the compact symmetric space associated to it. Suppose that Y has a cell decomposition with open cells $\{e_i : I = 1, \ldots, r\}$. Then, we have the following simple proposition.

Proposition 1.2. *With the preceding notation, the cell decomposition of F is given by*

$$\{e_i \cdot \mathrm{Sol}_m : I = 1, \ldots, r\}.$$

Moreover, if the closure $\bar{e}_i$ has a fundamental homology class (for Borel-Moore homology), then $\overline{e_i \cdot \mathrm{Sol}_m} = \bar{e}_i \cdot \mathrm{Sol}_m$ has a fundamental homology class with the same Poincaré dual.

Proof. By the Iwasawa decomposition $Y \times \mathrm{Sol}_m \simeq F$ via $(U, B) \mapsto U \cdot B$. Hence, if for $i \neq j$, $e_i \cap e_j = \emptyset$, then $(e_i \times \mathrm{Sol}_m) \cap (e_j \times \mathrm{Sol}_m) = \emptyset$ and $(e_i \cdot \mathrm{Sol}_m) \cap (e_j \cdot \mathrm{Sol}_m) = \emptyset$. Also, as $Y = \cup_i e_i$ is a disjoint union, so also is $F = \cup_i e_i \cdot \mathrm{Sol}_m$. Third, each $e_i \times \mathrm{Sol}_m$ is homeomorphic to a cell of dimension $\dim_{\mathbb{R}}(e_i) + 2\binom{m}{2} + m - 1$. Thus, F is a disjoint union of the cells $e_i \cdot \mathrm{Sol}_m$. Lastly, $\bar{e}_i = e_i \cup_{j_i} e_{j_i}$ where the last union is over cells of dimension less than $\dim e_i$. Hence, $e_i \cdot \bar{\mathrm{Sol}}_m = \bar{e}_i \cdot \mathrm{Sol}_m = (e_i \cdot \mathrm{Sol}_m) \cup_{j_i} (e_{j_i} \cdot \mathrm{Sol}_m)$. Hence this is a cell decomposition.

Then, $\bar{e}_i$ is a singular manifold with open smooth manifold e_i. If it has a Borel-Moore fundamental class, which restricts to that of e_i, then so does $\overline{e_i \cdot \mathrm{Sol}_m}$ have a fundamental class that restricts to that for $e_i \cdot \mathrm{Sol}_m \simeq e_i \times \mathrm{Sol}_m$. Then, as $\bar{e}_i$ is the pull-back of $\overline{e_i \cdot \mathrm{Sol}_m}$ under the map $i : Y \to Y \times \mathrm{Sol}_m \simeq F$ which is transverse to $\bar{e}_i \times \mathrm{Sol}_m \simeq \overline{e_i \cdot \mathrm{Sol}_m}$, by a fiber-square argument for Borel-Moore homology, the Poincaré dual of $\overline{e_i \cdot \mathrm{Sol}_m}$ pulls-back via i^* to the Poincaré dual of $\bar{e}_i$. As i is a homotopy equivalence, via the isomorphism i^* the Poincaré duals agree. $\square$

2. Cartan Models for the Symmetric Spaces

The General Cartan Model.

By Cartan, a symmetric space is defined by a Lie group G with an involution $\sigma : G \to G$ so that the symmetric space is given by the quotient space G/G^σ, where G^σ denotes the subgroup of G invariant under σ. Furthermore this space can be embedded into the Lie group G. The embedding is called the *Cartan model*. It is defined as follows, where we follow the approach of

Kadzisa-Mimura [KM] and the references therein. They introduce two subsets M and N of G defined by:

$$M \;=\; \{g\sigma(g^{-1}) : g \in G\} \qquad \text{and} \qquad N \;=\; \{g \in G : \sigma(g^{-1}) = g\}.$$

Then, we have $G/G^{\sigma} \simeq M \subset N$. The inclusion is the obvious one, and the homeomorphism is given by $g \mapsto g\sigma(g^{-1})$. Via this homeomorphism, we may identify the symmetric space G/G^{σ} with the subset $M \subset G$. The subspace N is closed in G, and it can be shown that M is the connected component of N containing the identity element. In the three cases we consider, it will be the case that $M = N$.

We also note that while M and N are subspaces of G, they are not preserved under products nor conjugacy; however they do have the following properties.

Further Properties of the Cartan Model:

i) there is an action of G on both M and N defined by $g \cdot h = gh\sigma(g^{-1})$ and on M it is transitive;
ii) the homeomorphism $G/G^{\sigma} \simeq M$ is G-equivariant under left multiplication on G/G^{σ} and the preceding action on M;
iii) both M and N are invariant under taking inverses; and
iv) if $g, h \in N$ commute then $gh \in N$.

For U_n, $g^* = g^{-1}$ so an alternative way to write the action in i) is given by $g \mapsto h \cdot g \cdot \sigma(h^*)$. We will refer to this action as *Cartan conjugacy.*

Then, Kadzisa-Mimura use the cell decompositions for various G to give the cell decompositions for M and hence the symmetric space G/G^{σ}. There is one key difference with what we will do versus what Kadzisa-Mimura do. They give the cell decomposition; however we also want to represent the closed cells where possible as the images of specific singular manifolds, specifically products of suspensions of projective spaces of various types and to relate the fundamental homology classes to corresponding classes in cohomology. Together with the reasoning in §1 and the identification of the global Milnor fibers with the Cartan models, we will then be able to give the Schubert decomposition for the global Milnor fibers and identify the Schubert homology classes with dual cohomology classes.

The Cartan Models for SU_m, SU_m/SO_m, and SU_{2m}/Sp_m.

For the three cases we consider: SU_m , SU_m/SO_m, SU_{2m}/Sp_m, we first observe that the exact sequence of groups (2.1) does not split

$$1 \longrightarrow SU_m \longrightarrow U_m \xrightarrow{\det} S^1 \longrightarrow 1\,. \tag{2.1}$$

However, it does split as manifolds $U_m \simeq S^1 \times SU_m$ sending

$$C \mapsto (\det(C), I_{1,m-1}(\det(C)) \cdot C),$$

where $I_{1,m-1}(\det(C)^{-1})$ is the $m \times m$ diagonal matrix with 1's on the diagonal except in the first position where it is $\det(C)^{-1}$. Thus, topological statements about U_m have corresponding statements about SU_m and conversely.

We first give the representation for the symmetric spaces. For SU_m we just use itself as a compact Lie group.

Next, for $SU(m)/SO(m)$ we let the involution σ on $SU(m)$ be defined by $C \mapsto \overline{C}$. We see that $\sigma(C) = C$ is equivalent to $C = \overline{C}$. Thus C is a real matrix which is unitary; and hence C is real orthogonal. As $\det(C) = 1$, we see that $SU_m^{\sigma} = SO_m$.

The third case is SU_{2n}/Sp_n for $m = 2n$. In this case, the involution σ on SU_{2n} sends $C \mapsto J_n\overline{C}J_n^*$ where J_n is the $2n \times 2n$ block diagonal matrix with 2×2 diagonal blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

As $J_n^* = J_n^T = -J_n = J_n^{-1}$, then $\sigma(C) = C$ is equivalent to $J_n\overline{C}J_n = -C$, or as $C^{-1} = \overline{C}^T$ we can rearrange to obtain $C^T J_n C = J_n$ (or alternatively $CJ_nC^T = J_n$), which implies that C leaves invariant the bilinear form $(v, w) = v^T J_n w$ (for column vectors v and w) and so is an element of $Sp_n(\mathbb{C})$, and so an element of $Sp_n = SU_{2n} \cap Sp_n(\mathbb{C})$.

The corresponding Cartan models are then given as follows. We denote the Cartan models by respectively: $\mathcal{C}_m$, $\mathcal{C}_m^{(sy)}$, and $\mathcal{C}_m^{(sk)}$.

First, for $G = SU_m$, which is itself a symmetric space, and we let $\mathcal{C}_m = SU_m$. In this case, Cartan conjugacy is replaced by left multiplication.

Second, for SU_m/SO_m we claim

$$\mathcal{C}_m^{(sy)} \overset{def}{=} \{C \cdot C^T : C \in SU_m\} = \{B \in SU_m : B = B^T\}. \tag{2.2}$$

The inclusion of the LHS in the RHS is immediate. For the converse, we note that if $B \in SU_m$ and $B = B^T$, then by the following Lemma given in [KM] there is an orthonormal basis of eigenvectors which are real vectors so we may write $B = ACA^{-1}$ with A an orthogonal matrix and C a diagonal matrix with diagonal entries λ_j so that $|\lambda_j| = 1$. Thus, $A^{-1} = A^T$, and so $B = ADA^T \cdot ADA^T$ with D a diagonal matrix with entries $\sqrt{\lambda_j}$.

Lemma 2.1. *If $B \in SU_m$ and $B = B^T$ then there is a real orthonormal basis of eigenvectors for B.*

This is a simple consequence of the eigenspaces being invariant under conjugation, which is easily seen to follow from the conditions. In this case, Cartan conjugacy by A on B is checked to be given by $B \mapsto A \cdot B \cdot A^T$.

Third, for SU_{2n}/Sp_n with $m = 2n$, we may directly verify

$$\mathcal{C}_m^{(sk)} \overset{def}{=} \{C \cdot J_n \cdot C^T \cdot J_n^* : C \in SU_{2n}\} = \{B \in SU_{2n} : (B \cdot J_n)^T = -B \cdot J_n\}. \tag{2.3}$$

Then, Cartan conjugacy by A on B is given by $B \mapsto A \cdot (B \cdot J_n) \cdot A^T \cdot J_n^{-1}$, with $B \cdot J_n$ skew-symmetric for $B \in \mathcal{C}_m^{(sk)}$.

Hence, from (2.2), we have the compact model for $F_m^{(sy)}$ as a subspace is given by

$$F_m^{(sy)\,c} = SU_m \cap Sym_m(\mathbb{C})$$

and the Cartan model for the symmetric space SU_m/SO_m is given by $F_m^{(sy)\,c}$ itself. Similarly, from (2.3), we have the compact model for $F_m^{(sk)}$ with $m = 2n$ as a subspace is given by $F_m^{(sk)\,c} = SU_m \cap Sk_m(\mathbb{C})$ and the Cartan model for the symmetric space SU_{2n}/Sp_n is given by $F_m^{(sk)\,c} \cdot J_n^{-1}$.

Remark 2.2. Frequently for all three cases, we will want to apply a Cartan conjugate for an element of U_n instead of SU_n. The formula for the Cartan conjugate remains the same and the corresponding symmetric spaces are U_n, U_n/O_n, and U_{2n}/Sp_n. By the properties of Cartan conjugacy, an iteration of Cartan conjugacy by elements $A_i \in U_n$ whose product belongs to SU_n will be a Cartan conjugate by an element of SU_n and preserve the Cartan models of interest to us.

Tower Structures of Global Milnor fibers and Symmetric Spaces by Inclusion.

Lastly, these global Milnor fibers, symmetric spaces and compact models form towers via inclusions: i) sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ for $SU_m \subset SU_{m+1}$, $F_m \subset F_{m+1}$, or $F_m^{(sy)} \subset F_{m+1}^{(sy)}$ which induce inclusions of the symmetric spaces SU_m and SU_m/SO_m and corresponding global Milnor fibers, or ii) sending $A \mapsto \begin{pmatrix} A & 0 \\ 0 & I_2 \end{pmatrix}$ for the 2×2 identity matrix I_2 for $SU_m \subset SU_{m+2}$ for $m = 2n$ and the corresponding symmetric spaces SU_{2n}/Sp_n and Milnor fibers $F_m^{(sk)} \subset F_{m+2}^{(sk)}$. The Schubert decompositions will satisfy the additional property that they respect the inclusions.

We summarize these results by the following.

Proposition 2.3. *For the varieties of singular $m \times m$ complex matrices which are either general, symmetric or skew-symmetric, their global Milnor fibers, representations as homogeneous spaces, compact models given as symmetric spaces and Cartan models are summarized in Table 1.*

Milnor Fiber $F_m^{(*)}$	Quotient Space	Symmetric Space	Compact Model $F_m^{(*)\,c}$	Cartan Model
F_m	$SL_m(\mathbb{C})$	SU_m	SU_m	F_m^c
$F_m^{(sy)}$	$SL_m(\mathbb{C})/SO_m(\mathbb{C})$	SU_m/SO_m	$SU_m \cap Sym_m(\mathbb{C})$	$F_m^{(sy)\,c}$
$F_m^{(sk)}, m = 2n$	$SL_{2n}(\mathbb{C})/Sp_n(\mathbb{C})$	SU_{2n}/Sp_n	$SU_m \cap Sk_m(\mathbb{C})$	$F_m^{(sk)\,c} \cdot J_n^{-1}$

Table 1. Global Milnor fiber, its representation as a homogenenous space, compact model as a symmetric space, compact model as subspace and Cartan model.

3. Schubert Decomposition for Compact Lie Groups

We recall the "Schubert decomposition"for compact Lie groups, concentrating on SU_n. The cell decompositions of certain compact Lie groups, especially SO_n and U_n and SU_n were carried out by C. E. Miller [Mi] and I. Yokota [Y], building on the work of J. H. C. Whitehead [W] for the cell decomposition of Stiefel varieties. In the case of Grassmannians, the Schubert decomposition is in terms of the dimensions of the intersections of the subspaces with a given fixed flag of subspaces. For these Lie groups, elements are expressed as ordered products of (complex) "pseudo rotations"about complex hyperplanes (or reflections about real hyperplanes in the case of SO_n). The cell decomposition is based on the subspaces of a fixed flag that contain the orthogonal lines to the hyperplane axes of rotation (or reflection). We will concentrate on the complex case which is relevant to our situation.

(Complex) Pseudo-Rotations.

We note that given a complex 1–dimensional subspace $L \subset \mathbb{C}^n$, we can define a "(complex) pseudo-rotation"about the orthogonal hyperplane $L^{\perp}$ as follows. Let $x \in L$ be a unit vector. As L is complex we have a positive sense of rotation through an angle θ given by $x \mapsto e^{i\theta}x$. We extend this to be the identity on $L^{\perp}$. This is given by the following formula for any $x' \in \mathbb{C}^n$:

$$A_{(\theta,x)}(x') \;=\; x' - ((1 - e^{i\theta}) < x', x >)\, x\,.$$

This is not a true rotation as a complex linear transformation so we refer to this as a "pseudo-rotation". Then, $A_{(\theta,x)}$ can be written in matrix form as $A_{(\theta,x)} = (I_n - (1 - e^{i\theta})\, x \cdot \bar{x}^T)$ for x an n-dimensional column vector.

Remark 3.1. In the special case that $A_{(\theta,x)}$ has finite order as an element of the group U_n, it is called a "complex reflection".

We observe a few simple properties of pseudo-rotations:

i) $A_{(\theta,x)}$ only depends on $L =< x >$, so we will also feel free to use the alternate notation $A_{(\theta,L)}$;

ii) $A_{(\theta,x)}$ is a unitary transformation with $\det(A_{(\theta,x)}) = e^{i\theta}$;

iii) if $B \in U_n$, then $B \cdot A_{(\theta,x)} \cdot B^{-1} = A_{(\theta,Bx)}$ is again a pseudo-rotation; and

iv) $\overline{A_{(\theta,x)}} = A_{(-\theta,\bar{x})}$; $A^{-1}_{(\theta,x)} = A_{(-\theta,x)}$; and $A^{T}_{(\theta,x)} = A_{(\theta,\bar{x})}$.

Ordered Factorizations in SU_m and Schubert Symbols.

Then, given any $B \in SU_n$, we may diagonalize B using an orthonormal basis $\{v_1, \dots, v_n\}$ so if C denotes the unitary matrix with the v_i as columns, then we may write $B = CDC^{-1}$ where D is a diagonal matrix with diagonal entries λ_i of unit length so that $\prod_{i=1}^{n} \lambda_i = 1$. This can be restated as saying that B is a product of pseudo-rotations about the hyperplanes $< v_j >^{\perp}$ with angles θ_j where $\lambda_j = e^{i\theta_j}$. Thus, $B = \prod_{j=1}^{n} A_{(\theta_j,v_j)}$. However, we note that as certain eigenspaces may have dimension > 1, the terms and their order in the product are not unique.

There is a method introduced by Whitehead and used by Miller and Yokota for obtaining a unique factorization leading to the Schubert decomposition in SU_n. The product is rewritten as a product of different pseudo-rotations whose lines satisfy certain inclusion relations for a fixed flag leading to an ordering of the pseudo-rotations. We let $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$ denote the standard flag. Then, if $L =< x > \subset \mathbb{C}^k$ and $L =< x > \not\subset \mathbb{C}^{k-1}$, we will say that x *and* L *minimally belong to* $\mathbb{C}^k$ and introduce the notation $x \in_{\min} \mathbb{C}^k$ or $L \subset_{\min} \mathbb{C}^k$. If $x = (x_1, x_2, \dots, x_n)$ then $x \in_{\min} \mathbb{C}^k$ iff $x_{k+1} = \cdots = x_n = 0$ and $x_k \neq 0$. We observe two simple properties: if $x \in_{\min} \mathbb{C}^k$ then $\bar{x} \in_{\min} \mathbb{C}^k$; and if $x' \in_{\min} \mathbb{C}^{k'}$ with $k' < k$, then $A_{(\theta,x')}(x) \in_{\min} \mathbb{C}^k$.

Then to rewrite the product in a different form, we proceed, as in the other papers, to follow Whitehead with the following lemma.

Lemma 3.2. *Suppose that we have two pseudo-rotations $A_{(\theta,x)}$ and $A_{(\theta',x')}$ with $x \in_{\min} \mathbb{C}^m$ and $x' \in_{\min} \mathbb{C}^{m'}$.*

1) *If $m > m'$, then*

$$A_{(\theta,x)} \cdot A_{(\theta',x')} \;=\; A_{(\theta',x')} \cdot A_{(\theta,\tilde{x})}, \tag{3.1}$$

where $\tilde{x} = A^{-1}_{(\theta',x')}(x)$.

2) *If $m = m'$, and $< x > \neq < x' >$ let $W =< x, x' >$, which has dimension 2, and let $L =< \tilde{x} > = W \cap \mathbb{C}^{m-1}$, with $\tilde{x} \in_{\min} \mathbb{C}^k$ for $k \leq m-1$. Then, there exist pseudo-rotations $A_{(\tilde{\theta},\tilde{x})}$ and $A_{(\tilde{\theta}',\tilde{x}')}$ with $\tilde{x} \in_{\min} \mathbb{C}^k$ and $\tilde{x}' \in_{\min} \mathbb{C}^m$ such that*

$$A_{(\theta,x)} \cdot A_{(\theta',x')} \;=\; A_{(\tilde{\theta},\tilde{x})} \cdot A_{(\tilde{\theta}',\tilde{x}')}\,. \tag{3.2}$$

Moreover, for generic $x, x' \in_{\min} \mathbb{C}^m$, $\tilde{x} \in_{\min} \mathbb{C}^{m-1}$.

Proof. For 1), by property iii) of pseudo-rotations, $A^{-1}_{(\theta',x')} \cdot A_{(\theta,x)} \cdot A_{(\theta',x')}$ is a pseudo-rotation of the form $A_{(\theta,\tilde{x})}$ with $\tilde{x} = A^{-1}_{(\theta',x')}(x)$. Also, both $A_{(\theta,x)}$ and $A_{(\theta',x')}$ are the identity on $\mathbb{C}^{m\perp}$; hence $\tilde{x} \in_{\min} \mathbb{C}^m$.

For 2), if $< x > = < x' >$, then the pseudo-rotations commute. Next, suppose these lines differ so the complex subspace W spanned by x and x' is 2-dimensional. Then, $\dim_{\mathbb{C}} W \cap \mathbb{C}^{m-1} = 1$. We denote it by L and let it be spanned by a unit vector $\tilde{x}$ with say $\tilde{x} \in_{\min} \mathbb{C}^k$ for $k \leq m-1$ (and generically $k = m-1$). We note that both pseudo-rotations are the identity on $W^{\perp}$. Also, $W \subset \mathbb{C}^m$. It is sufficient to consider the pseudo-rotations restricted to $W \simeq \mathbb{C}^2$ with $\tilde{x}$ denoted by e_2 and orthogonal unit vector e_1. Then, let $(A_{(\theta,x)} \cdot A_{(\theta',x')})^{-1}(e_1) = v$. Then, we want a pseudo-rotation on W that sends $e_1 \mapsto v$. If $v \neq -e_1$, then reflection about the complex line spanned by $e_1 + v$, is a pseudo-rotation by π and sends e_1 to v. If $v = -e_1$, then reflection about the complex line spanned by e_2 works instead. If we denote this reflection by $A_{(\pi,\tilde{x}')}$, then

$A_{(\theta,x)} \cdot A_{(\theta',x')} \cdot A_{(\pi,\tilde{x}')}$ is a unitary transformation which fixes e_1 and is hence a pseudo-rotation about the line $< e_1 >$ and so sends $e_2 = \tilde{x}$ to $e^{i\tilde{\theta}}\tilde{x}$ for some angle $\tilde{\theta}$. Thus,

$$A_{(\theta,x)} \cdot A_{(\theta',x')} \;=\; A_{(\tilde{\theta},\tilde{x})} \cdot A_{(\tilde{\theta}',\tilde{x}')}$$

giving the result. □

This allows us to rewrite a product of pseudo-rotations as a product where the lines are minimally contained in successively larger subspaces of the flag.

Whitehead Algorithm for ordered factorization of Unitary matrices. Given $B \in SU_n$, we may write $B = \prod_{j=1}^{k} A_{(\theta_j,x_j)}$, with the $\{x_j\}$ an orthonormal set of vectors with say $x_j \in_{\min} \mathbb{C}^{m_j}$. Note that k may be less than n as we may exclude the eigenvectors x'_j with eigenvalue 1, which give $A_{(0,x'_j)} = I_n$. Then, we may use Lemma 3.2 to reduce the product into a standard form as follows. For the sequence $(m_1, m_2, \dots, m_k)$, we find the largest j so that $m_j \geq m_{j+1}$. If $m_j > m_{j+1}$, then by 1) of Lemma 3.2, we may replace $A_{(\theta_j,x_j)} \cdot A_{(\theta_{j+1},x_{j+1})}$ by $A_{(\theta_{j+1},x_{j+1})} \cdot A_{(\theta_j,\tilde{x}_j)}$, with $\tilde{x}_j \in_{\min} \mathbb{C}^{m_j}$. If instead $m_j = m_{j+1}$, then by 2) of Lemma 3.2, we may instead replace the product by $A_{(\theta'_j,x'_j)} \cdot A_{(\theta'_{j+1},x'_{j+1})}$, where $x'_{j+1} \in_{\min} \mathbb{C}^{m_j}$ and $x'_j \in_{\min} \mathbb{C}^{\ell}$, where $\ell < m_j$ satisfies $(< x_j, x_{j+1} > \cap \mathbb{C}^{m_j}) \subset_{\min} \mathbb{C}^{\ell}$.

Then, we relabel the angles and vectors to be (θ_j, x_j), where now $m_j < m_{j+1} < \cdots < m_k$. Then, we may repeat the procedure until we obtain $m_1 < m_2 < \cdots < m_k$. We summarize the final result of this process.

Lemma 3.3. *Given $B \in SU_n$, it may be written as a product*

$$B \;=\; A_{(\theta_1,x_1)} \cdot A_{(\theta_2,x_2)} \cdots A_{(\theta_k,x_k)}\,, \tag{3.3}$$

with $x_j \in_{\min} \mathbb{C}^{m_j}$ and $1 \leq m_1 < m_2 < \cdots < m_k \leq n$, and each $\theta_i \not\equiv 0 \bmod 2\pi$.

If B has the form given in Lemma 3.3 with $m_1 > 1$, then we will say that B has Schubert type $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ and write $\mathbf{m}(B) = \mathbf{m}$. If instead $m_1 = 1$, then as $\det(B) = 1$

$$B = A_{(-\tilde{\theta},e_1)} \cdot A_{(\theta_2,x_2)} \cdot A_{(\theta_2,x_2)} \cdots A_{(\theta_k,x_k)},$$

where $\tilde{\theta} \equiv \sum_{j=2}^{k} \theta_j \bmod 2\pi$ and we instead denote $\mathbf{m}(B) = (m_2, \cdots, m_k)$. For the case of an empty sequence with $k = 0$, we associate the unique identity element I. We refer to the tuple $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ as the Schubert symbol of B. It will follow from Theorem 3.7 that this representation is unique.

There is also an alternative way to obtain a factorization (3.3) where instead $x_j \in_{\min} \mathbb{C}^{m'_j}$ with a decreasing sequence $m'_1 > m'_2 > \cdots > m'_k$. In fact, if we give a representation for B^{-1} as in (3.3) with the m_i increasing, then taking inverses gives a product of $A^{-1}_{(\theta_i,x_i)} = A_{(-\theta_i,x_i)}$ in decreasing order. There is a question for a given $B \in SU_n$ about the relation between the increasing and decreasing symbols. The relation between these is a consequence of the following lemma which is basically that given in [KM, Prop. 4.5] and is a consequence of the uniqueness of the Schubert symbol for one direction of ordering.

Lemma 3.4. *Suppose $x_i \in_{\min} \mathbb{C}^{m_i}$, for $1 \leq i \leq k$ and $m_1 < m_2 < \cdots < m_k$; and $y_j \in_{\min} \mathbb{C}^{m'_j}$, for $1 \leq j \leq k'$ and $m'_1 < m'_2 < \cdots < m'_k$. Also, suppose $\theta_i, \theta'_i \not\equiv 0 \bmod 2\pi$ for each i. Let $A_i = A_{(\theta_i,x_i)}$ and $B_j = A_{(\theta'_j,y_j)}$. If*

$$A_1 \cdot A_2 \cdots A_k \;=\; B_{k'} \cdot B_{k'-1} \cdots B_1,$$

then the following hold:

a) *$k = k'$ and $(m_1, m_2, \dots, m_k) = (m'_1, \dots, m'_{k'})$;*

b) $A_i \;=\; B_1^{-1} \cdot B_2^{-1} \cdots B_{i-1}^{-1} \cdot B_i \cdot B_{i-1} \cdots B_1$ *for* $1 \leq i \leq k$*; and*

c) $B_i \;=\; A_1 \cdot A_2 \cdots A_{i-1} \cdot A_i \cdot A_{i-1}^{-1} \cdots A_1^{-1}$ *for* $1 \leq i \leq k$.

In the cases of $k = 1$ *in b) and c), we let* $A_0 = B_0 = I_m$ *so they are understood to be* $A_1 = B_1$.

Proof. We let C_i denote the RHS of the equation in b) but for $1 \leq i \leq k'$. Since $B_{i-1} \cdot B_{i-2} \cdots B_1$ leaves pointwise invariant $(\mathbb{C}^{m'_i})^{\perp}$, we conclude $B_{i-1} \cdot B_{i-2} \cdots B_1(y_i) = y'_i \in_{min} \mathbb{C}^{m'_i}$; hence by property iii) for pseudo rotations, $C_i = A_{(\theta'_i, y'_i)}$. Thus, we have that A has two different Schubert factorizations with increasing Schubert symbols $(m_1, m_2, \ldots, m_k)$ and $(m'_1, \ldots, m'_{k'})$. By the uniqueness of the Schubert symbols, we obtain a).

Furthermore, by the uniqueness of the Schubert decomposition stated in Theorem 3.7 (for increasing Schubert decomposition) and Remark 3.8, it then furthermore follows that $A_i = C_i$ for all i so b) holds. Lastly, the uniqueness of the increasing order Schubert decomposition implies by taking inverses that we also have uniqueness of decreasing order Schubert decomposition. Then, the corresponding analogue of the argument for b) yields c). □

We then have the following corollary

Corollary 3.5. *If* $B \in SU_n$*, then*

$$\mathbf{m}(B) \;=\; \mathbf{m}(B^{-1}) \;=\; \mathbf{m}(\overline{B}) \;=\; \mathbf{m}(B^T)\,.$$

Proof. Given an increasing Schubert factorization $B = A_1 \cdot A_2 \cdots A_k$ for $A_i = A_{(\theta_i, x_i)}$ with Schubert symbol $\mathbf{m} = (m_1, m_2, \ldots, m_k)$, then $B^{-1} = A_k \cdot A_{k-1} \cdots A_1$ is a Schubert factorization for decreasing order. This has the decreasing Schubert symbol $(m_k, m_{k-1}, \ldots, m_1)$, and hence B^{-1} has the same increasing Schubert symbol $\mathbf{m}$.

Next, $\overline{B} = \overline{A_1} \cdot \overline{A_2} \cdots \overline{A_k}$, and by property iv) of pseudo-rotations $\overline{A_i} = A_{(-\theta_i, \bar{x_i})}$ so the Schubert Symbol is the same.

Lastly, as $B \in SU_n$, $B^T = \overline{B^{-1}}$, which combined with the two other properties implies that it has the same Schubert symbol. □

Remark 3.6. We will use the increasing order for the Schubert symbol to be in agreement with that used for the Schubert decomposition as in Milnor-Stasheff [MS]. In fact, if $A = A_1 \cdot A_2 \cdots A_k$ for $A_i = A_{(\theta_i, x_i)}$ with Schubert symbol $\mathbf{m} = (m_1, m_2, \ldots, m_k)$, and we let $V = \mathbb{C} < x_1, \ldots, x_k >$, then $\dim_{\mathbb{C}} V \cap \mathbb{C}^{m_i} = i$ so V as an element of the Grassmannian $G_k(\mathbb{C}^n)$ would also have Schubert symbol $\mathbf{m}$. In [KM], the decreasing order Schubert symbol is used; however, we easily change between the two.

We next state the form of the Schubert decomposition given in terms of the Schubert factorization giving the Schubert types for elements of SU_n.

Schubert Decomposition for SU_n.

In describing the Schubert decomposition for SU_n, we are giving a version of that contained in [W], [Mi], [Y] and summarized in [KM] (but using instead an increasing order).

Given an increasing sequence $m_1 < m_2 < \cdots < m_k$ with $1 < m_1$ and $m_k \leq n$, which we denote by $\mathbf{m} = (m_1, m_2, \ldots, m_k)$, we define a map

$$\psi_{\mathbf{m}} : S\mathbb{C}P^{m_1 - 1} \times S\mathbb{C}P^{m_2 - 1} \times \cdots \times S\mathbb{C}P^{m_k - 1} \longrightarrow SU_n\,,$$

where SX denotes the suspension of X. This is given as follows:

First, we define a simpler map for $m \leq n$, $I = [0, 1]$ and a complex line $L \subset \mathbb{C}^m$,

$$\tilde{\psi}_m : I \times \mathbb{C}P^{m-1} \to SU_n$$

defined by $\tilde{\psi}_m(t, L) = A_{(2\pi t, L)}$. Since $A_{(0,L)} = A_{(2\pi, L)} = I_n$ independent of L, this descends to a map $\psi_m : S\mathbb{C}P^{m-1} \to SU_n$. Then, we define

$$\begin{aligned} \psi_{\mathbf{m}}((t_1, L_1), \ldots, (t_k, L_k)) &= A_{(-2\pi\tilde{t}, e_1)} \cdot \psi_{m_1}(t_1, L_1) \cdot \psi_{m_2}(t_2, L_2) \cdots \psi_{m_k}(t_k, L_k) \\ &= A_{(-2\pi\tilde{t}, e_1)} \cdot A_{(2\pi t_1, L_1)} \cdot A_{(2\pi t_2, L_2)} \cdots A_{(2\pi t_k, L_k)} \,, \end{aligned} \tag{3.4}$$

where $\tilde{t} = \sum_{j=1}^{k} t_j$. We note that the first factor $A_{(-2\pi\tilde{t}, e_1)}$ ensures the product is in SU_n as in the splitting for (2.1).

We observe that each $I \times \mathbb{C}P^{m-1}$ has an open dense cell

$$E_m = (0, 1) \times \{x = (x_1, \ldots, x_m, 0, \ldots 0) : (x_1, \ldots, x_m) \in S^{2m-1} \text{ and } x_m > 0\},$$

which is of dimension $2m - 1$ (as $x_m = \sqrt{1 - \sum_{j=1}^{m-1} |x_j|^2}$). Also, if $x = (x_1, \ldots, x_m, 0, \ldots 0)$ with $x_m > 0$, then $x \in_{\min} \mathbb{C}^m$.

We now introduce some notation and denote

$$\tilde{S}_{\mathbf{m}} = S\mathbb{C}P^{m_1-1} \times S\mathbb{C}P^{m_2-1} \times \cdots \times S\mathbb{C}P^{m_k-1} ;$$

also, we consider the corresponding cell

$$E_{\mathbf{m}} = E_{m_1} \times E_{m_2} \times \cdots \times E_{m_k},$$

and the image $S_{\mathbf{m}} = \psi_{\mathbf{m}}(E_{\mathbf{m}})$ in SU_n. Then, $E_{\mathbf{m}}$ is an open dense cell in $\tilde{S}_{\mathbf{m}}$ with

$$\dim_{\mathbb{R}} E_{\mathbf{m}} = \sum_{j=1}^{k} (2m_j - 1) = 2|\mathbf{m}| - \ell(\mathbf{m})$$

for $|\mathbf{m}| = \sum_{j=1}^{k} m_j$ and $\ell(\mathbf{m}) = k$, which we refer to as the length of $\mathbf{m}$. Also, the image $S_{\mathbf{m}} = \psi_{\mathbf{m}}(E_{\mathbf{m}})$ consists of elements of SU_n of Schubert type $\mathbf{m}$. Furthermore, $\overline{S_{\mathbf{m}}} = \psi_{\mathbf{m}}(\tilde{S}_{\mathbf{m}})$. Then the results of Whitehead, Miller and Yokota together give the following Schubert decomposition of SU_n.

Theorem 3.7. *The Schubert decomposition of SU_n has the following properties:*

a) *SU_n is the disjoint union of the $S_{\mathbf{m}}$ as $\mathbf{m} = (m_1, \ldots, m_k)$ varies over all increasing sequences with $1 < m_1$, $m_k \leq n$, and $0 \leq k \leq n - 1$.*
b) *The map $\psi_{\mathbf{m}} : E_{\mathbf{m}} \to S_{\mathbf{m}}$ is a homeomorphism.*
c) *$(\overline{S_{\mathbf{m}}} \backslash S_{\mathbf{m}}) \subset \cup_{\mathbf{m}'} S_{\mathbf{m}'}$, where the union is over all $S_{\mathbf{m}'}$ with $\dim S_{\mathbf{m}'} < \dim S_{\mathbf{m}}$.*
d) *the Schubert cells $S_{\mathbf{m}}$ are preserved under taking inverses, conjugates, and transposes.*

We note that d) follows from Corollary 3.5.

Hence, the Schubert decomposition by the cells $S_{\mathbf{m}}$ is a cell decomposition of SU_n. The cells $S_{\mathbf{m}}$ are referred to as the *Schubert cells of SU_n*. We note that as $\overline{S_{\mathbf{m}}}$ is the image of the "singular manifold" $\tilde{S}_{\mathbf{m}}$ which has a Borel-Moore fundamental class, we can describe in §5 the homology of SU_n in terms of the images of these fundamental classes.

Remark 3.8. There is an analogous Schubert decomposition for U_n where the Schubert symbols can include $m_1 = 1$.

4. Schubert Decomposition for Symmetric Spaces

For the Milnor fibers for the varieties of singular matrices, we have compact models which are symmetric spaces. To give the Schubert decomposition of these, we use the results of Kadzisa and Mimura [KM] which modifies the Schubert decomposition given for SU_n to apply to the Cartan models for the symmetric spaces. We have given the Schubert decomposition for SU_n in the previous section so we will consider the form it takes for both SU_n/SO_n and SU_{2n}/Sp_n.

We again use the standard flag $0 \subset \mathbb{C} \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^n$ and the same notation for pseudo-rotations as in §3.

Schubert Decomposition for SU_n/SO_n.

We consider an element of the Cartan model $\mathcal{C}_n^{(sy)}$ for SU_n/SO_n. If $B \in \mathcal{C}_n^{(sy)}$ we have that $B \in SU_n$ and $B = B^T$. By Lemma 2.1, there is an orthonormal basis of real eigenvectors x_i for B. Hence, each $< x_i > \in \mathbb{R}P^{n-1}$. Then B can be written as a product of pseudo-rotations about complexifications of real hyperplanes $\mathbb{C} < x_i >^{\perp}$. We will refer to such a pseudo-rotation $A_{(\theta,x)}$ for a real vector x as an $\mathbb{R}$-pseudo-rotation. There are two problems in trying to duplicate the reasoning used for the Schubert decomposition for SU_n. First, there is no analogue of Lemma 3.2 for products of $\mathbb{R}$-pseudo-rotations. Second, it need not be true that the ordered product of $\mathbb{R}$-pseudo-rotations $A_{(\theta,x_i)}$ is an element of $\mathcal{C}_n^{(sy)}$ if the vectors x_i are not mutually orthogonal.

The solution obtained by Kadzisa-Mimura is to use instead "ordered symmetric factorizations" by $\mathbb{R}$-pseudo-rotations. Specifically it will be a product resulting from the successive application of Cartan conjugates by $\mathbb{R}$-pseudo rotations, which always yields elements of $\mathcal{C}_n^{(sy)}$.

Then, in describing the Schubert decomposition for SU_n/SO_n, we are giving a version of that contained in [KM], except we again define maps from products of cones on real projective spaces whose open cells give the cell decomposition.

Given an increasing sequence $m_1 < m_2 < \cdots < m_k$ with $1 < m_1$ and $m_k \leq n$, which we denote by $\mathbf{m} = (m_1, m_2, \ldots, m_k)$ we define a map

$$\psi_{\mathbf{m}}^{(sy)} : (C\mathbb{R}P^{m_1-1}) \times (C\mathbb{R}P^{m_2-1}) \times \cdots \times (C\mathbb{R}P^{m_k-1}) \longrightarrow SU_n\,,$$

with $CX = (I \times X)/(\{0\} \times X)$ for $I = [0,1]$, denoting the cone on X. This is given as follows:

First, we define a simpler map for $m \leq n$, $I = [0,1]$ and a real line $L \subset \mathbb{R}^m$,

$$\tilde{\psi}_m^{(sy)} : C\mathbb{R}P^{m-1} \to SU_n$$

defined by $\tilde{\psi}_m^{(sy)}(t, L) = A_{(\pi t, L_{\mathbb{C}})}$, with $L_{\mathbb{C}}$ denoting the complexification of the real line L. Note this factors through the cone as $A_{(0,L_{\mathbb{C}})} = Id$, independent of L. We will henceforth abbreviate this to $A_{(\pi t, L)}$. Then, we extend this to a map

$$\tilde{\psi}_{\mathbf{m}}^{(sy)} : \prod_{i=1}^{k} (C\mathbb{R}P^{m_i-1}) \;\longrightarrow\; SU_n$$

defined by

$$\begin{aligned}
\tilde{\psi}_{\mathbf{m}}^{(sy)}((t_1, L_1), \ldots, (t_k, L_k)) \;&=\; A_{(-\pi\tilde{t}, e_1)} \cdot \psi_{m_1}(t_1, L_1) \cdot \psi_{m_2}(t_2, L_2) \cdots \psi_{m_k}(t_k, L_k) \\
&=\; A_{(-\pi\tilde{t}, e_1)} \cdot A_{(\pi t_1, L_1)} \cdot A_{(\pi t_2, L_2)} \cdots A_{(\pi t_k, L_k)}\,. \qquad (4.1)
\end{aligned}$$

where $\tilde{t} = \sum_{j=1}^{k} t_j$. We note that the first factor $A_{(-\pi\tilde{t}, e_1)}$ ensures the product is in SU_n as in the splitting for (2.1). Then we define

(4.2)

$$\psi_{\mathbf{m}}^{(sy)}((t_1, L_1), \ldots, (t_k, L_k)) \;=\; \tilde{\psi}_{\mathbf{m}}^{(sy)}((t_1, L_1), \ldots, (t_k, L_k)) \cdot \left(\tilde{\psi}_{\mathbf{m}}^{(sy)}((t_1, L_1), \ldots, (t_k, L_k))\right)^T.$$

We note that the RHS is the Cartan conjugate of I by $\tilde{\psi}_{\mathbf{m}}((t_1, L_1), \ldots, (t_k, L_k)) \in SU_n$ and thus is in the Cartan model $\mathcal{C}_n^{(sy)}$. It can also be obtained by successively applying to I the Cartan conjugates by the $A_{(\pi t_j, L_j)}$, for $j = k, k-1, \ldots, 1, 0$, where we let $A_{(\pi t_0, L_0)}$ denote $A_{(-\pi\tilde{t}, e_1)}$ (each of these are, strictly speaking, Cartan conjugates for U_n but their product is in SU_n).

We observe that each $C\mathbb{R}P^{m-1}$ has an open dense cell

$$E_m^{(sy)} = (0,1) \times \{x = (x_1, \dots, x_m, 0, \dots 0) : (x_1, \dots, x_m) \in S^{m-1} \text{ and } x_m > 0\}$$

which is of dimension m. Also, if $x = (x_1, \dots, x_m, 0, \dots 0)$ with $x_m > 0$, then $x \in_{\min} \mathbb{C}^m$.

We now introduce some notation and denote

$$\tilde{S}_{\mathbf{m}}^{(sy)} = (C\mathbb{R}P^{m_1-1}) \times (C\mathbb{R}P^{m_2-1}) \times \cdots \times (C\mathbb{R}P^{m_k-1}),$$

the cell

$$E_{\mathbf{m}}^{(sy)} \;=\; E_{m_1}^{(sy)} \times E_{m_2}^{(sy)} \times \cdots \times E_{m_k}^{(sy)}\,,$$

and $S_{\mathbf{m}}^{(sy)} = \psi_{\mathbf{m}}(E_{\mathbf{m}}^{(sy)})$. Then, $E_{\mathbf{m}}^{(sy)}$ is an open dense cell in $\tilde{S}_{\mathbf{m}}^{(sy)}$ with

$$\dim_{\mathbb{R}} E_{\mathbf{m}}^{(sy)} = |\mathbf{m}| \stackrel{def}{=} \sum_{j=1}^{k} m_j .$$

Also, the image $S_{\mathbf{m}}^{(sy)} = \psi_{\mathbf{m}}(E_{\mathbf{m}}^{(sy)})$ consists of elements of SU_n of *real Schubert type* $\mathbf{m}$. Furthermore, $\overline{S_{\mathbf{m}}^{(sy)}} = \psi_{\mathbf{m}}^{(sy)}(\tilde{S}_{\mathbf{m}}^{(sy)})$. Then the results of Kadzisa-Mimura [KM, Thm 6.7] give the following Schubert decomposition of SU_n/SO_n.

Theorem 4.1. *The Schubert decomposition of SU_n/SO_n has the following properties:*

a) *SU_n/SO_n is the disjoint union of the $S_{\mathbf{m}}^{(sy)}$ as $\mathbf{m} = (m_1, \dots, m_k)$ varies over all increasing sequences with $1 < m_1$, $m_k \le n$, and $0 \le k \le n-1$.*
b) *The map $\psi_{\mathbf{m}}^{(sy)} : E_{\mathbf{m}}^{(sy)} \to S_{\mathbf{m}}^{(sy)}$ is a homeomorphism.*
c) *$(\overline{S_{\mathbf{m}}^{(sy)}} \backslash S_{\mathbf{m}}^{(sy)}) \subset \cup_{\mathbf{m}'} S_{\mathbf{m}'}^{(sy)}$, where the union is over all $S_{\mathbf{m}'}^{(sy)}$ with $\dim S_{\mathbf{m}'}^{(sy)} < \dim S_{\mathbf{m}}^{(sy)}$.*

Hence, the Schubert decomposition by the cells $S_{\mathbf{m}}^{(sy)}$ is a cell decomposition of SU_n/SO_n. We refer to the cells $S_{\mathbf{m}}^{(sy)}$ as the *symmetric Schubert cells of SU_n/SO_n*. We also refer to the factorization given by (4.2) for elements B of $S_{\mathbf{m}}^{(sy)}$ as the *ordered symmetric factorization* and the corresponding Schubert symbol is denoted by $\mathbf{m}^{(sy)}(B)$.

Remark 4.2. Unlike the case of SU_n, in general the $\tilde{S}_{\mathbf{m}}^{(sy)}$ do not carry a top-dimensional fundamental class. In the case of a simple Schubert symbol (m_1), since L is real, $A_{(\pi, L)}$ is the complexification of a real reflection about the real hyperplane $L_{\mathbb{C}}^{\perp}$ and hence it is its own inverse and transpose. This is independent of L. Then,

$$\begin{aligned} \psi_{(m_1)}^{(sy)}(\pi, L_1) \;&=\; A_{(-\pi, e_1)} \cdot A_{(\pi, L_1)} \cdot A_{(\pi, L_1)}^T \cdot A_{(-\pi, e_1)}^T \\ &=\; A_{(-\pi, e_1)} \cdot A_{(\pi, L_1)} \cdot A_{(\pi, L_1)}^{-1} \cdot A_{(-\pi, e_1)}^{-1} \;=\; Id \end{aligned} \tag{4.3}$$

Thus, $\psi_{(m_1)}^{(sy)}(\{1\} \times \mathbb{R}P^{m_1-1}) = Id$ and so factors to give a map $\psi_{(m_1)}^{(sy)} : S\mathbb{R}P^{m_1-1} \to \mathcal{C}_n^{(sy)}$. Hence, for the simple Schubert symbol (m_1), $\overline{E_{(m_1)}^{(sy)}} = \psi_{(m_1)}^{(sy)}(S\mathbb{R}P^{m_1-1})$ has a fundamental class which is the image of the fundamental class of $S\mathbb{R}P^{m_1-1}$.

For a general symmetric Schubert symbol $\mathbf{m} = \mathbf{m}^{(sy)} = (m_1, m_2, \dots, m_k)$, if $(SU_n/SO_n)^{(\ell)}$ denotes the ℓ-skeleton of SU_n/SO_n, then $\psi_{\mathbf{m}}^{(sy)}$ composed with the projection does factor through to give a map

$$\tilde{\psi}_{\mathbf{m}}^{(sy)\,\prime} : \prod_{i=1}^{k} S\mathbb{R}P^{m_i-1} \to (SU_n/SO_n)/(SU_n/SO_n)^{(|\mathbf{m}|-1)}\,.$$

The product again carries a fundamental class and in §5 we see how these images in homology correspond to generators.

Schubert Decomposition for SU_{2n}/Sp_n.

For the Schubert decomposition for SU_{2n}/Sp_n we will largely follow [KM, §7]; except that for the geometric properties of Milnor fibers we will emphasize the use of the quaternionic structure on $\mathbb{C}^{2n}$. We already have the complex structure giving multiplication by $\mathbf{i}$. We extend it to $\mathbb{H}$ by defining multiplication by $\mathbf{j}$ by $\mathbf{j}x = J_n\bar{x}$ for $x \in \mathbb{C}^{2n}$ with $\bar{x}$ complex conjugation (so $\mathbf{k}x = \mathbf{ij}x$). Then, it is a standard check (see e.g. [GW, §1.4.4]) that this defines a quaternionic action so $\mathbb{C}^{2n} \simeq \mathbb{H}^n$. For this quaternionic structure, each subspace $\mathbb{C}^{2m}$ spanned by $\{e_1, \dots, e_{2m}\}$ is a quaternionic subspace.

Let $< x, y >= x^T \cdot \bar{y}$ (for column vectors x and y) denote the Hermitian inner product on $\mathbb{C}^{2n}$. It has the following directly verifiable properties:

i) multiplication by J_n is $\mathbb{H}$-linear;
ii) $< \mathbf{j}x, \mathbf{j}y >= \overline{< x, y >}$; and
iii) (by ii)) both $< x, \mathbf{j}x >= 0$ and $< \mathbf{j}x, y >= -\overline{< x, \mathbf{j}y >}$.

An element B of the Cartan model for SU_{2n}/Sp_n is characterized from (2.3) by

$$(BJ_n)^T = -BJ_n.$$

so that BJ_n is an element of SU_{2n} and is skew-symmetric. This has the following consequence, which is basically equivalent to [KM, Thm 3.4].

Lemma 4.3. *If $B \in \mathcal{C}_{2n}^{(sk)}$, the Cartan model for SU_{2n}/Sp_n, then*
a) *$B\mathbf{j}x = \mathbf{j}B^*x$; and*
b) *if B satisfies the condition in a), then the eigenspaces of B are $\mathbb{H}$-subspaces.*

Proof. For a), this is a simple calculation.

$$B\mathbf{j}x \,=\, BJ_n\bar{x} \,=\, -(BJ_n)^T\bar{x} \,=\, -J_n^T B^T \bar{x} \,=\, J_n\overline{\bar{B}^T x} \,=\, J_n\overline{B^* x} \,=\, \mathbf{j}B^*x\,.$$

For b), we observe that if $Bx = \lambda x$, then as $B \in SU_{2n}$, $B^* = B^{-1}$ and $|\lambda| = 1$ so

$$B\mathbf{j}x \,=\, \mathbf{j}B^*x \,=\, \mathbf{j}B^{-1}x \,=\, \mathbf{j}\lambda^{-1}x \,=\, J_n\overline{\lambda^{-1}x} \,=\, \lambda J_n\bar{x} \,=\, \lambda\mathbf{j}x\,.$$

Thus, the λ-eigenspace of B is invariant under multiplication by $\mathbf{j}$. □

We will refer to a $B \in U_{2n}$ which satisfies the condition in a) of Lemma 4.3 as being $\mathbb{H}$*-linear. To factor such a matrix, we use a version of pseudo-rotation for $\mathbb{H}^n$. Given a quaternionic line $L \subset \mathbb{C}^{2n}$, let $L^\perp$ be the quaternionic hyperplane orthogonal to L. We define an $\mathbb{H}$-pseudo-rotation by an angle θ, $\tilde{A}_{(\theta,L)}$ which is the identity on $L^\perp$ and is multiplication by $e^{i\theta}$ on L. It is $\mathbb{C}$-linear and can be checked to be $\mathbb{H}$*-linear. If $x \in L$ is a unit vector, then by property iii), $\{x, \mathbf{j}x\}$ is an orthonormal basis for L. Then, $\tilde{A}_{(\theta,L)}$ can be written as a product of pseudo-rotations $A_{(\theta,x)}A_{(\theta,\mathbf{j}x)}$, which commute. By the properties of pseudo-rotations, we have the following properties of $\mathbb{H}$-pseudo-rotations.

i) $\tilde{A}^*_{(\theta,L)} = \tilde{A}^{-1}_{(\theta,L)} = \tilde{A}_{(-\theta,L)}$;
ii) $\overline{\tilde{A}_{(\theta,L)}} = \tilde{A}_{(-\theta,\bar{L})}$, where $\bar{L}$ is the $\mathbb{H}$-line generated by $\bar{x}$; and
iii) $\tilde{A}^T_{(\theta,L)} = \tilde{A}_{(\theta,\bar{L})}$;
iv) $\det(\tilde{A}_{(\theta,L)}) = e^{2i\theta}$;
v) If $L \perp L'$ then $\tilde{A}_{(\theta,L)}$ and $\tilde{A}_{(\theta,L')}$ commute;
vi) $\tilde{A}_{(\theta,L)}$ is $\mathbb{H}$*-linear.

Proof. All of i) - v) follow directly from the properties of pseudo-rotations. For vi) we observe that $\tilde{A}_{(\theta,L)}$ is characterized as a unitary matrix which has L for the eigenspace for $e^{i\theta}$ and $L^\perp$

as the eigenspace for the eigenvalue 1. Thus, for vi), as both L and $L^{\perp}$ are $\mathbb{H}$-subspaces we see $\tilde{A}_{(\theta,L)} \equiv Id$ on $L^{\perp}$ and for $x \in L$,

$$\tilde{A}_{(\theta,L)}(\mathbf{j}x) \;=\; e^{i\theta}\mathbf{j}x \;=\; \mathbf{j}e^{-i\theta}x \;=\; \mathbf{j}\tilde{A}^{-1}_{(\theta,L)}(x)\,.$$

As $\tilde{A}^{*}_{(\theta,L)} = \tilde{A}^{-1}_{(\theta,L)}$, we see that $\tilde{A}_{(\theta,L)}(\mathbf{j}x) = \mathbf{j}\tilde{A}^{*}_{(\theta,L)}(x)$ on each summand L and $L^{\perp}$; hence they are equal. □

In addition, we can give a unique representation of $\tilde{A}_{(\theta,L)}$ as an ordered product of pseudo-rotations.

Lemma 4.4. *Given an $\mathbb{H}$-line $L \subset_{\min} \mathbb{C}^{2m}$, there is a unique unit vector $x \in L \cap \mathbb{C}^{2m-1}$ of the form $x = (x_1, \dots, x_{2m-1}, 0)$ with $x_{2m-1} > 0$ so that $\mathbf{j}x = (\bar{x_2}, -\bar{x_1}, \bar{x_4}, -\bar{x_3}, \dots, 0, -x_{2m-1})$. Hence, $\tilde{A}_{(\theta,L)}$ can be uniquely written $A_{(\theta,x)} \cdot A_{(\theta,\mathbf{j}x)}$.*

Proof. As $\dim_{\mathbb{C}} L = 2$. $\dim_{\mathbb{C}}(L \cap \mathbb{C}^{2m-1}) = 1$. It is ≥ 1, and otherwise it would be 2, i.e. $L \subset \mathbb{C}^{2m-1}$. Then, under the $\mathbb{H}$-linear projection $p : \mathbb{C}^{2m} \to \mathbb{C}^{2m}/\mathbb{C}^{2m-2}$ the image of L, which is an $\mathbb{H}$-subspace would have $\mathbb{C}$-dimension 1, a contradiction.

As $\dim_{\mathbb{C}}(L \cap \mathbb{C}^{2m-1}) = 1$, and $L \not\subset \mathbb{C}^{2m-2}$, we may find a unit vector $x \in L$ of the form $x' = (x'_1, \dots, x'_{2m-1}, 0)$ with $x'_{2m-1} \neq 0$. Multiplying x' by an appropriate unit complex number we obtain x with $x_{2m-1} > 0$. Then, $\mathbf{j}x$ is as stated and so is $\tilde{A}_{(\theta,L)}$. □

Whitehead-Type Ordered Factorization.

For an $\mathbb{H}^*$-linear $B \in U_{2n}$, we may initially factor it as a product of $\mathbb{H}$-pseudo-rotations in a manner similar to the symmetric case as follows. Each eigenspace V_λ of B with $\lambda = e^{i\theta} \neq 1$ is an $\mathbb{H}$-subspace. We choose the smallest m'_1 so that $V_\lambda \cap \mathbb{C}^{2m'_1} \neq 0$, and hence is an $\mathbb{H}$-line $L_1^{(\lambda)}$. We successively repeat this for $(L_1^{(\lambda)})^{\perp} \cap V_\lambda$ and obtain an orthogonal decomposition $V_\lambda = L_1^{(\lambda)} \oplus L_2^{(\lambda)} \cdots L_{k'}^{(\lambda)}$ with $L_j^{(\lambda)} \subset_{\min} \mathbb{C}^{2m'_j}$ and $m'_1 < m'_2 < \cdots < m'_{k'}$. Each $L_j^{(\lambda)}$ gives an $\mathbb{H}$-pseudo-rotation $\tilde{A}_{(\theta, L_j^{(\lambda)})}$. We may do this for each eigenvalue $\lambda \neq 1$. Because different L_j are orthogonal, the corresponding $\mathbb{H}$-pseudo-rotations commute. Thus, we may factor B as a product of $\mathbb{H}$-pseudo-rotations

$$B \;=\; \tilde{A}_{(\theta_1,L_1)} \cdot \tilde{A}_{(\theta_2,L_2)} \cdots \tilde{A}_{(\theta_k,L_k)} \tag{4.4}$$

where $L_j \subset_{min} \mathbb{C}^{2m_j}$, $1 \leq m_1 \leq m_2 \leq \cdots \leq m_k$, and several θ_j may be equal. However, this is not an ordered factorization as some of the m_j may be equal.

We would like to apply an analogue of the Whitehead Lemma 3.2 to products of $\mathbb{H}$-pseudo-rotations. However, it is not possible to do so remaining in the category of $\mathbb{H}$-pseudo-rotations. For example, if $B \in U_{2n}$ then $B \cdot \tilde{A}_{(\theta,L)} \cdot B^{-1}$ is a unitary transformation with $B(L)$ as the eigenspace for $e^{i\theta}$ and $B(L^{\perp}) = (B(L))^{\perp}$ as the eigenspace for the eigenvalue 1. While $B(L)$ is a 2-dimensional complex space, it need not be an $\mathbb{H}$-subspace.

However, there is an alternate way to proceed which uses Lemma 4.4. We may uniquely decompose each $\mathbb{H}$-pseudo-rotation in (4.4) into a product of pseudo-rotations about orthogonal planes which thus all commute so that (4.4) may be rewritten

$$B \;=\; A_{(\theta_1,x_1)} \cdot A_{(\theta_2,x_2)} \cdots A_{(\theta_k,x_k)} \cdot A_{(\theta_k,\mathbf{j}x_k)} \cdots A_{(\theta_2,\mathbf{j}x_2)} \cdot A_{(\theta_1,\mathbf{j}x_1)} \tag{4.5}$$

Then, we can progressively apply Whitehead's Lemma to the factors $A_{(\theta_j,x_j)}$ beginning with the highest j and proceeding left to the lowest to obtain an ordered factorization for the product involving the $A_{(\theta_j,x_j)}$. Then for each application of Whitehead's Lemma for these, there is a corresponding application of it for the $A_{(\theta_j,\mathbf{j}x_j)}$ from the left proceeding to the right using the following lemma.

Lemma 4.5. *Given a relation between pseudo-rotations*

$$A_{(\theta,x)} \cdot A_{(\theta',x')} \;=\; A_{(\theta_1,x_1)} \cdot A_{(\theta_2,x_2)}\,, \tag{4.6}$$

there is a corresponding relation

$$A_{(\theta',\mathbf{j}x')} \cdot A_{(\theta,\mathbf{j}x)} \;=\; A_{(\theta_2,\mathbf{j}x_2)} \cdot A_{(\theta_1,\mathbf{j}x_1)}\,. \tag{4.7}$$

Proof. First, apply the transpose to each side of (4.6) and then conjugate with J_n to obtain

$$(J_n \cdot A^T_{(\theta',x')} \cdot J_n^{-1}) \cdot (J_n \cdot A^T_{(\theta,x)} \cdot J_n^{-1}) \;=\; (J_n \cdot A^T_{(\theta_2,x_2)} \cdot J_n^{-1}) \cdot (J_n \cdot A^T_{(\theta_1,x_1)} \cdot J_n^{-1}). \tag{4.8}$$

Then, for any pseudo-rotation $A_{(\theta,x)}$,

$$J_n \cdot A^T_{(\theta,x)} \cdot J_n^{-1} = J_n \cdot A_{(\theta,\bar{x})} \cdot J_n^{-1} = A_{(\theta,J_n\bar{x})} = A_{(\theta,\mathbf{j}x)}\,. \tag{4.9}$$

Thus, applying (4.9) to each product in (4.8) yields (4.7). □

Then, by applying Whitehead's Lemma successively to appropriate adjacent pairs $A_{(\theta_j,x_j)} \cdot A_{(\theta_{j'},x_{j'})}$ and Lemma 4.5 to the corresponding pairs $A_{(\theta_{j'},\mathbf{j}x_{j'})} \cdot A_{(\theta_j,\mathbf{j}x_j)}$ we may rewrite

$$B \;=\; A_{(\theta'_1,x'_1)} \cdot A_{(\theta'_2,x'_2)} \cdots A_{(\theta'_k,x'_k)} \cdot A_{(\theta'_k,\mathbf{j}x'_k)} \cdots A_{(\theta'_2,\mathbf{j}x'_2)} \cdot A_{(\theta'_1,\mathbf{j}x'_1)} \tag{4.10}$$

with the $A_{(\theta'_j,x'_j)}$ in increasing order and the $A_{(\theta'_j,\mathbf{j}x'_j)}$ in decreasing order.

Kadzisa-Mimura Ordered Skew-Symmetric Factorization.

In fact, this is the skew-symmetric factorization of $B \in \mathcal{C}_m^{(sk)}$ given by Kadzisa-Mimura. We further rewrite (4.10) using the properties of pseudo-rotations $\sigma(A_i^{-1}) = A_{(\theta_i,\mathbf{j}x_i)}$. Hence, B in (4.10) can be rewritten either as

$$B \;=\; \Big(A_{(\theta_1,x_1)} \cdot A_{(\theta_2,x_2)} \cdots A_{(\theta_k,x_k)} \cdot J_n \cdot A^T_{(\theta_k,x_k)} \cdots A^T_{(\theta_1,x_1)}\Big) \cdot J_n^{-1} \tag{4.11}$$

or alternatively for each $A_j = A_{(\theta_j,x_j)}$ as

$$B \;=\; A_1 \cdot A_2 \cdots A_k \cdot \sigma(A_k^{-1}) \cdots \sigma(A_1^{-1})\,, \tag{4.12}$$

which is a Cartan conjugate of I and hence belongs to $F_m^{(sk)\,c}$.

What we have not yet considered is the skew-symmetric Schubert symbol associated to this factorization. We shall do so in giving in the next section the Kadzisa-Mimura algorithm for obtaining the ordered skew-symmetric factorization from the full Whitehead ordered factorization.

We next define the maps for the cell decomposition of SU_{2n}/Sp_n via the Cartan Model $\mathcal{C}_{2n}^{(sk)}$. In describing the Schubert decomposition for SU_{2n}/Sp_n, we are giving a version that modifies that contained in [KM] to associate to the Borel-Moore fundamental classes of products of suspensions of quaternionic projective spaces the Borel-Moore fundamental classes of the "Schubert cycles" obtained as the closures of the Schubert cells. However, unlike the general and symmetric cases, we cannot directly do this by expressing the closures of Schubert cells as the images of the products of suspensions of quaternionic projective spaces. Instead we proceed through intermediate spaces which are products of suspensions of complex projective spaces.

For any $m > 0$, we define via the quaternionic structure on $\mathbb{C}^{2m} \simeq \mathbb{H}^m$ a map

$$\chi_m : \mathbb{C}P^{2m-2} \to \mathbb{H}P^{m-1}$$

by $\chi_m(L) = L + \mathbf{j}L$ for complex lines $L \subset \mathbb{C}^{2m-1}$. For a quaternionic line $Q \subset_{\min} \mathbb{H}^m$, Q has a unique element $x = (x_1, \dots, x_{4(m-1)}, x_{4m-3}, 0) \in S^{4m-3} \subset \mathbb{C}^{2m-1}$ with $x_{4m-3} > 0$. Then,

$$\mathbf{j}\,x = (\bar{x}_2, -\bar{x}_1, \bar{x}_4, -\bar{x}_3, \dots, \bar{x}_{4(m-1)}, -\bar{x}_{4m-5}, 0, -x_{4m-3})\,.$$

Hence, the set of such Q are parametrized by the cell E^{4m-4} in S^{4m-3} with $x_{4m-3} > 0$ (since $x_{4m-3} = \sqrt{1 - \sum_{j=1}^{4(m-1)} |x_j|^2}$). However, this cell also parametrizes the open dense subset of

$L \in \mathbb{C}P^{2m-2}$ with $L \subset_{\min} \mathbb{C}^{2m-1}$. The map χ_m acts as the identity on these parametrized cells of dimension $4m-4$, and the complements have lower dimensions. We may then take the suspension $S\chi_m : S\mathbb{C}P^{2m-2} \to S\mathbb{H}P^{m-1}$, which now is a homeomorphism on the cell $(0,1) \times E^{4m-4}$ of dimension $4m-3$. Thus, $S\chi_{m\,*}$ sends the Borel-Moore fundamental class of $S\mathbb{C}P^{2m-2}$ to that of $S\mathbb{H}P^{m-1}$.

Then, given an increasing sequence $1 < m_1 < m_2 < \cdots < m_k \le n$, which we denote by $\mathbf{m}^{(sk)} = (m_1, m_2, \ldots, m_k)$, we may form the product map

$$\tilde{\chi}^{(sk)}_{\mathbf{m}} \;=\; S\chi_{m_1} \times S\chi_{m_2} \times \cdots \times S\chi_{m_k}\,,$$

which again sends the Borel-Moore fundamental class of the product $S\mathbb{C}P^{2m_1-2}\times\cdots\times S\mathbb{C}P^{2m_k-2}$ to that of $S\mathbb{H}P^{m_1-1} \times \cdots \times S\mathbb{H}P^{m_k-1}$.

Then, the correspondence we give between the fundamental homology classes of

$$S\mathbb{H}P^{m_1-1} \times \cdots \times S\mathbb{H}P^{m_k-1}$$

and the Schubert cycles will be via the fundamental homology classes of

$$S\mathbb{C}P^{2m_1-2} \times \cdots \times S\mathbb{C}P^{2m_k-2}.$$

We do so by defining a map

$$\psi^{(sk)}_{\mathbf{m}} : S\mathbb{C}P^{2m_1-2} \times S\mathbb{C}P^{2m_2-2} \times \cdots \times S\mathbb{C}P^{2m_k-2} \longrightarrow \mathcal{C}^{(sk)}_{m}.$$

This is given as follows:

$$\tilde{\psi}^{(sk)}_{m} : (I \times \mathbb{C}P^{2m_1-2}) \times (I \times \mathbb{C}P^{2m_2-2}) \times \cdots \times (I \times \mathbb{C}P^{2m_k-2}) \longrightarrow SU_n$$

is defined by

$$\begin{aligned} \tilde{\psi}^{(sk)}_{\mathbf{m}}((t_1, L_1), \ldots, (t_k, L_k)) \;&=\; A_{(-2\pi\tilde{t}, e_1)} \cdot A_{(2\pi t_1, L_1)} \cdot A_{(2\pi t_2, L_2)} \cdots A_{(2\pi t_k, L_k)} \\ &\quad \cdot A_{(2\pi t_k, \mathbf{j}L_k)} \cdots A_{(2\pi t_2, \mathbf{j}L_2)} \cdot A_{(2\pi t_1, \mathbf{j}L_1)} \cdot A_{(-2\pi\tilde{t}, -e_3)}\,, \end{aligned} \tag{4.13}$$

where $\tilde{t} = \sum_{j=1}^{k} t_j$. We note that the product is of the form (4.10) and hence (4.12). Also, the first and last factors $A_{(-2\pi\tilde{t}, e_1)}$ and $A_{(-2\pi\tilde{t}, -e_3)}$ ensure the product is in SU_n as in the splitting for (2.1).

Since $A_{(0,L)} = A_{(2\pi,L)} = I_n$ independent of a complex line $L \subset \mathbb{C}^{2m-1}$, (4.13) descends to a map

$$\psi^{(sk)}_{m} : S\mathbb{C}P^{2m_1-2} \times S\mathbb{C}P^{2m_2-2} \times \cdots \times S\mathbb{C}P^{2m_k-2} \longrightarrow \mathcal{C}^{(sk)}_{m}\,.$$

As remarked above, each $S\mathbb{C}P^{2m_j-2}$ has an open dense cell of dimension $4m_j - 3$ which we denote by

$$\begin{aligned} E^{(sk)}_{m_j} \;&=\; (0,1) \times \{x = (x_1, \ldots, x_{4(m_j-1)}, x_{4m_j-3}, 0, \ldots 0) \\ &\qquad : (x_1, \ldots, x_{4(m_j-1)}, x_{4m_j-3}), 0) \in S^{4m_j-3} \text{ and } x_{4m_j-3} > 0\} \end{aligned}$$

and we conclude $\mathbb{H} < x > \subset_{\min} \mathbb{C}^{2m_j}$.

We now introduce some notation and denote

$$\tilde{S}^{(sk)}_{\mathbf{m}} = S\mathbb{C}P^{2m_1-2} \times S\mathbb{C}P^{2m_2-2} \times \cdots \times S\mathbb{C}P^{2m_k-2}\,.$$

Also, we consider the corresponding cell $E^{(sk)}_{\mathbf{m}} = E^{(sk)}_{m_1} \times E^{(sk)}_{m_2} \times \cdots \times E^{(sk)}_{m_k}$, and the image $S^{(sk)}_{\mathbf{m}} = \psi^{(sk)}_{\mathbf{m}}(E^{(sk)}_{\mathbf{m}})$ in $\mathcal{C}^{(sk)}_{2n}$. Then, $E^{(sk)}_{\mathbf{m}}$ is an open dense cell in $\tilde{S}^{(sk)}_{\mathbf{m}}$ with

$$\dim_{\mathbb{R}} E^{(sk)}_{\mathbf{m}} = \sum_{j=1}^{k} (4m_j - 3) = 4|\mathbf{m}^{(sk)}| - 3k = 4|\mathbf{m}^{(sk)}| - 3\ell(\mathbf{m}^{(sk)})$$

for $|\mathbf{m}^{(sk)}| = \sum_{j=1}^{k} m_j$ (and $\ell(\mathbf{m}^{(sk)}) = k$). Also, the image $S_{\mathbf{m}}^{(sk)} = \psi_{\mathbf{m}}^{(sk)}(E_{\mathbf{m}}^{(sk)})$ consists of elements of $\mathcal{C}_{2n}^{(sk)}$ of skew Schubert type $\mathbf{m}$. Furthermore, $\overline{S_{\mathbf{m}}^{(sk)}} = \psi_{\mathbf{m}}^{(sk)}(\tilde{S}_{\mathbf{m}}^{(sk)})$. Then the results of Kadzisa-Mimura [KM, Thm 8.7] give the following Schubert decomposition of SU_{2n}/Sp_n.

Theorem 4.6. *The Schubert decomposition of SU_{2n}/Sp_n has the following properties via the diffeomorphism $SU_{2n}/Sp_n \simeq \mathcal{C}_{2n}^{(sk)}$:*

a) *SU_{2n}/Sp_n is the disjoint union of the $S_{\mathbf{m}}^{(sk)}$ as $\mathbf{m} = \mathbf{m}^{(sk)} = (m_1, \dots, m_k)$ varies over all increasing sequences with $1 < m_1 < \cdots < m_k \leq n$, and $0 \leq k \leq n-1$.*
b) *The map $\psi_{\mathbf{m}}^{(sk)} : E_{\mathbf{m}}^{(sk)} \to S_{\mathbf{m}}^{(sk)}$ is a homeomorphism.*
c) *$(\overline{S_{\mathbf{m}}^{(sk)}} \backslash S_{\mathbf{m}}^{(sk)}) \subset \cup_{\mathbf{m}'} S_{\mathbf{m}'}^{(sk)}$, where the union is over all $S_{\mathbf{m}'}^{(sk)}$ with $\dim S_{\mathbf{m}'}^{(sk)} < \dim S_{\mathbf{m}}^{(sk)}$.*

Hence, the Schubert decomposition by the cells $S_{\mathbf{m}}^{(sk)}$ gives a corresponding cell decomposition of SU_{2n}/Sp_n. The cells $S_{\mathbf{m}}^{(sk)}$ will be referred to as the *skew-symmetric Schubert cells of SU_{2n}/Sp_n or $\mathcal{C}_{2n}^{(sk)}$*. We note that $\overline{S_{\mathbf{m}}^{(sk)}}$ has a Borel-Moore fundamental class which we refer to as a *skew-symmetric Schubert cycle*. It is the image of the Borel-Moore fundamental class of the "singular manifold" $\tilde{S}_{\mathbf{m}}^{(sk)}$. It corresponds to the Borel-Moore fundamental class of the associated product of suspensions of quaternionic projective spaces. We describe in §5 the homology of SU_{2n}/Sp_n and $\mathcal{C}_{2n}^{(sk)}$ in terms of these skew-symmetric Schubert cycles. Furthermore, for $m = 2n$ the relation of $\mathcal{C}_{m}^{(sk)}$ with $F_m^{(sk)\,c}$ allows us to give a Schubert decomposition for the Milnor fiber.

Remark 4.7. If in the initial factorization of $B \in \mathcal{C}_{2n}^{(sk)}$ given in (4.4) into a product of $\mathbb{H}$-pseudo-rotations, the orders for all of the $L_j^{(\lambda_\ell)}$ are all distinct then $1 < m_1 < m_2 < \cdots < m_k$. By the commutativity of the $\mathbb{H}$-pseudo-rotations, we may arrange them in increasing order and obtain (4.10) without using Whitehead's Lemma. Hence, the skew-symmetric Schubert symbol is given by $\mathbf{m}^{(sk)} = (m_1, m_2, \cdots, m_k)$, which would be the corresponding Schubert symbol in the quaternionic Grassmannian. In general, the use of Whitehead's Lemma has the effect of twisting the $\mathbb{H}$-lines which then again reappear from the form of the skew-symmetric factorization.

5. Schubert Decomposition for Milnor Fibers

In this section we apply the results giving the Schubert decomposition for the associated symmetric spaces providing compact models for the global Milnor fibers. We first give the form that the Schubert decomposition gives for the specific Cartan models, and extending these to the Milnor fibers themselves. Second, in doing this we give an algorithm due to Whitehead and Kadzisa-Mimura for identifying for a given matrix in the global Milnor fiber the Schubert cell to which it belongs. Third, we will see the form that the Schubert decomposition takes for the global Milnor fibers using Iwasawa decomposition.

Whitehead-Kadzisa-Mimura Algorithm for Identifying Schubert Cells.
The algorithm given by Kadzisa-Mimura [KM] for the ordered factorizations of matrices in the various Cartan models uses the ordered factorization for SU_m based on the work of Whitehead [W] as developed by Miller [Mi] and Yokota [Y]. They cleverly combine the uniqueness of the factorization for U_m (and SU_m) and the Cartan conjugacy for the Cartan models to give the symmetric, respectively skew-symmetric, factorizations for the cases of SU_m/SO_m and for $m = 2n$, SU_{2n}/Sp_n. We explain this algorithm as it will apply to the compact models for global Milnor fibers and then for the global Milnor fibers themselves.

An element of any of the Cartan models is a matrix $B \in SU_m$ for appropriate m. Thus, by Lemma 3.3 we may obtain an ordered factorization by pseudo-rotations except with decreasing order for B.

$$B \ = \ A_k \cdot A_{k-1} \cdots A_1 \, , \tag{5.1}$$

where $A_j = A_{(\theta_j, x_j)}$ with the $\{x_j\}$ a set of unit vectors with $x_j \in_{\min} \mathbb{C}^{m_j}$ and

$$1 \leq m_1 < m_2 < \cdots < m_k \leq m,$$

and $\theta_i \not\equiv 0 \bmod 2\pi$ for each i. In addition, if $m_1 = 1$ then the Schubert symbol is $\mathbf{m} = (m_2, \ldots, m_k)$. Now from (5.1) we describe how to obtain either the symmetric or skew-symmetric ordered factorizations as obtained by Kadzisa-Mimura.

Ordered Symmetric Factorizations for $\mathcal{C}^{(sy)}$. As $B \in \mathcal{C}^{(sy)}$, $\sigma(B^{-1}) = B$. Hence, as

$$\sigma(B^{-1}) = \overline{B^{-1}} = B^T,$$

we obtain from (5.1)

$$A_k \cdot A_{k-1} \cdots A_1 \ = \ A_1^T \cdot A_2^T \cdots A_k^T \, .$$

As each $A_j = A_{(\theta_j, x_j)}$, $A_j^T = A_{(\theta_j, \bar{x}_j)}$ is a pseudo-rotation with $\bar{x}_j \in_{\min} \mathbb{C}^{m_j}$. Thus, it follows by Lemma 3.4 that $A_1 = A_1^T$ and x_1 is real. Let $C_1 = A_{(\frac{\theta_1}{2}, x_1)}$. We can write $A_1 = C_1 \cdot C_1$, and as $A_{(\theta_1, x_1)}$ is a pseudo-rotation about a real hyperplane, so is C_1. Hence, $C_1 = C_1^T$ and $\sigma(C_1) = C_1^*$. Then, from (5.1) since

$$B \ = \ A_k \cdot A_{k-1} \cdots A_1 \, , \tag{5.2}$$

we have

$$\begin{aligned} C_1^* \cdot B \cdot \sigma(C_1) \ &= \ (C_1^* \cdot A_k \cdot A_{k-1} \cdots A_2 \cdot C_1) \cdot C_1 \cdot \sigma(C_1) \\ &= \ (C_1^* \cdot A_k \cdot C_1) \cdot (C_1^* \cdot A_{k-1} \cdot C_1) \cdots (C_1^* \cdot A_2 \cdot C_1) \\ &= \ A_k^{(2)} \cdot A_{k-1}^{(2)} \cdots A_2^{(2)}, \end{aligned} \tag{5.3}$$

where each $A_j^{(2)} = C_1^* \cdot A_j \cdot C_1$ is again a pseudo-rotation $A_{(\theta_j, x_j^{(2)})}$, with $x_j^{(2)} = C_1^{-1}(x_j)$ satisfying $x_j^{(2)} \in_{\min} \mathbb{C}^{m_j}$ as $C_1 \equiv Id$ on $(\mathbb{C}^{m_1})^{\perp}$.

Also, the LHS of (5.3) is the Cartan conjugate of the symmetric matrix B and so is still symmetric (and in SU_n), except now it is a product of $k-1$ pseudo-rotations with Schubert symbol $(m_k, \ldots, m_2)$. Thus we can inductively repeat the argument to write.

$$C_j^* \cdots C_2^* \cdot C_1^* \cdot B \cdot \sigma(C_1) \cdot \sigma(C_2) \cdots \sigma(C_j) \ = \ A_k^{(j+1)} \cdot A_{k-1}^{(j+1)} \cdots A_{j+1}^{(j+1)}$$

which has Schubert symbol $(m_{j+1}, \ldots, m_k)$. After $k-1$ steps we obtain

$$C_{k-1}^* \cdots C_2^* \cdot C_1^* \cdot B\sigma(C_1) \cdot \sigma(C_2) \cdots \sigma(C_{k-1}) \ = \ A_k^{(k)} \, , \tag{5.4}$$

with $A_k^{(k)} = A_{(\theta_k, x_k^{(k)})}$ for $x_k^{(k)} \in_{\min} \mathbb{C}^{m_k}$. The last step then allows us to rewrite (5.4) as

$$B \ = \ C_1 \cdots C_{k-1} \cdot C_k \cdot \sigma(C_k^*) \cdot \sigma(C_{k-1}^*) \cdots \sigma(C_1^*) \, , \tag{5.5}$$

which gives the ordered symmetric factorization.

We obtain as a corollary of the algorithm

Corollary 5.1. *If $B \in F_m^{(sy)\,c} = \mathcal{C}_m^{(sy)}$, and has increasing Schubert symbol $\mathbf{m} = (m_1, \ldots, m_k)$, then the symmetric factorization has the same Schubert symbol $\mathbf{m}^{(sy)} = \mathbf{m}$.*

Ordered Skew-symmetric Factorizations for $\mathcal{C}_m^{(sk)}$. The algorithm for $\mathcal{C}_m^{(sk)}$, with $m = 2n$, is very similar and depends on the following lemma, see [KM, Lemma 7.2].

Lemma 5.2. *If $B \in (U_{2n} \cap Sk_m(\mathbb{C})) \cdot J_n^{-1}$, with $m = 2n$, has a factorization as in (5.1), then: k is even, m_1 is odd, $m_2 = m_1 + 1$, and $A_2 = \sigma(A_1^*)$.*

Here $\sigma(A) = J_n \cdot \overline{A} \cdot J_n^{-1}$ and $A_1 = A_{(\theta_1, x_1)}$ with $x_1 \in_{\min} \mathbb{C}^{m_1}$, for which we may arrange $x_1 = (x_{1,1}, \ldots, x_{1,m_1})$ with $x_{1,m_1} > 0$. Then, by properties of pseudo-rotations

$$A_2 = \sigma(A_1^*) = A_{(\theta_1, \mathbf{j}x_1)}$$

(hence, $A_2 \cdot A_1$ is an $\mathbb{H}$-pseudo-rotation and A_1 and A_2 commute). We may then rewrite (5.1) as

$$\begin{aligned} A_1^* \cdot B \cdot \sigma(A_1) &= A_1^* \cdot A_k \cdot A_{k-1} \cdots A_3 \cdot A_1 \cdot \sigma(A_1^*) \cdot \sigma(A_1) \\ &= (A_1^* \cdot A_k \cdot A_1) \cdot (A_1^* \cdot A_{k-1} \cdot A_1) \cdots (A_1^* \cdot A_3 \cdot A_1) \\ &= A_k^{(2)} \cdot A_{k-1}^{(2)} \cdots A_3^{(2)}, \end{aligned} \tag{5.6}$$

where each $A_j^{(2)} = A_1^* \cdot A_j \cdot A_1$ is again a pseudo-rotation $A_{(\theta_j, x_j^{(2)})}$, with $x_j^{(2)} = A_1^{-1}(x_j)$ satisfying $x_j^{(2)} \in_{\min} \mathbb{C}^{m_j}$ as $A_1 \equiv Id$ on $(\mathbb{C}^{m_1})^{\perp}$.

Also, the LHS of (5.6) is the Cartan conjugate of B for which $B \cdot J_n$ is skew-symmetric (and in U_{2n}); and so it also has these properties, except now it is a product of $k-2$ pseudo-rotations with Schubert symbol $(m_k, \ldots, m_3)$. Thus we can inductively repeat the argument. After $\frac{k}{2}$ steps we obtain a factorization in the form

$$\begin{aligned} B &= A_{(\theta_1, x_1')} \cdots A_{(\theta_r, x_r')} \cdot \sigma(A^*_{(\theta_r, x_r')}) \cdots \sigma(A^*_{(\theta_1, x_1')}), \\ &= A_{(\theta_1, x_1')} \cdots A_{(\theta_r, x_r')} \cdot A_{(\theta_r, \mathbf{j}x_r')} \cdots A_{(\theta_1, \mathbf{j}x_1')}. \end{aligned} \tag{5.7}$$

Here $k = 2r$, and each $\mathbb{H} < x_r' > \subset_{\min} \mathbb{C}^{2m_j}$. This gives the ordered skew-symmetric factorization. By (4.9) we may write each $A_{(\theta_j, \mathbf{j}x_j')} = J_n \cdot A^T_{(\theta_j, x_j')} \cdot J_n^{-1}$, and then by (4.11) we may alternately write (5.7) in the form

$$B = A_{(\theta_1, x_1')} \cdots A_{(\theta_r, x_r')} \cdot J_n \cdot A^T_{(\theta_r, x_r')} \cdots A^T_{(\theta_1, x_1')} \cdot J_n^{-1}. \tag{5.8}$$

We obtain as a corollary of the algorithm.

Corollary 5.3. *If $B \in \mathcal{C}_m^{(sk)} = F_m^{(sk)\,c} \cdot J_n^{-1}$ (with $m = 2n$), then it has an increasing Schubert symbol of the form*

$$\mathbf{m} = (2m_1 - 1, 2m_1, 2m_2 - 1, 2m_2, \ldots, 2m_r - 1, 2m_r)$$

with $1 < m_1 < m_2, \cdots < m_r \leq n$. Then the ordered skew-symmetric factorization has the skew-symmetric Schubert symbol $\mathbf{m}^{(sk)} = (m_1, m_2, \ldots, m_r)$.

To use the preceding results for the global Milnor fibers, we use in each case the Iwasawa decomposition, which is given for SL_n by the Gram-Schmidt process, to determine the Schubert cell decomposition.

Global Milnor Fibers for the Variety of Singular $m \times m$-Matrices.

This is the simplest case and was essentially covered in Proposition 1.2. Given $B \in F_m$, the global Milnor fiber, we have $F_m = SL_m(\mathbb{C})$. To obtain its representation in the Iwasawa decomposition $SL_m(\mathbb{C}) = SU_m \cdot A_m \cdot N_m$ where A_m denotes the group of diagonal matrices with positive entries, and N_m is the nilpotent group of upper triangular complex matrices with 1'on the diagonal. We may apply the Gram-Schmidt process to the columns of B to obtain $B = A \cdot C$, where A is unitary and C is upper triangular with positive entries on the diagonal. As $\det(B) = 1$,

$\det(A)$ is a unit complex number, and $\det(C) > 0$; it follows that both $\det(A) = \det(C) = 1$; thus, C belongs to $\mathrm{Sol}_m = A_m \cdot N_m$. Then by applying the method of §3 for giving an ordered factorization for A gives the Schubert symbol for A, which we shall also use for B. Thus, we may describe the Schubert decomposition for the global Milnor fiber F_m as follows.

Theorem 5.4. *The Schubert decomposition of the global Milnor fiber F_m for the variety of $m \times m$ general complex matrices is given, via the diffeomorphism with $SL_m(\mathbb{C})$, by the disjoint union of the* Schubert cells $S_{\mathbf{m}} \cdot Sol_m$ *where the $S_{\mathbf{m}}$ are the Schubert cells of SU_m for all Schubert symbols $\mathbf{m} = (m_1, \ldots, m_k)$ with $1 < m_1 < \cdots < m_k \leq m$.*

Global Milnor Fibers for the Variety of Singular $m \times m$-Symmetric Matrices.

If $B \in F_m^{(sy)}$, then we want to relate B to a matrix $C \in F_m^{(sy)\,c} = SU_m \cap Sym_m(\mathbb{C}) = \mathcal{C}_m^{(sy)}$. As B is symmetric and $\det(B) = 1$, as in [D3, Table 1] we may diagonalize the quadratic form $X^T \cdot B \cdot X$, for column vectors X so there is a $C \in SL_m(\mathbb{C})$ so that $(CX)^T \cdot B \cdot CX = X^T \cdot X$. Thus, $C^T \cdot B \cdot C = I_m$ or $B = (C^{-1})^T \cdot C^{-1}$. Then, by Iwasawa decomposition $C^{-1} = A \cdot E$, with $A \in SU_m$ and $E \in Sol_m$. Then, $B = E^T \cdot (A^T \cdot A) \cdot E$, and $A^T \cdot A \in \mathcal{C}_m^{(sy)}$. If

$$\mathbf{m} = (m_1, m_2, \ldots, m_k)$$

is the Schubert symbol for $\tilde{A} = A^T \cdot A$, it is also the symmetric Schubert symbol and so

$$\tilde{A} = A^T \cdot A \in S_{\mathbf{m}}^{(sy)}$$

and conversely.

We let Sol_m^T denote the group of lower triangular complex matrices E with positive entries on the diagonal and $\det(E) = 1$. Then, there is the action of Sol_m^T on $\mathcal{C}_m^{(sy)}$ as follows:

$$Sol_m^T \times \mathcal{C}_m^{(sy)} \to \mathcal{C}_m^{(sy)} \qquad \text{sending} \qquad (E, \tilde{A}) \mapsto E \cdot \tilde{A} \cdot E^T .$$

Then, the action applied to each Schubert cell $S_{\mathbf{m}}^{(sy)}$ gives by Proposition 1.2 the Schubert cell for $F_m^{(sy)}$ which we denote by $Sol_m^T \cdot (S_{\mathbf{m}}^{(sy)})$. Combining this with Theorem 4.1 we obtain

Theorem 5.5. *The Schubert decomposition of the global Milnor fiber $F_m^{(sy)}$ for the variety of $m \times m$ symmetric complex matrices is given by the disjoint union of the* symmetric Schubert cells $Sol_m^T \cdot (S_{\mathbf{m}}^{(sy)})$ *for $S_{\mathbf{m}}^{(sy)}$ the symmetric Schubert cells of SU_m/SO_m for all symmetric Schubert symbols $\mathbf{m}^{(sy)} = (m_1, \ldots, m_k)$ with $1 < m_1 < \cdots < m_k \leq m$.*

Furthermore, the preceding algorithm using ordered factorization gives the symmetric Schubert symbol for a given matrix in $F_m^{(sy)}$.

Global Milnor Fibers for the Variety of Singular $m \times m$ Skew-Symmetric Matrices.

For the case of $B \in F_m^{(sk)}$ with $m = 2n$, we follow an analogous argument to the preceding. We first want to relate B to a matrix $C \in F_m^{(sk)\,c} = SU_m \cap Sk_m(\mathbb{C})$, and then use the relation $F_m^{(sk)\,c} \cdot J_n^{-1} = \mathcal{C}_m^{(sk)}$ to determine the skew-symmetric factorization for $C \cdot J_n^{-1}$ to determine its skew-symmetric Schubert type.

As B is skew-symmetric with $\mathrm{Pf}(B) = 1$, as in [D3, Table 1] we may block diagonalize the quadratic form $X^T \cdot B \cdot X$, for column vectors X so there is a $C \in SL_m(\mathbb{C})$ so that

$$(CX)^T \cdot B \cdot CX = X^T \cdot J_n \cdot X.$$

Thus, $C^T \cdot B \cdot C = J_n$ or $B = (C^{-1})^T \cdot J_n \cdot C^{-1}$. Then, we again apply Iwasawa decomposition $C^{-1} = A \cdot E$, with $A \in SU_m$ and $E \in Sol_m$. Then,

$$B = E^T \cdot (A^T \cdot J_n \cdot A) \cdot E,$$

and

$$\tilde{A} = A^T \cdot J_n \cdot A \in SU_m \cap Sk_m(\mathbb{C}).$$

It follows $\tilde{A} \cdot J_n^{-1} \in \mathcal{C}_m^{(sk)}$. The Schubert symbol

$$\mathbf{m} = (2m_1 - 1, 2m_1, 2m_2 - 1, 2m_2, \ldots, 2m_k - 1, 2m_k)$$

for $\tilde{A} \cdot J_n^{-1}$ is obtained from the ordered factorization of $\tilde{A} \cdot J_n^{-1}$. By (5.8), this may be alternatively written as a skew-symmetric factorization of $\tilde{A}$

$$\tilde{A} \; = \; A_{(\theta_1, x_1')} \cdots A_{(\theta_k, x_k')} \cdot J_n \cdot A^T_{(\theta_k, x_k')} \cdots A^T_{(\theta_1, x_1')} \, . \tag{5.9}$$

By Corollary 5.3, $\mathbf{m}^{(sk)} = (m_1, m_2, \ldots, m_k)$ is the skew-symmetric Schubert symbol. Then, under the map $\mathcal{C}_m^{(sk)} \to F_m^{(sk)}$ given by right multiplication by J_n, i.e.

$$\tilde{A} \cdot J_n^{-1} \mapsto \tilde{A} \in SU_m \cap Sk_m(\mathbb{C}) = F_m^{(sk)},$$

we have $S_{\mathbf{m}}^{(sk)}$ mapping diffeomorphically to $S_{\mathbf{m}}^{(sk)} \cdot J_n \subset F_m^{(sk)}$. Hence, we again use the action of Sol_m^T but on $F_m^{(sk)}$ given by :

$$Sol_m^T \times F_m^{(sk)} \to F_m^{(sk)} \qquad \text{sending} \qquad (E, \tilde{A}) \mapsto E \cdot \tilde{A} \cdot E^T \, .$$

Then, from the action applied to each Schubert cell $S_{\mathbf{m}}^{(sk)}$ after right multiplication by J_n gives by Proposition 1.2 the Schubert cell for $F_m^{(sk)}$ which we denote by $Sol_m^T \cdot (S_{\mathbf{m}}^{(sk)} \cdot J_n)$. Combining this with Theorem 4.1 we obtain

Theorem 5.6. *The Schubert decomposition of the global Milnor fiber $F_m^{(sk)}$ for the variety of $m \times m$ skew-symmetric complex matrices (with $m = 2n$) is given by the disjoint union of the* skew-symmetric Schubert cells $Sol_m^T \cdot (S_{\mathbf{m}}^{(sk)} \cdot J_n)$ *corresponding to the skew-symmetric Schubert cells $S_{\mathbf{m}}^{(sk)}$ of $\mathcal{C}_m^{(sk)}$, for all skew-symmetric Schubert symbols $\mathbf{m}^{(sk)} = (m_1, \ldots, m_k)$ with $1 < m_1 < \cdots < m_k \leq n$.*

Furthermore, the preceding algorithm using ordered factorization gives the associated skew-symmetric Schubert symbol for a given matrix in $F_m^{(sk)}$.

6. Representation of the Dual Classes in Cohomology

Having given the Schubert decomposition for the global Milnor fibers in terms of the corresponding Cartan models, we now consider how the Schubert decomposition corresponds to the (co)homology of the global Milnor fibers as given in [D3], which was deduced from that of the corresponding symmetric spaces. We will refer to the closures of the Schubert cells in each case as *Schubert cycles* of the appropriate type. We shall see that for both the general and skew-symmetric cases the Schubert cycles are cycles whose fundamental classes define $\mathbb{Z}$-homology classes. For the symmetric case, the symmetric Schubert cycles are only mod 2-cycles which define unique $\mathbb{Z}/2\mathbb{Z}$-homology classes. The situation is somewhat similar to that for real Grassmannians where the $\mathbb{Z}/2\mathbb{Z}$-cohomology classes correspond to real Schubert cycles, while the rational classes are more difficult to identify in terms of the Schubert decomposition.

This identification is made using the standard method (see e.g. [Ma, Chap. IX, §4]) for computing the (co)homology of a finite CW-complex X with skeleta $\{X^{(k)}\}$ with coefficient ring R using the finite algebraic complex $C_k(\{X^{(k)}\}) = H_k(X^{(k)}, X^{(k-1)}; R)$, with boundary map given by the boundary map for the exact sequence of a triple. Then, $\mathrm{rk}_R(C_k(\{X^{(k)}\}))$ equals the number of cells q_k of dimension k. Thus, $\mathrm{rk}_R H_k(X; R) \leq q_k$ with equality iff the closures of the cells of dimension k give a free set of generators for $H_k(X; R)$. Likewise the cohomology is computed from the complex $C^k(\{X^{(k)}\}) = H^k(X^{(k)}, X^{(k-1)}; R)$ using the coboundary map for the exact sequence of a triple in cohomology.

Milnor Fiber for the Variety of Singular $m \times m$-Matrices.

We consider the Schubert decomposition for F_m obtained from that for the compact model $F_m^c = SU_m$ as a result of Theorem 5.4. Then, the homology of SU_m can be computed from the algebraic complex with basis formed from the Schubert cells $S_{\mathbf{m}}$. By a result of Hopf, the homology of SU_m (which is isomorphic as a graded $\mathbb{Z}$-module to its cohomology) is given as a graded $\mathbb{Z}$-module by

$$H_*(SU_m; \mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}\langle s_3, s_5, \ldots, s_{2m-1}\rangle\,,$$

where s_{2j-1} has degree $2j-1$. Then, a count shows that $H_q(SU_n; \mathbb{Z})$ is spanned by $s_{2m_1-1} \cdot s_{2m_2-1} \cdots s_{2m_k-1}$ where $1 < m_1 < m_2 < \cdots < m_k \leq m$ and $q = \sum_{j=1}^{k}(2m_j - 1)$. This equals the number of Schubert cells $S_{\mathbf{m}}$ of real dimension q. Thus, each $\overline{S_{\mathbf{m}}}$ defines a $\mathbb{Z}$-homology class of dimension $\dim_{\mathbb{R}} S_{\mathbf{m}}$. Together they form a basis for $H_q(SU_m; \mathbb{Z})$. Also, $\psi_{\mathbf{m}}(\tilde{S}_{\mathbf{m}}) = \overline{S_{\mathbf{m}}}$ and $\tilde{S}_{\mathbf{m}}$ has a top homology class in $H_q(\tilde{S}_{\mathbf{m}}; \mathbb{Z})$ for $q = \dim_{\mathbb{R}}(\tilde{S}_{\mathbf{m}})$, which we can view as a fundamental class for $\tilde{S}_{\mathbf{m}}$ for Borel-Moore homology. We have a similar dimension count in cohomology, so that the duals of the classes $\overline{S_{\mathbf{m}}}$ via the Kronecker pairing give a $\mathbb{Z}$-basis for cohomology.

Then, as $F_m^c = SU_m$ and the inclusion $i_m : F_m^c \hookrightarrow F_m$ is a homotopy equivalence, we obtain the following

Theorem 6.1. *The homology $H_*(F_m; \mathbb{Z})$ has for a free $\mathbb{Z}$-basis the fundamental classes of the Schubert cycles, given as images $i_{m\,*} \circ \psi_{\mathbf{m}\,*}([\tilde{S}_{\mathbf{m}}]) = \psi_{\mathbf{m}\,*}(\tilde{S}_{\mathbf{m}}) = \overline{S_{\mathbf{m}}}$ as we vary over the Schubert decomposition of SU_m. The Kronecker duals of these classes give the $\mathbb{Z}$-basis for the cohomology*

$$H^*(SU_m; \mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}\langle e_3, e_5, \ldots, e_{2m-1}\rangle\,.$$

Moreover, the Kronecker duals of the simple Schubert classes $S_{(m_1)}$ *are homogeneous generators of the exterior algebra cohomology.*

Proof. The preceding discussion establishes all of the theorem except for the last statement about the generators of the cohomology algebra. We prove this by induction on m. It is trivially true for $m = 1, 2$. Suppose it is true for $m < n$ and let $i_{n-1} : SU_{n-1} \hookrightarrow SU_n$ denote the natural inclusion $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & 1 \end{smallmatrix}\right)$. The Schubert decomposition preserves the inclusion so that any $S_{\mathbf{m}}$ for $\mathbf{m} = (m_1, m_2, \cdots, m_k)$ with $m_k < n$ is contained in the image of i_{n-1} and so is also a Schubert cell for SU_{n-1}; while if $m_k = n$, then $S_{\mathbf{m}}$ is in the complement of the image of SU_{n-1}. Thus, if the result is true for SU_{n-1}, the Kronecker duals to the simple $S_{(m_1)}$ with $m_1 < n$ restrict via i_{n-1}^* to the Kronecker duals of the $S_{(m_1)}$ with $m_1 < n$ viewed as Schubert cells of SU_{n-1}. Thus, they map to the generators of the exterior algebra $\Lambda^*\mathbb{Z} < e_3, e_5, \cdots e_{2n-3} >$. Also, the Kronecker dual to any $S_{\mathbf{m}}$ with $m_k = n$ is zero on any Schubert cell of SU_{n-1} so by a counting argument the kernel of i_{n-1}^*, which is the ideal generated by e_{2n-1}, is spanned by the Kronecker duals of the Schubert cells with $m_k = n$.

Now there is a unique Schubert class of this type of degree $2n-1$, and hence its Kronecker dual is the added generator which together with the others for $S_{(m_1)}$ with $m_1 < n$ generate $H^*(SU_n; \mathbb{Z})$. □

There is also the question of identifying the Kronecker dual of the Schubert cycle $[\overline{S}_{\mathbf{m}}]$ for $\mathbf{m} = (m_1, m_2, \cdots, m_k)$, which we denote by $e_{\mathbf{m}}$. We claim it is given up to sign by the cohomology class $e_{2m_1-1} \cdot e_{2m_2-1} \cdots e_{2m_k-1}$ (where the products denote cup-products). We show this using the product structure of the group SU_m to give a product representation for the closures of Schubert cells together with the Hopf algebra structure of $H^*(SU_m)$.

We let $\overline{S_{\mathbf{m}}} \cdot \overline{S_{\mathbf{m}'}}$ denote the group product in SU_m of the closures of Schubert cells $\overline{S_{\mathbf{m}}}$ and $\overline{S_{\mathbf{m}'}}$. We also use the simpler notation S_{m_1} to denote the Schubert cell $S_{\mathbf{m}}$ when $\mathbf{m} = (m_1)$. In particular, we emphasize that

$$S_{m_1} \;=\; \{A_{(-\theta,e_1)} \cdot A_{(\theta,x_1)} : \theta \in (0,2\pi), x_1 \in_{\min} \mathbb{C}^{m_1}\}\,.$$

First, as result of Lemma 3.2, we obtain the following version of a lemma due to Whitehead (see e.g. [KM, Lemma 4.2] or [Mi, Lemma 2.2]).

Lemma 6.2. *For Schubert cells in $\mathcal{C}_m$ for SU_m,*

1) *If $1 < m_1 < m_2 \le m$, then*
$$\overline{S_{m_2}} \cdot \overline{S_{m_1}} \;=\; \overline{S_{m_1}} \cdot \overline{S_{m_2}} \;=\; \overline{S_{(m_1,m_2)}}\,.$$
2) *If $1 < m' \le m$, then*
$$\overline{S_{m'}} \cdot \overline{S_{m'}} \;\subseteq\; \overline{S_{(m'-1,m')}}\,.$$

We note that this differs slightly from the above referred to lemmas as each element in S_{m_1} is a product of two pseudo-rotations, one of which is $A_{(-\theta,e_1)}$. However, by the lemma, this pseudo-rotation can also be interchanged with other $A_{(\theta,x_j)}$, and combined via multiplication with other $A_{(-\theta',e_1)}$. We also note in the lemma that $\dim_{\mathbb{R}} S_{(m'-1,m')} \le 2\cdot \dim_{\mathbb{R}} S_{m'} - 2$.

We can inductively repeat this to obtain

Lemma 6.3. *For Schubert cells S_{m_j} in $\mathcal{C}_m$ (for SU_m):*

1) *If $\mathbf{m} = (m_1, m_2, \dots, m_r)$ then*
$$\overline{S_{\mathbf{m}}} \;=\; \overline{S_{m_1}} \cdot \overline{S_{m_2}} \cdots \overline{S_{m_r}}\,.$$
2) *If $\mathbf{m} = (m_1, m_2, \dots, m_r)$ and $\mathbf{m}' = (m'_1, m'_2, \dots, m'_{r'})$ with*
$$\{m_1, m_2, \dots, m_r\} \cap \{m'_1, m'_2, \dots, m'_{r'}\} = \emptyset,$$
then
$$\overline{S_{\mathbf{m}}} \cdot \overline{S_{\mathbf{m}'}} \;=\; \overline{S_{\mathbf{m}''}}\,,$$
where $\mathbf{m}''$ is the union of $\mathbf{m}$ and $\mathbf{m}'$ in increasing order.
3) *If $\mathbf{m} = (m_1, m_2, \dots, m_r)$ and $\mathbf{m}' = (m'_1, m'_2, \dots, m'_{r'})$ with*
$$\{m_1, m_2, \dots, m_r\} \cap \{m'_1, m'_2, \dots, m'_{r'}\} \neq \emptyset,$$
then
$$\overline{S_{\mathbf{m}}} \cdot \overline{S_{\mathbf{m}'}} \;\subset\; \mathcal{C}_m^{(q)}\,,$$
where $q \le \dim_{\mathbb{R}} S_{\mathbf{m}} + \dim_{\mathbb{R}} S_{\mathbf{m}'} - 2$.

Proof. For 1) we consider a product in $S_{m_1} \cdot S_{m_2} \cdots S_{m_r}$ which has the form

$$B \;=\; (A_{(-\theta_1,e_1)} \cdot A_{(\theta_1,x_1)}) \cdot (A_{(-\theta_2,e_1)} \cdot A_{(\theta_2,x_2)} \cdots (A_{(-\theta_r,e_1)} \cdot A_{(\theta_r,x_r)}), \tag{6.1}$$

where each $x_j \in_{\min} \mathbb{C}^{m_j}$. Then, we may repeatedly apply the Whitehead Lemma to move each $A_{(-\theta_j,e_1)}$ to the left and obtain a factorization in the form

$$B \;=\; A_{(-\tilde{\theta},e_1)} \cdot A_{(\theta_1,x'_1)} \cdot A_{(\theta_2,x'_2)} \cdots A_{(\theta_r,x'_r)}), \tag{6.2}$$

where $\tilde{\theta} = \sum_{j=1}^{r} \theta_j$ and each $x'_j \in_{\min} \mathbb{C}^{m_j}$. Hence, $B \in S_{\mathbf{m}}$. Conversely we can reverse the process beginning with B in (6.2) and obtain a factorization as in (6.1). This gives the equality for the Schubert cells. Since the closures are compact, we obtain the equality of 1) by taking closures of the Schubert cells.

Given 1) we may write

$$S_{\mathbf{m}} \cdot S_{\mathbf{m}'} \;=\; (S_{m_1} \cdot S_{m_2} \cdots S_{m_r}) \cdot (S_{m'_1} \cdot S_{m'_2} \cdots S_{m'_{r'}}). \tag{6.3}$$

If $\{m_1, m_2, \dots, m_r\} \cap \{m'_1, m'_2, \dots, m'_{r'}\} = \emptyset$, then we can repeatedly apply a) of the Whitehead Lemma to move an element of $S_{m'_j}$ across an element of S_{m_i} when $m_i > m'_j$ while preserving the order of the m_i's and m'_j's. We arrive at an ordered factorization with increasing order $\mathbf{m}''$, which is the union of $\mathbf{m}$ and $\mathbf{m}'$ in increasing order. Taking closures of the Schubert cells then gives 2).

Finally, for 3), we may begin with (6.3). There are smallest $m_\ell = m'_k$. Then, if $m'_j < m'_k$ then it differs from all m_i. Hence, we can first move the elements in $S_{m'_j}$ across all of those in S_{m_i} as in the previous case by 2) of Lemma 6.3. Next, we can move elements in $S_{m_{k'}}$ across those in S_{m_j} as long as $m_j > m_\ell$. Then, we arrive at a factorization where we have successive terms in S_{m_ℓ} and $S_{m_{k'}}$ with $m_\ell = m'_k$. Then, we may apply b) of the Whitehead lemma (or 2) of Lemma 6.2) and obtain a new pair in $S_{\tilde{m}}$ and S_{m_ℓ} with $\tilde{m} \leq m_\ell - 1$. This has the effect of reducing the sum of the Schubert symbol values in the product by at least 1. Also, further application of the Whitehead Lemma will not increase the sum. Hence, by further application of the Whitehead Lemma we obtain a product in the union of Schubert cells of dimension $q \leq \dim_{\mathbb{R}} S_{\mathbf{m}} + \dim_{\mathbb{R}} S_{\mathbf{m}'} - 2$. Thus, it lies in the q-skeleton of $\mathcal{C}_m$. This gives 3) when we take closures. $\square$

Now we will use the Hopf structure of $H^*(SU_n)$ to relate the fundamental classes from the Schubert decomposition with the cohomology classes via the Kronecker pairing. Let

$$\mu : SU_n \times SU_n \to SU_n$$

denote the multiplication map. Then, we can use Lemma 6.3 to determine the effect of μ_* for homology using the complex $C_k(\{X^{(k)}\})$ and then the coproduct map μ^* for the Hopf algebra. We obtain as a corollary of Lemma 6.3.

Corollary 6.4. *We let $s_{\mathbf{m}}$ denote the homology class obtained from $\psi_{\mathbf{m}\,*}([\tilde{S}_{\mathbf{m}}])$ with restriction to positive orientation for $E_{\mathbf{m}}$. For $\mathbf{m} = (m_1, m_2, \dots, m_r)$ and $\mathbf{m}' = (m'_1, m'_2, \dots, m'_{r'})$ we let $m = \{m_1, m_2, \dots, m_r\} \cap \{m'_1, m'_2, \dots, m'_{r'}\}$ and let $\mathbf{m}'' = (m''_1, m''_2, \dots, m''_{r''})$ denote the union of the elements of $\mathbf{m}$ and $\mathbf{m}'$ written in increasing order. Then,*

$$\mu_*(s_{\mathbf{m}} \otimes s_{\mathbf{m}'}) \;=\; \begin{cases} \varepsilon_{\mathbf{m},\mathbf{m}'} \cdot s_{\mathbf{m}''} & \text{if } m = \emptyset, \\ 0 & \text{if } m \neq \emptyset\,, \end{cases} \tag{6.4}$$

where $\varepsilon_{\mathbf{m},\mathbf{m}'}$ is the sign of the permutation which moves $(\mathbf{m}, \mathbf{m}')$ to increasing order.

The reason for the factor $\varepsilon_{\mathbf{m},\mathbf{m}'}$ is that each interchange of two factors $S_{(m_1)}$ and $S_{(m_2)}$ will change the orientation by a factor $(-1)^{(2m_1-1)(2m_2-1)} = -1$.

From the corollary we obtain a formula for the coproduct μ^* in terms of the (Kronecker) dual basis $\{e_{\mathbf{m}}\}$ in cohomology to Schubert basis for homology $\{s_{\mathbf{m}}\}$.

$$\mu^*(e_{\mathbf{m}}) \;=\; \sum (-1)^{\deg(e_{\mathbf{m}'})\deg(e_{\mathbf{m}''})} \varepsilon_{\mathbf{m}',\mathbf{m}''} \cdot e_{\mathbf{m}'} \otimes e_{\mathbf{m}''}\,, \tag{6.5}$$

where the sum is over all disjoint $\mathbf{m}'$ and $\mathbf{m}''$ whose union in increasing order gives $\mathbf{m}$ (and the terms $(-1)^{\deg(e_{\mathbf{m}'})\deg(e_{\mathbf{m}''})}$ arise from the property $(\varphi \otimes \psi)(\sigma \otimes \nu) = (-1)^{\deg(\varphi)\deg(\psi)}\varphi(\sigma)\psi(\nu)$). Since $S_{\mathbf{m}}$ is a product of odd dimensional cells, $\deg(e_{\mathbf{m}'})(= \dim_{\mathbb{R}} S_{\mathbf{m}}) \equiv \ell(\mathbf{m}) \bmod 2$ and the sign in (6.5) equals $(-1)^{\ell(\mathbf{m}')\ell(\mathbf{m}'')}$. Also, note the sum includes the empty symbol which denotes the Schubert cell consisting of just I_n. In the case of the simple Schubert symbol (m_1) we obtain

$$\mu^*(e_{(m_1)}) \;=\; e_{(m_1)} \otimes 1 \,+\, 1 \otimes e_{(m_1)}\,.$$

Hence, all of the $e_{(m_1)}$ are independent primitive classes. Then there is the following relation between the generators of $H^*(SU_n)$ and the Schubert classes.

Theorem 6.5. *$H^*(SU_n)$ is a free exterior algebra with generators $e_{(m)}$ of degrees $2m-1$, for $m = 2, \dots, n$. Moreover the Kronecker dual to $s_{\mathbf{m}}$ for $\mathbf{m} = (m_1, m_2, \dots, m_r)$ is*

$$e_{\mathbf{m}} = (-1)^{\beta(\mathbf{m})} e_{(m_1)} e_{(m_2)} \dots e_{(m_r)},$$

where $\beta(\mathbf{m}) = \binom{\ell(\mathbf{m})}{2}$ *(where we denote* $\binom{1}{2} = 0$*).*

Proof. We already have established the first statement about the algebra generators in Theorem 6.1. We note that it also follows from the Hopf algebra structure. Since the $e_{(m)}$, for $m = 2, \dots, n$ are primitive generators of degree $2m-1$, and $H^*(SU_n)$ is a Hopf algebra which is a free exterior algebra on generators of degrees $2m-1$ for $m = 2, \dots, n$, it follows by a theorem of Hopf-Samuelson that $H^*(SU_n)$ is the free exterior algebra generated by the primitive elements $e_{(m)}$, for $m = 2, \dots, n$.

We furthermore claim that the Kronecker dual to the Schubert class $s_{\mathbf{m}}$ for

$$\mathbf{m} = (m_1, m_2, \dots, m_r)$$

is given by $(-1)^{\beta(\mathbf{m})} e_{(m_1)} e_{(m_2)} \dots e_{(m_r)}$, which will follow from $e_{\mathbf{m}} = (-1)^{\ell(\mathbf{m}')} e_{(m_1)} e_{\mathbf{m}'}$ for

$$\mathbf{m}' = (m_2, m_3, \dots m_r).$$

We prove this by induction on r. It is already true for $r = 1$. Next, consider the case of $\mathbf{m} = (m_1, m_2)$; then $\varepsilon_{(m_1),(m_2)} = 1$, $\varepsilon_{(m_2),(m_1)} = -1$ and $(-1)^{\ell(m_1)\ell(m_2)} = -1$. Then, from (6.5)

$$\mu^*(e_{(m_1,m_2)}) \;=\; e_{(m_1,m_2)} \otimes 1 - e_{(m_1)} \otimes e_{(m_2)} + e_{(m_2)} \otimes e_{(m_1)} + 1 \otimes e_{(m_1,m_2)} \,. \tag{6.6}$$

Also, as μ^* is an algebra homomorphism,

$$\begin{aligned} \mu^*(e_{(m_1)} \cdot e_{(m_2)}) &= \mu^*(e_{(m_1)}) \cdot \mu^*(e_{(m_2)}) \\ &= \big(e_{(m_1)} \otimes 1 + 1 \otimes e_{(m_1)}\big) \cdot \big(e_{(m_2)} \otimes 1 + 1 \otimes e_{(m_2)}\big) \\ &= e_{(m_1)} \cdot e_{(m_2)} \otimes 1 + e_{(m_1)} \otimes e_{(m_2)} - e_{(m_2)} \otimes e_{(m_1)} + 1 \otimes e_{(m_1)} \cdot e_{(m_2)} \,, \end{aligned} \tag{6.7}$$

where the signs on the RHS result from both $e_{(m_1)}$ and $e_{(m_2)}$ having odd degree. Adding (6.7) and (6.6), we obtain

$$\mu^*(e_{(m_1,m_2)} + e_{(m_1)} \cdot e_{(m_2)}) = \big(e_{(m_1,m_2)} + e_{(m_1)} \cdot e_{(m_2)}\big) \otimes 1 + 1 \otimes \big(e_{(m_1,m_2)} + e_{(m_1)} \cdot e_{(m_2)}\big) \,. \tag{6.8}$$

This implies that if $e_{(m_1,m_2)} + e_{(m_1)} \cdot e_{(m_2)} \neq 0$, then it is a primitive element independent from the other primitive elements $e_{(m)}$. This contradicts the Hopf-Samuelson theorem. Thus, $e_{(m_1,m_2)} = -e_{(m_1)} \cdot e_{(m_2)}$.

Suppose by induction the result holds for $k < r$. Then, for $\mathbf{m} = (m_1, \dots, m_r)$, let $\mathbf{m}' = (m_2, \dots, m_r)$. First, by (6.5) we have

$$\mu^*(e_{\mathbf{m}}) \;=\; e_{\mathbf{m}} \otimes 1 + 1 \otimes e_{\mathbf{m}} + \sum (-1)^{\ell(\mathbf{m}')\ell(\mathbf{m}'')} \varepsilon_{\mathbf{m}',\mathbf{m}''} \cdot e_{\mathbf{m}'} \otimes e_{\mathbf{m}''} \,, \tag{6.9}$$

where the sum is over all $\mathbf{m}' = (m'_1, m'_2, \dots, m'_k)$ and $\mathbf{m}'' = (m''_1, m''_2, \dots, m''_{k'})$ which are both nonempty, disjoint, and whose union in increasing order is $\mathbf{m}$. Then, by induction we obtain

$$\begin{aligned} \mu^*(e_{(m_1)} \cdot e_{\mathbf{m}'}) &= \mu^*(e_{(m_1)}) \cdot \mu^*(e_{\mathbf{m}'}) \\ &= \big(e_{(m_1)} \otimes 1 + 1 \otimes e_{(m_1)}\big) \cdot \Big(e_{\mathbf{m}'} \otimes 1 + 1 \otimes e_{\mathbf{m}'} + \\ &\qquad \sum (-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes e_{\mathbf{m}'''}\Big) \,, \end{aligned} \tag{6.10}$$

where the sum is over $\mathbf{m}''$ and $\mathbf{m}'''$ which are nonempty, disjoint and whose union in increasing order is $\mathbf{m}'$. In the sum on the RHS of (6.9), we have in addition to the terms $e_{\mathbf{m}} \otimes 1$ and $1 \otimes e_{\mathbf{m}}$ the four following types of terms :

Four Types of Terms in (6.9):

i) $(-1)^{\ell(\mathbf{m}')} \varepsilon_{(m_1),\mathbf{m}'} \cdot e_{(m_1)} \otimes e_{\mathbf{m}'} \;=\; (-1)^{\ell(\mathbf{m}')} e_{(m_1)} \otimes e_{\mathbf{m}'}$

ii) $(-1)^{\ell(\mathbf{m}')} \varepsilon_{\mathbf{m}',(m_1)} \cdot e_{\mathbf{m}'} \otimes e_{(m_1)} \;=\; e_{\mathbf{m}'} \otimes e_{(m_1)}$

iii) $(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes e_{\mathbf{m}'''}$ with m_1 in $\mathbf{m}''$

iv) $(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes e_{\mathbf{m}'''}$ with m_1 in $\mathbf{m}'''$

For comparison, we have in addition to the terms $(e_{(m_1)}e_{\mathbf{m}'}) \otimes 1$ and $1 \otimes (e_{(m_1)}e_{\mathbf{m}'})$ the corresponding terms from (6.10) which have the following types:
Corresponding Four Types of Terms in (6.10):

i) $e_{(m_1)} \otimes e_{\mathbf{m}'}$
ii) $(-1)^{\ell(\mathbf{m}')} e_{\mathbf{m}'} \otimes e_{(m_1)}$
iii) $(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot (e_{(m_1)}e_{\mathbf{m}''}) \otimes e_{\mathbf{m}'''}$
iv) $(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes (e_{(m_1)}e_{\mathbf{m}'''})$

In the first two cases for (6.10), we can view them as a decomposition of $\mathbf{m}$ either as $(\{m_1\}, \mathbf{m}')$ or $(\mathbf{m}', \{m_1\})$. We see that the corresponding coefficients for i) and ii) for (6.10) and (6.9) differ by a factor $(-1)^{\ell(\mathbf{m}')}$. The corresponding terms in iii) and iv) for (6.10) can be viewed as a decomposition either as $(\{m_1\} \cup \mathbf{m}'', \mathbf{m}''')$ or $(\mathbf{m}'', \{m_1\} \cup \mathbf{m}''')$. The corresponding coefficients will also be shown to differ by the same factor $(-1)^{\ell(\mathbf{m}')}$.

For example, for iv) let $\tilde{\mathbf{m}}''' = \{m_1\} \cup \mathbf{m}'''$. Then,

$$\varepsilon_{\mathbf{m}'',\tilde{\mathbf{m}}'''} = (-1)^{\ell(\mathbf{m}'')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''}, \quad \ell(\tilde{\mathbf{m}}''') = \ell(\mathbf{m}''') + 1;$$

and by the induction hypothesis $e_{\tilde{\mathbf{m}}'''} = (-1)^{\ell(\mathbf{m}''')} e_{(m_1)} \cdot e_{\mathbf{m}'''}$. Then, substituting these values in iv) for (6.10) yields

$$\begin{aligned}
&(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}'')\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\mathbf{m}'''} \cdot e_{\mathbf{m}''} \otimes (e_{(m_1)}e_{\mathbf{m}'''}) \ = \\
&(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}'')\ell(\tilde{\mathbf{m}}''')}(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\tilde{\mathbf{m}}'''} \cdot e_{\mathbf{m}''} \otimes e_{\tilde{\mathbf{m}}'''} \\
&= (-1)^{\ell(\mathbf{m}'')\ell(\tilde{\mathbf{m}}''')}(-1)^{\ell(\mathbf{m}'')}(-1)^{\ell(\mathbf{m}''')} \varepsilon_{\mathbf{m}'',\tilde{\mathbf{m}}'''} \cdot e_{\mathbf{m}''} \otimes e_{\tilde{\mathbf{m}}'''} \\
&= (-1)^{\ell(\mathbf{m}')} \left((-1)^{\ell(\mathbf{m}'')\ell(\tilde{\mathbf{m}}''')} \varepsilon_{\mathbf{m}'',\tilde{\mathbf{m}}'''} \cdot e_{\mathbf{m}''} \otimes e_{\tilde{\mathbf{m}}'''} \right)
\end{aligned} \tag{6.11}$$

A similar, but somewhat simpler, argument works for the terms iii).

Then, we proceed as in the previous case. We compute $\mu^*(e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'})$ from (6.10) and (6.9) and by the above all terms of types i) - iv) cancel so we obtain

$$\mu^*(e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}) \ = \ (e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}) \otimes 1 \ + \ 1 \otimes (e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}). \tag{6.12}$$

This again implies that $e_{\mathbf{m}} - (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}$ is a primitive element if it is nonzero. Hence, it is zero and so $e_{\mathbf{m}} = (-1)^{\ell(\mathbf{m}')} e_{(m_1)}e_{\mathbf{m}'}$. Repeated inductive application of this implies that for $\mathbf{m} = (m_1, m_2, \dots, m_r)$

$$e_{\mathbf{m}} \ = \ (-1)^{\beta(\mathbf{m})} e_{(m_1)} \cdot e_{(m_2)} \cdots e_{(m_r)} \, .$$

with $\beta(\mathbf{m}) = 1 + 2 + \cdots + (r-1) = \binom{\ell(\mathbf{m})}{2}$. ∎

As a consequence we have determined the Poincaré duals to the Schubert classes.

Corollary 6.6. *For each Schubert symbol* $\mathbf{m} = (m_1, m_2, \dots, m_r)$ *let the ordered complement in* $\{2, 3, \dots, n\}$ *be denoted by* $\mathbf{m}' = (m'_1, m'_2, \dots, m'_{n-1-r})$.

i) *The Poincaré dual to the Schubert class* $\overline{[S_{\mathbf{m}}]}$ *in* F_n^c *and to the Schubert class* $\overline{[S_{\mathbf{m}} \cdot Sol_n]}$ *in* F_n *is given by*

$$(-1)^{(\beta(\mathbf{n})+\beta(\mathbf{m}))} \varepsilon_{\mathbf{m},\mathbf{m}'}\, e_{(m'_1)} \cdot e_{(m'_2)} \cdots e_{(m'_{n-1-r})}$$

for $\mathbf{n} = (2, 3, \dots, n)$.

ii) *For Schubert symbols* $\mathbf{m}$ *and* $\mathbf{m}'$ *such that* $\ell(\mathbf{m})+\ell(\mathbf{m}')=n-1$, *the intersection pairing satisfies*

$$\langle[\overline{S_{\mathbf{m}}}],[\overline{S_{\mathbf{m}'}}]\rangle \;=\; \begin{cases} (-1)^{(\beta(\mathbf{n})+\beta(\mathbf{m})+\beta(\mathbf{m}'))}\varepsilon_{\mathbf{m},\mathbf{m}'} & \textit{if } \mathbf{m}' \textit{ is the ordered} \\ & \textit{complement to } \mathbf{m}, \\ 0 & \textit{otherwise} \;. \end{cases} \tag{6.13}$$

Proof. By Theorem 6.5, the Kronecker dual to $\left[\overline{S_{\mathbf{m}}}\right]$ is given by

$$e_{\mathbf{m}}=(-1)^{\beta(\mathbf{m})}e_{(m_1)}\cdot e_{(m_2)}\cdots e_{(m_r)}.$$

Also, the fundamental class for $[SU_n]$ with orientation given by $\left[\overline{S_{\mathbf{n}}}\right]$ has Kronecker dual

$$(-1)^{\beta(\mathbf{n})}e_{(2)}\cdot e_{(3)}\cdots e_{(n)}.$$

Then, the Poincaré dual to $\left[\overline{S_{\mathbf{m}}}\right]$ is given by a cohomology class ν such that

$$e_{\mathbf{m}}\cup\nu=(-1)^{\beta(\mathbf{n})}e_{(2)}\cdot e_{(3)}\cdots e_{(n)}.$$

This is satisfied by

$$\nu=(-1)^{(\beta(\mathbf{n})+\beta(\mathbf{m}))}\varepsilon_{\mathbf{m},\mathbf{m}'}\,e_{(m'_1)}\cdot e_{(m'_2)}\cdots e_{(m'_{n-1-r})}\,.$$

In the case of the Schubert class $\left[\overline{S_{\mathbf{m}}}\cdot Sol_n\right]$ in F_n, we note that $\overline{S_{\mathbf{m}}}$ is the transverse intersection of $F_n^c=SU_n$ with $\overline{S_{\mathbf{m}}}\cdot Sol_n$ in F_n and that the inclusion $i_n: F_n^c\hookrightarrow F_n$ is a homotopy equivalence. Hence, by a fiber square argument, the Poincaré dual in $H^*(F_n;\mathbb{Z})$ to the fundamental class of $\overline{S_{\mathbf{m}}}\cdot Sol_n$ for Borel-Moore homology, agrees via i_n^* with that for the fundamental class of $\overline{S_{\mathbf{m}}}$ in $H^*(F_n^c;\mathbb{Z})$.

The consequence for the intersection pairing follows from the above and

$$\langle[\overline{S_{\mathbf{m}}}],[\overline{S_{\mathbf{m}'}}]\rangle \;=\; \langle e_{\mathbf{m}}\cup e_{\mathbf{m}'},\left[\overline{S_{\mathbf{n}}}\right]\rangle. \tag{6.14}$$

□

Milnor Fiber for the Variety of Singular $m\times m$-Skew-Symmetric Matrices.

We second consider the case of the global Milnor fiber $F_m^{(sk)}$ for skew-symmetric matrices with $m=2n$. Then, the homology of SU_{2n}/Sp_n can be computed from the algebraic complex with basis formed from the Schubert cells $S_{\mathbf{m}}^{(sk)}$. By a result of Cartan (see e.g. Mimura-Toda [MT, Theorem 6.7]) the homology of SU_{2n}/Sp_n (which is isomorphic as a graded $\mathbb{Z}$-module to its cohomology) is given as a graded $\mathbb{Z}$-module by

$$H_*(SU_{2n}/Sp_n;\mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}\langle s_5,s_9,\ldots,s_{4n-3}\rangle\,. \tag{6.15}$$

where s_{4j-3} has degree $4j-3$. By the universal coefficient theorem this holds as well as a vector space over a field $\mathbf{k}$ of characteristic zero.

Theorem 6.7. *The homology* $H_*(F_m^{(sk)\,c};\mathbb{Z})$ *for* $m=2n$ *has for a free* $\mathbb{Z}$*-basis the fundamental classes of the skew-symmetric Schubert cycles,* $i_{m\,*}\circ\psi_{\mathbf{m}\,*}^{(sk)}([\tilde{S}_{\mathbf{m}}^{(sk)}])=\psi_{\mathbf{m}\,*}^{(sk)}(\tilde{S}_{\mathbf{m}}^{(sk)})=\overline{S_{\mathbf{m}}^{(sk)}}$ *as we vary over the Schubert decomposition of* $\mathcal{C}_m^{(sk)}\simeq SU_{2n}/Sp_n$*. Moreover, the Kronecker duals of the simple skew-symmetric Schubert cycles* $\overline{S_{(m_1)}^{(sk)}}$ *give homogeneous exterior algebra generators for the cohomology.*

This likewise extends to $H_*(F_m^{(sk)};\mathbb{Z})$ *(*$m=2n$*) for Borel-Moore homology with basis given by the fundamental classes of the global skew-symmetric Schubert cycles* $Sol_m^T\cdot(\overline{S_{\mathbf{m}}^{(sk)}}\cdot J_n)$ *for* $F_m^{(sk)}$*. The Poincaré duals of these classes form a* $\mathbb{Z}$*-basis for the cohomology*

$$H^*(F_m^{(sk)};\mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}\langle e_5,e_9,\ldots,e_{4n-3}\rangle\,.$$

Proof. The proof follows the same lines as that of Theorem 6.1. Then, a count from (6.15) shows that $H_q(SU_{2n}/Sp_n;\mathbb{Z})$ is spanned by $s_{4m_1-3} \cdot s_{4m_2-3} \cdots s_{4m_k-3}$, where

$$1 < m_1 < m_2 < \cdots < m_k \leq n$$

and $q = \sum_{j=1}^{k}(4m_j - 3)$. By Theorem 5.6 this equals the number of skew-symmetric Schubert cells $S_{\mathbf{m}}^{(sk)}$ of real dimension q. Thus, each $\psi_{\mathbf{m}}^{(sk)}(\tilde{S}_{\mathbf{m}}^{(sk)}) = \overline{S_{\mathbf{m}}^{(sk)}}$ defines a $\mathbb{Z}$-homology class of dimension $\dim_{\mathbb{R}} S_{\mathbf{m}}^{(sk)}$. Together they form a basis for $H_q(SU_{2n}/Sp_n;\mathbb{Z})$. That the Kronecker duals of the simple Schubert cycles $S_{(m_1)}^{(sk)}$ give algebra generators for the cohomology follows by the same argument used in Theorem 6.1.

As $\tilde{S}_{\mathbf{m}}^{(sk)}$ has a top homology class in $H_q(\tilde{S}_{\mathbf{m}}^{(sk)};\mathbb{Z})$ for $q = \dim_{\mathbb{R}}(\tilde{S}_{\mathbf{m}}^{(sk)})$, we can view it as a fundamental class for $\tilde{S}_{\mathbf{m}}^{(sk)}$ for Borel-Moore homology. As $F_m^{(sk)\,c} \simeq \mathcal{C}_m^{(sk)} \simeq SU_{2n}/Sp_n$ by multiplication by J_n and the inclusion $i_m : F_m^{(sk)\,c} \hookrightarrow F_m^{(sk)}$ is a homotopy equivalence, we conclude that these classes form a $\mathbb{Z}$-basis for the cohomology via $H^*(F_m^{(sk)};\mathbb{Z}) \simeq H^*(F_m^{(sk)\,c};\mathbb{Z})$. Their Poincaré duals then form a $\mathbb{Z}$-basis for the Borel-Moore homology. $\square$

Again there is the question of explicitly identifying the Kronecker dual of the fundamental class $\psi_{\mathbf{m}\,*}^{(sk)}([\tilde{S}_{\mathbf{m}}^{(sk)}])$ with a cohomology class as a polynomial in the cohomology algebra generators e_{4j-3}, $j = 2, \ldots, n$, and as a consequence explicitly identifying the generators for the cohomology algebra. We shall comment on this after next considering the symmetric case.

Milnor Fiber for the Variety of Singular $m \times m$-Symmetric Matrices.

We next consider the case of $F_m^{(sy)}$. Again the line of reasoning will be similar to the two preceding cases with the crucial difference that the (co)homology has two different forms for coefficients $\mathbb{Z}/2\mathbb{Z}$ or a field of characteristic zero. There is the compact model

$$F_n^{(sy)\,c} \simeq \mathcal{C}_n^{(sy)} \simeq SU_n/SO_n$$

for $F_n^{(sy)}$. Then, the homology of SU_n/SO_n can be computed from the algebraic complex with basis formed from the Schubert cells $S_{\mathbf{m}}^{(sy)}$. By a result of Borel and Hopf, see e.g. [Bo] and see [KM], the homology of SU_n/SO_n with $\mathbb{Z}/2\mathbb{Z}$-coefficients (which is isomorphic as a graded $\mathbb{Z}/2\mathbb{Z}$-vector space to its cohomology) is given as a graded vector space over the field $\mathbb{Z}/2\mathbb{Z}$

$$H_*(SU_n/SO_n;\mathbb{Z}/2\mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}/2\mathbb{Z}\langle s_2, s_3, \ldots, s_n\rangle\,,$$

where s_j has degree j. A count shows that

$$\dim_{\mathbb{Z}/2\mathbb{Z}} H_*(SU_n/SO_n;\mathbb{Z}/2\mathbb{Z}) \;=\; 2^{n-1}\,.$$

This is the same as the number of Schubert cells $S_{\mathbf{m}}^{(sy)}$, for

$$1 < m_1 < \cdots < m_k \leq n$$

in the cell decomposition of SU_n/SO_n. Thus, the Schubert cycles $\overline{S_{\mathbf{m}}^{(sy)}}$, which are mod 2-homology cycles, give a $\mathbb{Z}/2\mathbb{Z}$-basis for the homology $H_*(SU_n/SO_n;\mathbb{Z}/2\mathbb{Z})$. In particular the mod 2-homology cycles $\overline{S_{\mathbf{m}}^{(sy)}}$ for which $|\mathbf{m}| = q$ give a $\mathbb{Z}/2\mathbb{Z}$-basis for $H_q(SU_n/SO_n;\mathbb{Z}/2\mathbb{Z})$ for each $q \geq 0$.

Thus, we conclude by an analogous argument to that used in the preceding two cases

Theorem 6.8. *The homology $H_*(F_n^{(sy)\,c};\mathbb{Z}/2\mathbb{Z})$ has for a $\mathbb{Z}/2\mathbb{Z}$-basis the $\mathbb{Z}/2\mathbb{Z}$ fundamental classes of the symmetric Schubert cycles $\left[\overline{S_{\mathbf{m}}^{(sy)}}\right]$ as we vary over the Schubert decomposition of*

$\mathcal{C}_n^{(sy)} \simeq SU_n/SO_n$ for all symmetric Schubert symbols $\mathbf{m}^{(sy)} = (m_1, \dots, m_k)$ with

$$1 < m_1 < \cdots < m_k \leq n.$$

Moreover, the Kronecker duals of the simple symmetric Schubert cycles $\overline{S_{(m_1)}^{(sy)}}$ are algebra generators for the exterior cohomology algebra with $\mathbb{Z}/2\mathbb{Z}$-coefficients.

This extends to $H_*(F_n^{(sy)}; \mathbb{Z}/2\mathbb{Z})$ with $\mathbb{Z}/2\mathbb{Z}$-basis given by the Borel-Moore mod 2*-cycles given by the global symmetric Schubert cycles $\left[Sol_m^T \cdot (S_{\mathbf{m}}^{(sy)})\right]$ for $S_{\mathbf{m}}^{(sy)}$ over the symmetric Schubert symbols $\mathbf{m}^{(sy)}$. The Poincaré duals of these classes form a $\mathbb{Z}/2\mathbb{Z}$-basis for the cohomology.*

$$H^*(F_m^{(sy)}; \mathbb{Z}/2\mathbb{Z}) \;\simeq\; \Lambda^*\mathbb{Z}/2\mathbb{Z}\langle e_2, e_3, \dots, e_n\rangle\,.$$

There are several points to be made regarding this result and that for skew-symmetric matrices.

First, unlike the cases of SU_n and SU_{2n}/Sp_n, the closure of the Schubert cells are not the images of Borel-Moore homology classes of singular manifolds. As mentioned earlier, if we consider instead the quotient space $F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)}$, and $|\mathbf{m}| = q$, then the composition of the map

$$\tilde{\psi}_{\mathbf{m}}^{(sy)} : \prod_{i=1}^{k}(C\mathbb{R}P^{m_i-1}) \;\longrightarrow\; SU_n/SO_n \simeq F_m^{(sy)\,c}$$

with the quotient map $pr_q : F_m^{(sy)\,c} \to F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)}$ factors through to give a map

$$pr_q \circ \tilde{\psi}_{\mathbf{m}}^{(sy)} : \prod_{i=1}^{k} S\mathbb{R}P^{m_i-1} \;\longrightarrow\; F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)}\,.$$

As

$$pr_q : (F_m^{(sy)\,c}, (F_m^{(sy)\,c})^{(q-1)}) \to (F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)}, *),$$

for $*$ the point representing $(F_m^{(sy)\,c})^{(q-1)}$ in the quotient, is a relative homeomorphism,

$$pr_{q\,*} : H_q(F_m^{(sy)\,c}, (F_m^{(sy)\,c})^{(q-1)}; \mathbb{Z}/2\mathbb{Z}) \;\simeq\; H_q(F_m^{(sy)\,c}/(F_m^{(sy)\,c})^{(q-1)}, *; \mathbb{Z}/2\mathbb{Z})\,.$$

Then, the closure $\overline{S_{\mathbf{m}}^{(sy)}}$ corresponds via the isomorphism to the image of the fundamental class of $\prod_{i=1}^{k}(S\mathbb{R}P^{m_i-1})$ under $pr_{q\,*} \circ \tilde{\psi}_{\mathbf{m}\,*}^{(sy)}$.

Moreover, as noted earlier for the simple Schubert symbol (m_1), there is a factored map $\tilde{\psi}_{(m_1)}^{(sy)} : S\mathbb{R}P^{m_i-1} \to SU_n/SO_n \simeq F_m^{(sy)\,c}$ with image $\overline{S_{(m_1)}^{(sy)}}$, giving it a Borel-Moore fundamental homology class for $\mathbb{Z}/2\mathbb{Z}$-coefficients.

However, for cohomology with rational coefficients, see e.g. [MT, Chap. 3, Thm 6.7 (2)] or Table 1 in [D3], many of these Schubert cells do not contribute homology classes. This is similar to the situation for oriented Grassmannians for $\mathbb{Z}/2\mathbb{Z}$ versus rational coefficients. This relation extends further. Over SU_n/SO_n is a natural n-dimensional real oriented vector bundle $E_n = (SU_n \times_{SO_n} \mathbb{R}^n)$ where $\mathbb{R}^n$ has the natural representation of SO_n. This bundle can be viewed geometrically as the set of oriented real subspaces $V \subset \mathbb{C}^n$ with $\dim_{\mathbb{R}} V = n$ such that $\mathbb{C}\langle V\rangle = \mathbb{C}^n$. Then, by e.g. [MT, Chap. 3, Thm 6.7 (3)] the cohomology of SU_n/SO_n, already quoted in Theorem 6.8 has $e_j = w_j(E_n)$, the j-th Stiefel-Whitney class. This bundle pulls-back by the homotopy equivalence $SU_n/SO_n \simeq F_n^{(sy)\,c} \simeq F_n^{(sy)}$ to give an n-dimensional real oriented vector bundle, which we denote by $\tilde{E}_n$ and then

$$H^*(F_n^{(sy)}; \mathbb{Z}/2\mathbb{Z}) \;\simeq \Lambda^*\mathbb{Z}/2\mathbb{Z} < w_2, w_3, \dots, w_n >$$

where $w_j = w_j(\tilde{E}_n)$ for each $j = 2, 3, \dots, n$. We will see in the next section that this algebra naturally pulls back to a characteristic subalgebra of Milnor fibers for general symmetric matrix singularities generated by the Stiefel-Whitney classes of the pull-back of $\tilde{E}_n$ to the Milnor fiber.

Although both

$$H^*(F_n^{(sy)}; \mathbb{Z}/2\mathbb{Z}) \simeq H^*(SU_n/SO_n; \mathbb{Z}/2\mathbb{Z}) \quad \text{and} \quad H^*(F_{2n}^{(sk)}; \mathbb{Z}) \simeq H^*(SU_{2n}/Sp_n; \mathbb{Z})$$

are exterior algebras, neither is a Hopf algebra. Hence, the full argument given for $H^*(F_n; \mathbb{Z})$ for the relation between the cohomology and the Schubert decomposition cannot be given using Hopf algebra methods. However, it does suggest the following conjecture is true and constitutes work in progress.

Conjecture: For both $F_n^{(sk)\,c}$ and $F_n^{(sy)\,c}$, the Kronecker duals to the Schubert classes $S_{(\mathbf{m})}^{(sk)}$, resp. $S_{(\mathbf{m})}^{(sy)}$ for Schubert symbols $\mathbf{m}^{(sk)}$, or $\mathbf{m}^{(sy)} = (m_1, m_2, \dots, m_r)$ are given up to sign by $e_{(m_1)} \cdot e_{(m_2)} \cdots e_{(m_r)}$ in the corresponding cohomology algebra.

7. Characteristic Subalgebra in the Cohomology of General Matrix Singularities

In the preceding section we have identified for the Milnor fibers F_m, $F_m^{(sy)}$, and $F_m^{(sk)}$ (for m = 2n), their cohomology and the decomposition of their homology using the Schubert decomposition. We see how this applies to the structure of Milnor fibers of general matrix singularities of each of these types.

Let M denoting any one of the three spaces of complex $m \times m$ matrices which are general $M_{m,m}(\mathbb{C})$, symmetric $Sym_m(\mathbb{C})$, or skew-symmetric $Sk_m(\mathbb{C})$ with $m = 2n$. Also, let $\mathcal{D}_m$, resp. $\mathcal{D}_m^{(sy)}$, or $\mathcal{D}_m^{(sk)}$ denote the variety of singular matrices of the corresponding type. We suppose that each type is defined by $H : M \to \mathbb{C}$, which denotes either the determinant det for $\mathcal{D}_m$ or $\mathcal{D}_m^{(sy)}$, or the Pfaffian Pf for $\mathcal{D}_m^{(sk)}$.

Matrix Singularities of a Given Type.

A matrix singularity of any of the given types is defined by a holomorphic germ

$$f_0 : \mathbb{C}^s, 0 \to M, 0,$$

and the singularity is defined by $X_0 = f_0^{-1}(\mathcal{V}), 0$ where $\mathcal{V}$ denotes the appropriate variety of singular matrices. We impose an additional condition on f which can take several forms based on forms of $\mathcal{K}$-equivalence preserving $\mathcal{V}$. There is the equivalence defined using the parametrized action by points in $\mathbb{C}^s$ of the group $G = GL_m(\mathbb{C})$ acting by $C \mapsto A \cdot C \cdot A^T$ in the symmetric or skew-symmetric cases. For the general $m \times m$ matrix case, the action of $G = GL_m(\mathbb{C})$ acting by left multiplication suffices for studying the Milnor fiber. However, for the general equivalence studying the pull-back of $\mathcal{D}_m$ the action is given by $G = GL_m(\mathbb{C}) \times GL_m(\mathbb{C})$ acting by $C \mapsto A \cdot C \cdot B^{-1}$. We denote the equivalence for any of the general, symmetric, or skew-symmetric cases as $\mathcal{K}_M$-equivalence. The second equivalence allows the action of germs of diffeomorphisms of $\mathbb{C}^s \times M, (0,0)$ of the form $\varphi(x, y) = (\varphi_1(x), \varphi_2(x, y))$ which preserve $\mathbb{C}^s \times \mathcal{V}$, and is denoted $\mathcal{K}_{\mathcal{V}}$ equivalence. The third is a subgroup of $\mathcal{K}_{\mathcal{V}}$ which preserves the defining equation of $\mathbb{C}^s \times \mathcal{V}$, $H \circ pr_M$, with pr_M denoting projection onto M. It is denoted $\mathcal{K}_H$. See for example [DP2], [D2], or [D1] for more details about the groups of equivalence and their relations and the properties of germs which have finite codimension for one of these equivalences. In particular, for the three classes of varieties of singular matrices, $\mathcal{K}_{\mathcal{V}}$ and $\mathcal{K}_M$ equivalences agree.

If f_0 has finite $\mathcal{K}_{\mathcal{V}}$-codimension, then it may be deformed to f_t which is transverse to $\mathcal{V}$ in a neighborhood $B_\varepsilon(0)$ of $0 \in \mathbb{C}^s$ for $t \neq 0$. Then it is shown in [DM] that one measure of the vanishing topology of X_0 is by the"singular Milnor fiber" $\tilde{X}_t = f_t^{-1}(\mathcal{V}) \cap B_\varepsilon(0)$. It is homotopy equivalent to a bouquet of real spheres of dimension $s-1$. If $s < \mathrm{codim}_M(\mathrm{sing}(\mathcal{V}))$, then this is the usual Milnor fiber of $\mathcal{V}_0$. This condition requires $s < 4$, resp. 3, resp. 6, for the three types of matrices.

In the special case that $\mathcal{V}$ is a free divisor and holonomic in the sense of Saito [Sa] and satisfies a local weighted homogeneity condition [DM] or is a free divisor and H-holonomic [D1], then the singular Milnor number is given by the length of the normal space $N\mathcal{K}_{H\,e}f_0$, which is a determinantal module.

For the three classes of varieties of singular matrices, the varieties are not free divisors. Nonetheless, when $s \leq \mathrm{codim}_M(\mathrm{sing}(\mathcal{V}))$, Goryunov and Mond [GM] give a formula for the Milnor number which adds a correction term for the lack of freeness given by an Euler characteristic of a Tor complex. Instead, Damon-Pike [DP3] give a formula valid for all s but which is presently restricted to a limited range of matrices. It is given by a sum of terms which are lengths of determinantal modules, based on placing the varieties in a tower of free divisors [DP2].

Cohomology Structure of Milnor Fibers of General Matrix Singularities.

We explain how the results in earlier sections provide information about the cohomology of the Milnor fiber for a matrix singularity X_0 for all s.

We consider the defining equation $H : \mathbb{C}^N, 0 \to \mathbb{C}, 0$ for $\mathcal{V}$, where $M \simeq \mathbb{C}^N$ for each case. For $\mathcal{V}$ there exists $0 < \delta << \eta$ such that for balls $B_\delta \subset \mathbb{C}$ and$B_\eta \subset \mathbb{C}^N$ (with all balls centered 0), we let $\mathcal{F}_\delta = H^{-1}(B_\delta) \cap B_\eta$ so $H : \mathcal{F}_\delta \to B_\delta$ is the Milnor fibration of H, with Milnor fiber $\mathcal{V}_w = H^{-1}(w) \cap B_\eta$ for each $w \in B_\delta$. By continuity, there is an $\varepsilon > 0$ so that $f_0(B_\varepsilon) \subset \mathcal{F}_\delta$. By possibly shrinking all three values, $H \circ f_0 : f_0^{-1}(\mathcal{F}_\delta) \cap B_\varepsilon \to B_\delta$ is the Milnor fibration of $H \circ f_0$. Also, by the parametrized transversality theorem, for almost all $w \in B_\delta$, f_0 is transverse to $\mathcal{V}_w$ and so the Milnor fiber of $H \circ f_0$ is given by

$$X_w \;=\; (H \circ f_0)^{-1}(w) \cap B_\varepsilon \;=\; f_0^{-1}(\mathcal{V}_w) \cap B_\varepsilon \,.$$

Thus, if we denote $f_0 | X_w = f_{0,w}$, then in cohomology with coefficient ring R,

$$f_{0,w}^* : H^*(\mathcal{V}_w; R) \to H^*(X_w; R).$$

For any of the three types of matrices with $(*)$ denoting $()$ for general matrices, (sy) for symmetric matrices, or (sk) for skew-symmetric matrices, we let

$$\mathcal{A}^{(*)}(f_0; R) \;\stackrel{def}{=}\; f_{0,w}^*(H^*(\mathcal{V}_w; R)) \,,$$

which we refer to as the *characteristic subalgebra* of the cohomology of the Milnor fiber $H^*(X_w; R)$ of X_0. This is an algebra over R, and the cohomology of the Milnor fiber of the matrix singularity X_0 is a graded module over $\mathcal{A}^{(*)}(f_0; R)$ (both with coefficients R).

By Theorems 6.1 and 6.7 for the $m \times m$ general case or skew-symmetric case (with $m = 2n$), for $R = \mathbb{Z}$-coefficients (and hence for any coefficient ring R) $\mathcal{A}^{(*)}(f_0; R)$ is the quotient ring of a free exterior R-algebra on generators e_{2j-1}, for $j = 2, 3, \dots, m$, resp. e_{4j-3} for $j = 2, 3, \dots, n$. For the $m \times m$ symmetric case there are two important cases where either $R = \mathbb{Z}/2\mathbb{Z}$ or is a field of characteristic zero. In the first case, by Theorem 6.8, $\mathcal{A}^{(*)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is the quotient ring of a a free exterior algebra on generators $e_j = w_j(\tilde{E}_m)$, for $j = 2, 3, \dots, m$, for $w_j(\tilde{E}_m)$ the Stiefel-Whitney classes of the real oriented m-dimensional vector bundle $\tilde{E}_m$ on the Milnor fiber of $\mathcal{D}_m^{(sy)}$. Hence, $\mathcal{A}^{(*)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is a subalgebra generated by the Stiefel-Whitney classes of the pull-back vector bundle $f_{0,w}^*(\tilde{E}_m)$ on X_w.

For the coefficient ring $R = \mathbf{k}$ a field of characteristic zero, the symmetric case breaks-up into two cases depending on whether m is even or odd (see [MT, (2),Thm. 6.7, Chap. 3] or Table 1 of [D3]).

$$H^*(F_m^{(sy)}; \mathbf{k}) \;\simeq\; \begin{cases} \Lambda^*\mathbf{k}\langle e_5, e_9, \dots, e_{2m-1}\rangle & \text{if } m = 2k+1 \\ \Lambda^*\mathbf{k}\langle e_5, e_9, \dots, e_{2m-3}\rangle\{1, e_m\} & \text{if } m = 2k \end{cases} \tag{7.1}$$

where e_m is the Euler class of $\tilde{E}_m$. Hence, in both cases they are graded modules over an exterior algebra. Hence, the Milnor fiber of X_0 has cohomology over a field of characteristic zero which, via the characteristic subalgebra is a graded module over the exterior algebra in either case of (7.1).

We summarize these cases with the following.

Theorem 7.1. *Let $f_0 : \mathbb{C}^s, 0 \to M, 0$ be a matrix singularity of finite $\mathcal{K}_M$-codimension for M the space of $m \times m$ matrices which are either general, symmetric, or skew-symmetric (with $m = 2n$). Let $\mathcal{V}$ denote the variety of singular matrices. Then,*

i) *The cohomology (with coefficients in a ring R) of the Milnor fiber of $X_0 = f_0^{-1}(\mathcal{V})$ has a graded module structure over the characteristic subalgebra $\mathcal{A}^{(*)}(f_0; R)$ of f_0.*
ii) *In the general and skew-symmetric cases, $\mathcal{A}^{(*)}(f_0; R)$ is a quotient of the free R-exterior algebra with generators given in Theorems 6.1 and 6.7 .*
iii) *In the symmetric case with $R = \mathbb{Z}/2\mathbb{Z}$, $\mathcal{A}^{(sy)}(f_0; \mathbb{Z}/2\mathbb{Z})$ is the quotient of the free exterior algebra over $\mathbb{Z}/2\mathbb{Z}$ on the Stiefel-Whitney classes of the real oriented vector bundle $\tilde{E}_m$ on the Milnor fiber of $\mathcal{V}$.*
iv) *In the symmetric case with $R = \mathbf{k}$, a field of characteristic zero, $\mathcal{A}^{(sy)}(f_0; \mathbf{k})$ is a quotient of the $\mathbf{k}$-algebras in each of the cases in (7.1).*

In light of this theorem there are several problems to be solved for determining the cohomology of the Milnor fiber of the matrix singularity X_0 for coefficients R.

Questions for the Cohomology of the Milnor Fibers of Matrix Singularities

1) Determine the characteristic subalgebras as the images of the exterior algebras by determining which monomials map to nonzero elements in $H^*(X_w; R)$.
2) Find the non-zero monomials in the image by geometrically identifying the pull-backs of the Schubert classes.
3) For the symmetric case with $\mathbb{Z}/2\mathbb{Z}$-coefficients, compute the Stiefel-Whitney classes of the pull-back of the vector bundle $\tilde{E}_m$.
4) Determine a set of module generators for the cohomology of the Milnor fibers as modules over the characteristic subalgebras.

Transversality to Schubert Cycles.

We can give a first step for these using transversality. We let M denote one of the spaces of $m \times m$ matrices with variety of singular matrices denoted by $\mathcal{V}$. There is a transitive action on $SL_m(\mathbb{C})$ on the global Milnor fibers of the varieties of singular matrices in all three cases. We let $S_{\mathbf{m}}^{(*)}$ denote the Schubert cell in the global Milnor fiber of the corresponding type. For each Schubert class $S_{\mathbf{m}}^{(*)}$ and $A \in SL_m(\mathbb{C})$, we let $A \cdot S_{\mathbf{m}}^{(*)}$ denote the image under the action of A. Also, we let the germ $f_1 = A^{-1} \cdot f_0$ denote the germ obtained by applying the constant matrix A^{-1} to $f_0(x)$ independent of x. This action preserves the global Milnor fibers of $\mathcal{V}$. Then, deforming either the Schubert cells or f_0 by multiplication by A yields the following.

Lemma 7.2. *Given $f_0 : \mathbb{C}^s, 0 \to M, 0$ of finite $\mathcal{K}_M$-codimension, for almost all $A \in SL_m(\mathbb{C})$ the germ f_0 is transverse to $A \cdot S_{\mathbf{m}}^{(*)}$ for all Schubert cells $S_{\mathbf{m}}^{(*)}$ in a Milnor fiber $\mathcal{V}_w$ of $\mathcal{V}$. Then, for $f_1 = A^{-1} \cdot f_0$ and $e'_{\mathbf{m}}$ the Poincaré dual to $[\overline{S_{\mathbf{m}}^*}]$, $f_1^*(e'_{\mathbf{m}})$ is the Poincaré dual of $[f_1^{-1}(\overline{S_{\mathbf{m}}^{(*)}})]$.*

Then, f_1 is $\mathcal{K}_M$-equivalent to f_0, and $f_{0w}^* = f_{1w}^*$.

Proof. As $SL_m(\mathbb{C})$ is path-connected, the action of A is homotopic to the identity. Let A_t be such a path from I_m to A. Hence, $[A_t \cdot \overline{S_{\mathbf{m}}^{(*)}}] = [\overline{S_{\mathbf{m}}^{(*)}}]$ for all t.

Next, by the parametrized transversality theorem and the transitive acton of $SL_m(\mathbb{C})$ on the global Mlnor fiber, it follows that f_0 is transverse to $A \cdot \overline{S_{\mathbf{m}}^{(*)}}$ for almost all $A \in SL_m(\mathbb{C})$. As there are only a finite number of Schubert cells, then for almost all A this simultaneously holds for all of the Schubert cells $S_{\mathbf{m}}^{(*)}$. For such an A with $f_1 = A^{-1} \cdot f_0$, it follows that $f_1 = A \cdot f_0$ is transverse to all of the Schubert cells. If $e'_{\mathbf{m}}$ denotes the Poincaré dual to $[\overline{S_{\mathbf{m}}^{(*)}}]$, it is also the Poincaré dual to $[A \cdot \overline{S_{\mathbf{m}}^{(*)}}]$. Thus, by a fiber square argument $f_{1w}^*(e'_{\mathbf{m}})$ is the Poincaré dual to $[f_{1w}^{-1}(A \cdot \overline{S_{\mathbf{m}}^{(*)}})]$.

Lastly, the family $f_t = A_t^{-1} \cdot f_0$ is a $\mathcal{K}_M$-constant family so that $f_1 = A^{-1} \cdot f_0$ is $\mathcal{K}_M$-equivalent to f_0 and $f_{0w}^* = f_{1w}^*$. $\square$

Remark 7.3. As a simple consequence of this lemma, we may replace f_0 by the $\mathcal{K}_M$-equivalent $f_1 = A^{-1} \cdot f_0$ transverse to $\overline{S_{\mathbf{m}}^{(*)}}$. If $s < \frac{1}{2}\text{codim}_{\mathbb{R}}(S_{\mathbf{m}}^{(*)}) + 1$, then $f_{1w}^{-1}(A \cdot \overline{S_{\mathbf{m}}^{(*)}})$ is empty. Hence $f_{0w}^*(e'_{\mathbf{m}}) = 0$.

Module Structure for the Milnor Fibers.

We make several remarks regarding these questions concerning the module structure. These involve two cases at opposite extremes, namely $s < \text{codim}_M(\text{sing}(X_0))$ or f_0 is the germ of a submersion. In the first case when $s < \text{codim}_M(\text{sing}(\mathcal{V}))$, X_0 has an isolated singularity, and the singular Milnor fiber for f_0 is the Milnor fiber for X_0, so the Milnor number and singular Milnor number agree. Also, $f_{0w}^*(e'_{\mathbf{m}}) = 0$ for all $e'_{\mathbf{m}}$ of positive degree; thus

$$\mathcal{A}^{(*)}(f_0, R) = H^0(X_w; R) \simeq R.$$

As the Minor fiber is homotopy equivalent to a CW-complex of real dimension $s - 1$, the corresponding classes which occur for the Milnor fiber will have a trivial module structure over $\mathcal{A}^{(*)}(f_0, R)$.

Second, if f_0 is the germ of a submersion, then the Milnor fiber has the form $\mathcal{V}_w \times \mathbb{C}^k$, where $k = s - \dim_{\mathbb{C}} M$ and so has the same cohomology, so we conclude that

$$f_0^* : H^*(\mathcal{V}_w; R) \simeq H^*(X_w; R)$$

so $\mathcal{A}^{(*)}(f_0, R) = H^*(X_w; R)$. Also, there are no singular vanishing cycles. Thus, for these two cases there is the following expression for the cohomology of the Milnor fiber, where the second summand has trivial module structure shifted by degree $s - 1$.

$$H^*(X_w; R) \simeq \mathcal{A}^{(*)}(f_0, R) \oplus R^{\mu}[s-1], \tag{7.2}$$

where $\mu = \mu_{\mathcal{V}}(f_0)$ for $\mathcal{V} = \mathcal{D}_m^{(*)}$ the corresponding variety of singular matrices.

We ask whether this holds in general or at least for a large class of matrix singularities.

Question: How generally valid is (7.2) *for matrix singularities of the three types?*

For this question, we note that for the case of 2×3 complex matrices with $\mathcal{V}$ denoting the variety of singular matrices and $s = 5$, the matrix singularities define Cohen-Macaulay 3-fold singularities. A stabilization of these singularities gives a smoothing and Milnor fiber. In [DP3, Thm. 8.4] is given an algebraic formula for the vanishing Euler characteristic, which becomes the difference of the Betti numbers $b_3 - b_2$ of the Milnor fiber. While specific calculations in the Appendix of [DP3] show that the vanishing Euler characteristic typically increases in families with the $\mathcal{K}_V$-codimension, it is initially not clear how this increase is distributed as changes of b_3 and b_2. Surprisingly, Frühbis-Krüger and Zach [FZ], [Z] show that for a large class of such singularities that $b_2 = 1$. This suggests it may be possible to identify certain classes of $m \times m$ matrix singularities for which there are contributions from $\mathcal{A}^{(*)}(f_0, R)$ for the topology of the Milnor fiber. This is a fundamental question whose answer along with the preceding ones will clarify our understanding of the full cohomology of the Milnor fibers of matrix singularities.

8. Extensions to Exceptional Orbit Varieties, Complements, and Links

We indicate in this section how the methods of the previous sections can be extended to exceptional orbit hypersurfaces for prehomogeneous vector spaces in the sense of Sato, see [So] and [SK]. This includes equidimensional prehomogeneous spaces, see [D3], in the cases of both block representations of solvable linear algebraic groups [DP2] and the discriminants for quivers of finite type in the sense of Gabriel, see [G], [G2], represented as linear free divisors by Buchweitz-Mond [BM].

Second, we can also apply the preceding methods to the complements of exceptional orbit hypersurfaces arising as the varieties of singular $m \times m$ matrices just considered and the equidimensional prehomogeneous spaces just described. Third, in [D3], the cohomology of the link of one of these singularities is computed as a shift of the (co)homology of the complement. Thus, the Schubert classes for the complement correspond to cohomology classes in the link. However, we explain how the multiplicative cohomology structure of the complement contains more information than the cohomology of the link.

Exceptional Orbit Hypersurfaces for the Equidimensional Cases.

Block Representations of Linear Solvable Algebraic Groups.

First, for the case of block representations of solvable linear algebraic groups, in [DP, Thm 3.1] the complement was shown to be a $K(\pi, 1)$-space where π is a finite extension of $\mathbb{Z}^n$ (for n the rank of the solvable group) by the finite isotropy group of the action on the open orbit. The solvable group is an extension of an algebraic torus by a unipotent group which is contractible. The resulting cell decomposition follows from that for the torus times the unipotent group. Thus, the decomposition is that modulo the finite group. In important cases of (modified) Cholesky-type factorization for the three types of matrices and also $m \times (m+1)$ matrices the finite group is either the identity or $(\mathbb{Z}/2\mathbb{Z})^n$ and the resulting quotient is shown, see [DP, Thm 3.4], to still be the extension of a torus by a (contractible) unipotent group.

Thus, for these cases the cell decomposition follows from the product decomposition for the complex torus times the unipotent group, which has as a compact model a compact torus of the same rank. Moreover, by [DP, Thm 4.1], the cohomology with complex coefficients is an exterior algebra which has as generators 1-forms defined from the defining equation of the exceptional orbit hypersurface.

Also, by [DP, Thm 3.2] the Milnor fiber is again a $K(\pi', 1)$-space with π' a subgroup of π (for the complement) with quotient $\mathbb{Z}$. Again, by [DP, Thm 3.4] for the cases of (modified) Cholesky-type factorization of matrices, it is also true that the Milnor fiber for these cases is the extension of a torus, except of one lower rank, by the unipotent group. Likewise the cohomology with complex coefficients of the Milnor fiber is again an exterior algebra which has one fewer generator, as the result of a quotient by a single specified relation.

Discriminants of Quivers of Finite Type.

The quivers are defined by a finite ordered graph Γ having for each vertex v_i a space $\mathbb{C}^{n_i}$ and for each directed edge from v_i to v_j a linear map $\varphi_{ij} : \mathbb{C}^{n_i} \to \mathbb{C}^{n_j}$. Those quivers of finite type were classified by Gabriel [G], [G2]. The discriminants for the quiver spaces of finite type were shown by Buchweitz-Mond [BM] to be linear free divisors. As such these discriminants are exceptional orbit hypersurfaces for the action of the group $G = (\prod_{i=1}^{k} GL_{n_i}(\mathbb{C}))/\mathbb{C}^*$, where $k =$

$|\Gamma|$. Since each $GL_{n_i}(\mathbb{C})$ topologically factors as $SL_{n_i}(\mathbb{C}) \times \mathbb{C}^*$, the complement is diffeomorphic to $(\prod_{i=1}^{k} SL_{n_i}(\mathbb{C})) \times (\mathbb{C}^*)^{k-1}$. The earlier results for the Schubert decomposition for each $SL_n(\mathbb{C})$ via its maximal compact subgroup SU_n and the product cell decomposition for $(\mathbb{C}^*)^{k-1}$ gives a *product Schubert cell decomposition* for the complement.

The Milnor fiber has an analogous form $(\prod_{i=1}^{k} SL_{n_i}(\mathbb{C})) \times (\mathbb{C}^*)^{k-2}$, and a *product Schubert cell decomposition* for the Milnor fiber.

The cohomology of the complement is given by [D3, (5.11)] as an exterior algebra on a specific set of generators. The cohomology of the Milnor fiber is also an exterior algebra except with one fewer degree 1 generator, see [D3, (Thm 5.4)]. Furthermore, by Theorem 6.1 relating the Schubert decomposition for $SL_n(\mathbb{C})$ via its maximal compact subgroup SU_n with the cohomology classes, we conclude that for both the complement and the Milnor fiber of the discriminant of the space of quivers, the closures of the product Schubert cells provide a set of generators for the homology.

Complements of the Varieties of Singular Matrices.

We can likewise give a Schubert decomposition for the complements of the varieties of $m \times m$ matrices which are general, symmetric or skew-symmetric. We note that in [D3] the complements were given as $GL_m(\mathbb{C})$ for the general matrices, $GL_m(\mathbb{C})/O_m(\mathbb{C})$ for the symmetric matrices, and $GL_{2n}(\mathbb{C})/Sp_n(\mathbb{C})$ for the skew-symmetric case with $m = 2n$. These have as compact models the symmetric spaces U_m, resp. U_m/O_m, resp. U_{2n}/Sp_n. Each of these has a Schubert decomposition given in [KM]. As remarked in §3, U_m has a Schubert decomposition by cells $S_{\mathbf{m}}$ for $\mathbf{m} = (m_1, m_2, \dots, m_r)$, where m_1 may equal 1 and it is not required that $\sum_{i=1}^{r} \theta_i \equiv 0 \bmod 2\pi$.

Second, in [KM, §5] is given a Schubert decomposition for U_m/O_m using for the symmetric Schubert cell $S_{\mathbf{m}}^{(sy)}$ the symmetric factorization into pseudo-rotations except again

$$\mathbf{m}^{(sy)} = (m_1, m_2, \dots, m_r),$$

where m_1 may equal 1 and it is not required that $\sum_{i=1}^{r} \theta_i \equiv 0 \bmod \pi$.

Third, in [KM, §7] is given a Schubert decomposition for U_{2n}/Sp_n using for the skew-symmetric Schubert cell $S_{\mathbf{m}}^{(sk)}$ the skew-symmetric factorization into pseudo-rotations except again $\mathbf{m}^{(sk)} = (m_1, m_2, \dots, m_r)$, where m_1 may equal 1 and it is not required that

$$\sum_{i=1}^{r} \theta_i \equiv 0 \bmod 2\pi.$$

In the case of U_m and U_{2n}/Sp_n the cohomology with integer coefficients is an exterior algebra with an added generator of degree 1; and for U_m/O_m the cohomology with $\mathbb{Z}/2\mathbb{Z}$ coefficients is an exterior algebra with an added generator of degree 1. Hence, a counting argument analogous to that for the Milnor fibers show that the closure of each Schubert class gives a homology generator for the complement.

Complements of the Varieties of Singular $m \times n$ Matrices.

The varieties of singular $m \times n$ complex matrices, $\mathcal{V}_{m,n}$, with $m \neq n$ were not considered earlier because they do not have Milnor fibers. However, the methods do apply to the complement and link as a result of work of J. H. C. Whitehead [W]. Let $M = M_{m,n}(\mathbb{C})$ denote the space of $m \times n$ complex matrices. We consider the case where $m > n$. The other case $m < n$ is equivalent by taking transposes. The left action of $GL_m(\mathbb{C})$ acts on M with an open orbit consisting of the matrices of rank n. This is the complement to the variety $\mathcal{V}_{m,n}$ of singular matrices and can be described as the ordered set of n independent vectors in $\mathbb{C}^m$. Then, the Gram-Schmidt procedure replaces them by an orthonormal set of n vectors in $\mathbb{C}^m$. This is the Stiefel variety $V_n(\mathbb{C}^m)$ and the Gram-Schmidt procedure provides a strong deformation retract of the complement $M\backslash\mathcal{V}_{m,n}$ onto the Stiefel variety $V_n(\mathbb{C}^m)$. Thus, the Stiefel variety is a compact model for the complement. Whitehead [W] computes both the (co)homology of the Stiefel variety using a Schubert decomposition which he gives. The cohomology for integer coefficients of the complement of the variety $\mathcal{V}_{m,n}$ is given by:

$$H^*(M_{m,n}\backslash\mathcal{V}_{m,n};\mathbb{Z}) \simeq \Lambda^*\mathbb{Z}\langle e_{2(m-n)+1}, e_{2(m-n)+3}, \dots, e_{2m-1}\rangle \tag{8.1}$$

with degree of e_j equal to j. Again the Schubert decomposition gives for the closure of each Schubert cell a homology generator.

Cohomology of the Links and Schubert Decomposition of the Complement.

Consider an exceptional orbit variety $\mathcal{E}$ of a prehomogeneous vector space V of $\dim_{\mathbb{C}}V = N$. Suppose there is a compact manifold $K \subset V\backslash\mathcal{E}$ oriented for a coefficients field $\mathbf{k}$, which is a compact model for the complement $V\backslash\mathcal{E}$. Then the cohomology of the link $L(\mathcal{E})$ is given, see [D3, Prop. 1.9], by the following formula

Cohomology of the Link $L(\mathcal{E})$:

$$\widetilde{H}^*(L(\mathcal{E});\mathbf{k}) \simeq \widetilde{H^*(K;\mathbf{k})}\,[2N-2-\dim_{\mathbb{R}}K]\ , \tag{8.2}$$

where the graded vector space $\widetilde{H^*(X;\mathbf{k})}\,[r]$ will denote the vector space $H^*(X;\mathbf{k})$, truncated at the top degree and shifted upward by degree r. Furthermore, to a basis of vector space generators of $H_q(K;\mathbf{k})$, $q < \dim_{\mathbb{R}}K$, there corresponds by Alexander duality a basis of vector space generators of $H^{2N-2-q}(K;\mathbf{k})$.

As a consequence of this and the preceding established relations between the Schubert decomposition (or product Schubert decomposition) of the complement and the homology, we obtain the following conclusions.

Theorem 8.1. *For the following exceptional orbit varieties $\mathcal{E}$ there are the following relations between the Schubert (or product Schubert) decomposition for a compact model of the complement and the cohomology of the link obtained by shifting the cohomology of the compact model (for coefficients a field of characteristic zero $\mathbf{k}$ unless otherwise stated).*

1) *For the equidimensional solvable case for (modified) Cholesky-type factorizations of $m\times m$ matrices of all three types or $(m+1)\times m$ matrices, the cohomology of the link is given by the shifted cohomology of the compact model torus, see* [D3, Thm 4.5]*. The closures of the cells of the product cell decomposition of nonmaximal dimension give a homology basis which correspond to the cohomology basis of the link after the shift.*
2) *For the discriminant of the quiver space for a quiver of finite type, the cohomology of the link is the shifted cohomology of the compact model described above with shift given by* [D3, Thm. 5.4]*. The closures of cells of the product Schubert decomposition of nonmaximal dimension for the complement give a homology basis which correspond after the shift to the cohomology basis for the link.*
3) *For the varieties of singular $m\times m$ complex matrices, in the general case or the skew-symmetric case with m even, the cohomology of the link is the shifted cohomology of the compact symmetric spaces U_m, resp. U_{2n}/Sp_n $(m = 2n)$ given above with shift given in* [D3, Table 2]*. The closures of the Schubert cells of nonmaximal dimension in each case give a homology basis which corresponds to the cohomology basis of the link after the shift.*
4) *For the varieties of singular $m\times m$ complex symmetric matrices, the shifted cohomology of $H^*(U_m/O_m;\mathbb{Z}/2\mathbb{Z})$, described above, gives the cohomology of the link for $\mathbb{Z}/2\mathbb{Z}$-coefficients, where the shift is $\binom{m+1}{2}-2$. The closures of the Schubert cells of nonmaximal*

dimension in the Schubert decomposition give a basis of $\mathbb{Z}/2\mathbb{Z}$*-homology classes corresponding to the cohomology basis of the link after the shift. For coefficients in a field* $\mathbf{k}$ *of characteristic zero, the cohomology of* U_m/O_m*, is an exterior algebra which depends on whether* m *is even or odd and the shifts are given in* [D3, Table 2]*, without a direct relation with the Schubert decomposition.*

5) *For the variety of singular* $m \times n$ *complex matrices,* $\mathcal{V}_{m,n}$ *(with* $m > n$*), the cohomology of the compact model, the Stiefel variety* $V_n(\mathbb{C}^m)$*, for the complement is given by* (8.1)*. The cohomology of the link is given in* (8.2) *as the upper truncated and cohomology* $H^*(M_{m,n} \backslash \mathcal{V}_{m,n}, \mathbf{k})$ *shifted by* $n^2 - 2$ *(as a graded vector space). The closures of the Schubert cells of nonmaximal dimension give a homology basis for the cohomology of the link after the shift.*

Complements of the Varieties of Matrix Singularities.

Given a matrix singularity $f_0 : \mathbb{C}^s, 0 \to M, 0$ with $\mathcal{V} \subset M$ the variety of singular matrices and $X_0 = f_0^{-1}(\mathcal{V})$. Here M can denote any of the spaces of matrices and of any sizes. In the preceding, we indicated how the cohomology of the link $L(\mathcal{V})$ is expressed as an upper truncated and shifted cohomology of the complement $M \backslash \mathcal{V}$. Because of the shift, we showed in [D3] that the cohomology product structure is essentially trivial. Thus, the link is a stratified real analytic set whose structure depends upon much more than just the group structure of the (co)homology. On the other hand, we showed in [D3] that the cohomology structure of the complement is an exterior algebra, and hence contributes considerably more that just the vector space structure of the cohomology of the link. This extra cohomology structure captures part of the additional structure.

Consequently, for the matrix singularity, using the earlier notation, we note that there is a map of complements $f_0 : (B_\varepsilon \backslash X_0) \to (B_\delta \backslash \mathcal{V})$. Also, $B_\delta \backslash \mathcal{V} \simeq M \backslash \mathcal{V}$, which has a compact model given by either a symmetric space or a Stiefel manifold. Thus, the cohomology of the complement $H^*(B_\varepsilon \backslash X_0; R)$ is a module over the characteristic subalgebra which is the image of $H^*(B_\delta \backslash \mathcal{V}; R)$ under f_0^*. In turn, this is an exterior algebra. Hence, the multiplicative structure considerably adds to the group structure that would result from the link. This is just as for the Milnor fiber described earlier.

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