

A POLYNOMIAL INVARIANT FOR PLANE CURVE COMPLEMENTS: KRAMMER POLYNOMIALS

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ABSTRACT. We use the Krammer representation of the braid group in Libgober’s invariant and construct a new multivariate polynomial invariant for curve complements: Krammer polynomial. We show that the Krammer polynomial of an essential braid is equal to zero. We also compute the Krammer polynomials of some certain n -gonal curves.

1. INTRODUCTION

The study of the topology of algebraic curves has a long history. The main question which has been worked on is “If C is an algebraic curve in a complex projective plane P^2 , what is the fundamental group of $P^2 - C$?”.

The fundamental group of the complement of a projective plane curve (henceforth referred to as “the fundamental group of the curve”) can be studied in terms of a generic projection of the complement to P^1 and the braid monodromies around the singular fibers. The Zariski-van Kampen theorem gives a way to compute a presentation of the fundamental group of a plane curve from the braid monodromies [10].

In general, computing the fundamental group of a curve complement is not an easy task and it is also hard to distinguish two fundamental groups by their presentations. In the early 80’s, A. Libgober defined the Alexander polynomial as an invariant of the fundamental group [7]. He, for example, showed that in Zariski’s famous example, the sextic with 6 cusps has the Alexander polynomial $t^2 - t + 1$ if all cusps are one a conic and 1 if they are not.

In 1989, Libgober defined a more general polynomial invariant based on the braid monodromies using the representations of the braid group [8]. He also showed that this invariant coincides with the Alexander polynomial when the Burau representation of the braid group is used. He proposed to use other representations to get other polynomial invariants, possibly multivariate polynomials [9].

1.1. Main Results. In this paper, we use the Krammer representation of the braid group in Libgober’s invariant and construct a new multivariate polynomial invariant, *Krammer polynomials*. Our first result is about the Krammer polynomial corresponding to an essential braid:

Theorem 1. The Krammer polynomial of an essential braid $b \in B_n$ is equal to zero.

We also study the Krammer polynomials of the n -gonal curves. An n -gonal curve is an algebraic curve equipped with a pencil of degree n (see Section 2.1 for more information). Using Theorem 1, we compute the local Krammer polynomials of the n -gonal curves around a special type singular fiber:

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Theorem 2. Let C be a completely reducible n -gonal curve and F be its singular fiber where only m components intersect with $m < n$. Then the local Krammer polynomial for the monodromy around F is equal to zero.

We show that the Krammer polynomial of an n -gonal curve is not always zero:

Theorem 3. The Krammer polynomial $\mathfrak{k}(t, q)$ of a completely reducible n -gonal curve C that has one singular fiber only is given by

$$(t^{2d}q^{6d} - 1)^{\binom{n}{2}}$$

where d is maximum degree of the irreducible components of C .

Organization of the paper. The paper is structured as follows: In Section 2, we briefly define the n -gonal curves and the braid monodromy of the n -gonal curves. In Section 3, we define the Krammer representation of the braid group and Libgober invariant. We introduce the Krammer polynomials of curve complements in this section. In Section 4, we present some results on the Krammer polynomial of essential braids. In Section 5, we study the Krammer polynomials of the n -gonal curves and prove our last two main results. We conclude our work in Section 6.

2. PRELIMINARIES

In this chapter, we define the n -gonal curves and the braid monodromy of the n -gonal curves which is the important tool for computing the invariants of curve complements.

2.1. The n -gonal curves . Let $\Sigma = P^1 \times P^1$ and let $p : \Sigma \rightarrow P^1$ be a projection of Σ to one of its components. Let E be a section of p and for each b in P^1 , let F_b be the fiber over b .

Definition 1. An n -gonal curve is a curve $C \subset \Sigma$ not containing E or a fiber of Σ as a component such that the restriction $p : C \rightarrow P^1$ is a map of degree n , i.e., each fiber intersects with C in at most n points. In the affine part, C is defined by $F(x, y) = 0$ with $F(x, y) \in \mathbb{C}[x, y]$ where $\deg_y F = n$.

As we understand from previous definition, fibers do not have to intersect with $C \cup E$ at $n + 1$ points. A *singular fiber* of an n -gonal curve $C \in \Sigma$ is a fiber F of Σ intersecting $C \cup E$ geometrically fewer than $n + 1$ points.

In this paper, we sometimes narrow our studies for a special subset of the n -gonal curves.

Definition 2. An n -gonal curve C is *completely reducible* if it is defined by

$$F(x, y) = (y - y_1(x)) \cdots (y - y_n(x)) = 0,$$

where $y_i \in \mathbb{C}[x]$ for all $i \in \{1, \dots, n\}$ in the affine part.

2.2. The Braid Monodromy of the n -gonal Curves. Let C be an n -gonal curve. Let F_1, F_2, \dots, F_r be the singular fibers of C and E be the distinguished section. Pick a nonsingular fiber F and let $F^\sharp = F \setminus (C \cup E)$. Clearly, F^\sharp is equal to $F \setminus E$ with n punctures, i.e., it is isomorphic to n -punctured complex disk D_n . Let $B^\sharp = P^1 \setminus \{p_1, p_2, \dots, p_r\}$ where p_i is the image under the ruling of the corresponding singular fiber F_i .

We know that $\pi_1(F^\sharp) = \langle \alpha_1, \dots, \alpha_n \rangle$ where α_i is the loop which covers i -th intersection of the fiber F^\sharp and the n -gonal curve C and $\pi_1(B^\sharp) = \langle \gamma_1, \dots, \gamma_r \rangle$ where γ_j is the loop which covers p_j . For each $j = 1, \dots, r$, dragging the fiber F along γ_j and keeping the base point results in a certain automorphism $\mathfrak{m}_j : \pi_1(F^\sharp) \rightarrow \pi_1(F^\sharp)$, which is called *the local braid monodromy of γ_j* . The set of all local braid monodromies $\{\mathfrak{m}_1, \dots, \mathfrak{m}_r\}$ is called the *global braid monodromy*.

3. THE KRAMMER POLYNOMIAL AS A TOPOLOGICAL INVARIANT

In this section, we will first present the matrices for the Krammer representation and then construct the Krammer polynomial. (For more information about the representation, see Appendix A.)

3.1. Matrices for the Krammer representation. The Krammer representation $K(t, q)$ is a representation of the braid group B_n in $GL_m(\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]) = \text{Aut}(F_m)$ where $m = \binom{n}{2}$ and F_m is the free module of rank m over $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ [5]. The representation can be formulated as follows:

$$K(\sigma_k)(e_{i,j}) = \begin{cases} tq^2 e_{k,k+1} & i = k, j = k + 1; \\ (1 - q)e_{i,k} + qe_{i,k+1} & j = k, i < k; \\ e_{i,k} + tq^{k-i+1}(q-1)e_{k,k+1} & j = k + 1, i < k; \\ tq(q-1)e_{k,k+1} + qe_{k+1,j}, & i = k, k + 1 < j; \\ e_{k,j} + (1 - q)e_{k+1,j} & i = k + 1, k + 1 < j; \\ e_{i,j}, & i < j < k \text{ or } k + 1 < i < j; \\ e_{i,j} + tq^{k-i}(q-1)^2 e_{k,k+1} & i < k < k + 1 < j \end{cases}$$

where $\{e_{i,j}\}_{1 \leq i < j \leq n}$ is the free basis of F_m .

For example, for the braid group B_3 , here is the matrix representation of its Artin generators:

$$K(\sigma_1) = \begin{pmatrix} tq^2 & 0 & 0 \\ tq(q-1) & 0 & q \\ 0 & 1 & 1-q \end{pmatrix}$$

$$K(\sigma_2) = \begin{pmatrix} 1-q & q & 0 \\ 1 & 0 & tq^2(q-1) \\ 0 & 0 & tq^2 \end{pmatrix}$$

3.2. The Krammer Polynomial. In [8], Libgober defined a polynomial invariant employing the representations of the braid group as follows: Let C be an algebraic curve and $\{p_1, \dots, p_r\}$ be the set of its singularities. Let ρ be a d dimensional linear representation of the braid group B_n over the ring A of Laurent polynomials $\mathbb{Q}[t_1^{\pm 1}, \dots, t_k^{\pm 1}]$ for $k \in \mathbb{N}$.

Definition 3. The *Libgober invariant*, $P(C, \rho)$, is the greatest common divisor of the order d minors in the $N \times d$ matrix of the map $\bigoplus(\rho(\mathbf{m}(\gamma_i)) - \text{Id})$, where $N = rd$, γ_i is the loop encloses p_i and $\mathbf{m}(\gamma_i)$ is the braid monodromy of the loop γ_i . We call this matrix the *Libgober matrix*.

Now, we can define the Krammer polynomial.

Definition 4. If we take ρ as the Krammer representation K of the braid group in the Libgober's invariant, then $P(C, K)$ gives the **Krammer polynomial** $\mathfrak{k}_C(t, q)$ of C .

Here is an example for a Krammer polynomial of a trigonal curve.

Example 1. Let $C : (y - x^3)(y + x^3)(y - 4x)$ be a trigonal curve. It has the singular fiber $x = 0$. Using the algorithm in [1], we find the corresponding braid monodromy as follows:

$$\sigma_1 \sigma_2 \sigma_1 (\sigma_2)^4 \sigma_1 \sigma_2 \sigma_1$$

The corresponding Libgober matrix L_C is

$$\begin{pmatrix} t^6 q^{14} - 1 & 0 & 0 \\ t^3 q^8 (q-1)(t^3 q^5 + tq^2 - q + 1) & t^2 q^9 - t^2 q^8 + t^2 q^7 - 1 & -t^2 q^7 \frac{q^4 - 1}{q+1} \\ t^3 q^7 (q-1)(t^2 q^4 - tq^3 + tq^2 + q^2 - q + 1) & t^2 q^6 (q-1)(q^2 + 1) & t^2 q^6 \frac{q^5 + 1}{q+1} - 1 \end{pmatrix}$$

The Krammer polynomial is given by the greatest common divisor of the order 3 minors in L_C . Since we just compute the local Krammer polynomial around the singular fiber $x = 0$, it is equal to the determinant of L_C which gives

$$\mathfrak{k}_C(t, q) = (t^6 q^{14} - 1)(t^2 q^{10} - 1)(t^2 q^6 - 1).$$

4. KRAMMER POLYNOMIAL OF ESSENTIAL BRAIDS

In this section, we present some results on the Krammer polynomials of essential braids. Our main result here is that the Krammer polynomial of an essential braid is zero.

We first define what an essential braid is.

Definition 5. A braid element b in the braid group B_n is an *essential braid* if it does not have at least one of the generators in it. For example, $b = \sigma_1 \sigma_2 \sigma_4 \in B_5$ is essential since σ_3 is not in b .

Now, we introduce some useful observations about the Krammer representations of essential braids. Here σ_i^n denotes the Artin generator σ_i of the braid group B_n for $1 \leq i \leq n-1$.

Observation 1. For $2 \leq i \leq n-1$,

$$K(\sigma_i^n) = \left(\begin{array}{c|c} \beta_i^{n-1} & \gamma_i^{n-1} \\ \hline 0 & K(\sigma_{i-1}^{n-1}) \end{array} \right)$$

where γ_i^{n-1} is an $(n-1) \times \binom{n-1}{2}$ matrix with all zero entries but last $n-i$ entries of the \hat{k} th column, with $\hat{k} = 1 + \sum_{j=3}^i (n-j+1)$, are

$$\begin{pmatrix} tq^i(q-1) \\ tq^{i-1}(q-1)^2 \\ \vdots \\ tq^{i-1}(q-1)^2 \end{pmatrix}$$

and

$$\beta_i^{n-1} = \left(\begin{array}{c|cc|c} I_{i-2} & 0 & 0 & 0 \\ \hline 0 & 1-q & q & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & I_{n-i-1} \end{array} \right).$$

Example 2. Krammer representation of $\sigma_2^4 \in B_4$ is

$$K(\sigma_2^4) = \left(\begin{array}{ccc|ccc} 1-q & q & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & tq^2(q-1) & 0 & 0 \\ 0 & 0 & 1 & tq(q-1)^2 & 0 & 0 \\ \hline 0 & 0 & 0 & tq^2 & 0 & 0 \\ 0 & 0 & 0 & tq(q-1) & 0 & q \\ 0 & 0 & 0 & 0 & 1 & 1-q \end{array} \right).$$

Now we have a proposition about β_i^{n-1} of $K(\sigma_i^n)$ in Observation 1:

Proposition 1. Let β_i^{n-1} of $K(\sigma_i^n)$ be defined as in Observation 1 and let $\beta = \prod_{k=1}^r \beta_{i_k}^{n-1}$ with $i_k \in \{2, \dots, n-1\}$ for all k . Then $\det(\beta - I_{n-1}) = 0$.

Proof. In general, the equation $\det(\beta - \lambda I) = 0$ is equivalent to the statement that λ is an eigenvalue of β . In our case $\lambda = 1$. To prove $\lambda = 1$ is an eigenvalue, we only need to show that there exists a vector v such that $\beta v = v$. Take v be the all ones vectors, i.e., the vector whose entries are all equal to 1. Then it is clear that $\beta_i v = v$ for all i . Hence, $\det(\beta - I_{n-1}) = 0$. \square

TABLE 1. The non-trivial columns in the Krammer representation of the Artin generators of B^6

σ_1	σ_2	σ_3	σ_4	σ_5
tq^2	0	0	0	0
$tq(q-1)$	$tq^2(q-1)$	0	0	0
$tq(q-1)$	$tq(q-1)^2$	$tq^3(q-1)$	0	0
$tq(q-1)$	$tq(q-1)^2$	$tq^2(q-1)^2$	$tq^4(q-1)$	0
$tq(q-1)$	$tq(q-1)^2$	$tq^2(q-1)^2$	$tq^3(q-1)^2$	$tq^5(q-1)$
0	tq^2	0	0	0
0	$tq(q-1)$	$tq^2(q-1)$	0	0
0	$tq(q-1)$	$tq(q-1)^2$	$tq^3(q-1)$	0
0	$tq(q-1)$	$tq(q-1)^2$	$tq^2(q-1)^2$	$tq^4(q-1)$
0	0	tq^2	0	0
0	0	$tq(q-1)$	$tq^2(q-1)$	0
0	0	$tq(q-1)$	$tq(q-1)^2$	$tq^3(q-1)$
0	0	0	tq^2	0
0	0	0	$tq(q-1)$	$tq^2(q-1)$
0	0	0	0	tq^2

Observation 2. For $2 \leq i \leq n-1$, the non-zero columns of γ_i^{n-1} in Observation 1 add up into the \hat{k} th column of $K(\sigma_i)$ with $\hat{k} = 1 + \sum_{j=1}^{k-1} (n-j)$. This column is called the *non-trivial* column of $K(\sigma_i)$.

Example 3. In B_6 , the non-trivial columns in $K(\sigma_i)$ for $i \in \{1, 2, 3, 4, 5\}$ are given in Table 1.

Observation 3. For $1 \leq i \leq n-2$,

$$K(\sigma_i^n) = E \left(\begin{array}{c|c} \alpha_i^{n-1} & \eta_i^{n-1} \\ \hline 0 & K(\sigma_i^{n-1}) \end{array} \right) E$$

where η_i^{n-1} is an $(n-1) \times \binom{n-1}{2}$ matrix, E is a fixed elementary unitary matrix that switches $[\sum_{i=1}^k (n-i)]'$ th row and column with k th row and column respectively for all $k \in \{1, \dots, n-1\}$ and

$$\alpha_i^{n-1} = \left(\begin{array}{c|cc|c} I_{i-1} & 0 & 0 & 0 \\ \hline 0 & 0 & q & 0 \\ 0 & 1 & 1-q & 0 \\ \hline 0 & 0 & 0 & I_{n-i-2} \end{array} \right).$$

Example 4. As in Example 2, we work on Krammer representation of $\sigma_2^4 \in B_4$. We first color the rows and columns to be shifted as

$$K(\sigma_2^4) = \begin{pmatrix} 1-q & q & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & tq^2(q-1) & 0 & 0 \\ 0 & 0 & 1 & tq(q-1)^2 & 0 & 0 \\ 0 & 0 & 0 & tq^2 & 0 & 0 \\ 0 & 0 & 0 & tq(q-1) & 0 & q \\ 0 & 0 & 0 & 0 & 1 & 1-q \end{pmatrix}.$$

In the matrix above, the uncolored parts actually gives $K(\sigma_2^3)$, hence we just transform these parts to the lower right corner by shifting rows and columns, i.e., we obtain

$$K(\sigma_2^4) = E \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & tq(q-1)^2 \\ 0 & 0 & q & 0 & 0 & tq(q-1) \\ 0 & 1 & 1-q & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1-q & q & 0 \\ 0 & 0 & 0 & 1 & 0 & tq^2(q-1) \\ 0 & 0 & 0 & 0 & 0 & tq^2 \end{array} \right) E.$$

where E is the corresponding elementary matrix.

Proposition 2. Let α_i^{n-1} of $K(\sigma_i^n)$ be defined as in Observation 3 and let $\alpha = \prod_{k=1}^r \alpha_{i_k}^{n-1}$ with $i_k \in \{2, \dots, n-1\}$ for all k . Then $\det(\alpha - I_{n-1}) = 0$.

Proof. Similar to the Proposition 1, to prove that $\det(\alpha - I_{n-1}) = 0$, it suffices to show that there exists a vector v such that $\alpha v = v$ or $\alpha^t v = v$. Let v be the all ones vectors. Then it is clear that $\alpha_i^t v = v$ for all i . Hence, $\det(\alpha^t - I_{n-1}) = \det(\alpha - I_{n-1}) = 0$. \square

Observation 4. For $1 \leq i \leq n-2$,

$$K(\sigma_i^n) = \left(\begin{array}{c|c|c|c} \diamond & 0 & 0 & 0 \\ * & 0 & qI_{n-i-1} & 0 \\ \hline 0 & I_{n-i-1} & (1-q)I_{n-i-1} & 0 \\ \hline 0 & 0 & 0 & I_{n'} \end{array} \right)$$

where \diamond is an $n_1 \times n_1$ matrix with $n_1 = 1 + \sum_{j=1}^{i-1} (n-j)$, $*$ is an $(n-i-1) \times n_1$ matrix and $n' \geq 0$.

Example 5. *Krammer representation of $\sigma_1^4 \in B_4$ is*

$$K(\sigma_1^4) = \left(\begin{array}{c|cc|cc|c} tq^2 & 0 & 0 & 0 & 0 & 0 \\ tq(q-1) & 0 & 0 & q & 0 & 0 \\ tq(q-1) & 0 & 0 & 0 & q & 0 \\ \hline 0 & 1 & 0 & 1-q & 0 & 0 \\ 0 & 0 & 1 & 0 & 1-q & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array} \right).$$

Now, we prove our main result in this section.

Theorem 1. The Krammer polynomial of an essential braid $b \in B_n$ is equal to zero.

Proof. We will divide the proof into three cases.

- (1) b does not have the generator σ_1 .
- (2) b does not have the generator σ_{n-1} .
- (3) b does not have the generator σ_i for $1 < i < n-1$.

We prove each case separately.

Case (1): Assume that b does not have the generator σ_1 , then $b \in \langle \sigma_2^n, \dots, \sigma_{n-1}^n \rangle \subset B_n$. Let $b = \sigma_{i_1}^n \dots \sigma_{i_r}^n$ be an essential braid in B_n where $2 \leq i_k \leq n-1$ for all $k \in \{1, \dots, r\}$. From Observation 1,

$$\begin{aligned}
K(b) &= K(\sigma_{i_1}^n \cdots \sigma_{i_r}^n) = K(\sigma_{i_1}^n) \cdots K(\sigma_{i_r}^n) \\
&= \left(\begin{array}{c|c} \beta_{i_1}^{n-1} & \gamma_{i_1}^{n-1} \\ \hline 0 & K(\sigma_{i_1-1}^{n-1}) \end{array} \right) \cdots \left(\begin{array}{c|c} \beta_{i_r}^{n-1} & \gamma_{i_r}^{n-1} \\ \hline 0 & K(\sigma_{i_r-1}^{n-1}) \end{array} \right) \\
&= \left(\begin{array}{c|c} \prod_{k=1}^r \beta_{i_k}^{n-1} & \gamma \\ \hline 0 & \prod_{k=1}^r K(\sigma_{i_k-1}^{n-1}) \end{array} \right).
\end{aligned}$$

Hence, the Krammer polynomial of b is

$$\begin{aligned}
\mathfrak{k}(b) &= \det(K(b) - I) = \det \left(\begin{array}{c|c} \prod_{k=1}^r \beta_{i_k}^{n-1} - I & \gamma \\ \hline 0 & \prod_{k=1}^r K(\sigma_{i_k-1}^{n-1}) - I \end{array} \right) \\
&= \det \left(\prod_{k=1}^r \beta_{i_k}^{n-1} - I \right) \det \left(\prod_{k=1}^r K(\sigma_{i_k-1}^{n-1}) - I \right) \\
&= 0
\end{aligned}$$

since $\det(\prod_{k=1}^r \beta_{i_k}^{n-1} - I) = 0$ from Proposition 1.

Case (2): Assume that b does not have the generator σ_{n-1} , then $b \in \langle \sigma_1^n, \dots, \sigma_{n-2}^n \rangle \subset B_n$. Let $b = \sigma_{i_1}^n \cdots \sigma_{i_r}^n$ be an essential braid in B_n where $1 \leq i_k \leq n-2$ for all $k \in \{1, \dots, r\}$. In Observation 3, since E is an elementary unitary matrix, we have

$$\begin{aligned}
K(b) &= K(\sigma_{i_1}^n \cdots \sigma_{i_r}^n) = K(\sigma_{i_1}^n) \cdots K(\sigma_{i_r}^n) \\
&= E \left(\begin{array}{c|c} \alpha_{i_1}^{n-1} & \eta_{i_1}^{n-1} \\ \hline 0 & K(\sigma_{i_1-1}^{n-1}) \end{array} \right) E \cdots E \left(\begin{array}{c|c} \alpha_{i_r}^{n-1} & \eta_{i_r}^{n-1} \\ \hline 0 & K(\sigma_{i_r-1}^{n-1}) \end{array} \right) E \\
&= E \left(\begin{array}{c|c} \prod_{k=1}^r \alpha_{i_k}^{n-1} & \eta \\ \hline 0 & \prod_{k=1}^r K(\sigma_{i_k-1}^{n-1}) \end{array} \right) E.
\end{aligned}$$

Hence, the Krammer polynomial of b is

$$\begin{aligned}
\mathfrak{k}(b) &= \det(K(b) - I) = \det \left(E \left(\begin{array}{c|c} \prod_{k=1}^r \alpha_{i_k}^{n-1} & \eta \\ \hline 0 & \prod_{k=1}^r K(\sigma_{i_k-1}^{n-1}) \end{array} \right) E - I \right) \\
&= \det(E) \det \left(\left(\begin{array}{c|c} \prod_{k=1}^r \alpha_{i_k}^{n-1} & \eta \\ \hline 0 & \prod_{k=1}^r K(\sigma_{i_k-1}^{n-1}) \end{array} \right) - I \right) \det(E) \\
&= \det \left(\left(\begin{array}{c|c} \prod_{k=1}^r \alpha_{i_k}^{n-1} & \eta \\ \hline 0 & \prod_{k=1}^r K(\sigma_{i_k-1}^{n-1}) \end{array} \right) - I \right) \\
&= \det \left(\prod_{k=1}^r \alpha_{i_k}^{n-1} - I \right) \det \left(\prod_{k=1}^r K(\sigma_{i_k-1}^{n-1}) - I \right) \\
&= 0
\end{aligned}$$

since $\det(\prod_{k=1}^r \alpha_{i_k}^{n-1} - I) = 0$ from Proposition 2.

Case (3): Assume that b does not have the generator σ_i for $1 < i < n-1$, i.e.,

$$b \in \langle \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_{n-1} \rangle \subset B_n.$$

Again, similar to the proof of Proposition 1 and 2, we only need to show that there exists a vector v such that $vK(b) = v$ (i.e., v is a left-eigenvector with size $m = \binom{n}{2}$). Hence, it is enough

to show that there exist a vector v such that $vK(\sigma_k) = v$ for all $k \in \{1, \dots, i-1, i+1, \dots, n-1\}$. Let $v = [v_1, \dots, v_m]$ be an m -dimensional vector with $v_j \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ for $1 \leq j \leq m$. Let us further partition v as

$$v = [v_1, \dots, v_m] = [\delta_1, \dots, \delta_{n-1}]$$

where δ_k is a sub-vector of v with $n-k$ elements for $1 \leq k \leq n-1$. This partitioning reveals the relationship among the elements of v and makes its structure more clear. Moreover, we use the notation δ_k^j for the j th element in δ_k .

From the identity block matrix of $K(\sigma_k)$ in Observation 4, we notice that some elements of v are equal to each other. That is,

$$(1) \quad \delta_k^j = \delta_{k+1}^{j-1}$$

for $k \in \{1, \dots, i-1, i+1, \dots, n-1\}$ and $j \in \{2, \dots, n-1\}$. Moreover, from the block matrix $\begin{pmatrix} 1-q & q \\ 1 & 0 \end{pmatrix}$ in Observation 1, we get

$$(2) \quad \delta_k^{j+1} = q\delta_k^j$$

for all $k \in \{1, \dots, n-2\}$ and $j \leq n-k$.

Combining the relations (1) and (2), we obtain 3 different cases for δ_k ,

$$(3) \quad \delta_k = \begin{cases} [xq^{k-1}, \dots, xq^{i-2}, 1, q, \dots, q^{n-i-1}] & k < i; \\ [1, q, q^2, q^3, \dots, q^{n-i-1}] & k = i; \\ [yq^{k-i-1}, \dots, yq^{n-i-2}] & k > i \end{cases}$$

where $x, y \in \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$. Although $\delta_k^{i-k+1} = \delta_i^1$ for $k < i$ can be any non-zero integer, 1 is chosen for simplicity.

Example 6. For $n = 6$ and $i = 3$ (i.e., the braid is in B_6 and does not have the generator σ_3), the vector v is given by

$$v = [\underbrace{x, xq, 1, q, q^2}_{\delta_1}, \underbrace{xq, 1, q, q^2}_{\delta_2}, \underbrace{1, q, q^2}_{\delta_3}, \underbrace{y, yq}_{\delta_4}, \underbrace{yq}_{\delta_5}].$$

Furthermore, the non-trivial column of $K(\sigma_k)$ also provides extra relation, call the relation as f_k , that can be employed to find x and y . Indeed, there exist $i-1$ relations with entries in $[\delta_1, \dots, \delta_{i-1}]$ that include x and $n-i-1$ linear relations with entries in $[\delta_{i+1}, \dots, \delta_{n-1}]$ that include y . More specifically, f_k has entries from the last $n-k$ elements of δ_k for all

$$k \in \{1, \dots, i-1, i+1, \dots, n-1\}.$$

For example, in B_6 , f_2 has entries $\{\delta_1^2, \dots, \delta_1^5, \delta_2^1, \dots, \delta_2^4\}$ where δ_k^j is the j -th element in δ_k .

Note that all other rows in $K(\sigma_k)$ give redundant relations, i.e., the relations (1), (2) and f_k for $k \in \{1, \dots, i-1, i+1, \dots, n-1\}$ are all relations that the elements of the eigenvector v need to satisfy.

Lemma 1. There exist unique solutions for x and y in (3) such that the relations f_k for

$$k \in \{1, \dots, i-1, i+1, \dots, n-1\}$$

are satisfied.

Proof. First, it is easy to check that f_k with $k < i$ gives the relations for x and with $k > i$ for y . For $k < i$, f_k gives

$$xtq^{i+k-1} - tq^{k-1}(1 - q^{n-i}) = xq^{k-1}.$$

This yields to the following solution:

$$x = \frac{tq^{k-1}(1-q^{n-i})}{q^{k-1}(tq^i-1)} = \frac{t(1-q^{n-i})}{tq^i-1}$$

which is independent from k , i.e., there exists a unique solution for x for all f_k with $k < i$.

Similarly, for $k > i$, f_k gives

$$ytq^{k-1} - tq^{k-1}(1-q^{n-i}) = yq^{k-i-1}.$$

Hence, we have the following solution for y :

$$y = \frac{tq^{k-1}(1-q^{n-i})}{q^{k-i-1}(tq^i-1)} = \frac{tq^{i+1}(1-q^{n-i})}{tq^i-1}.$$

which is again independent from k , i.e., there exists a unique solution of y for all f_k with $k > i$. \square

Now, we give an example for the construction in the lemma above.

Example 7. Let $n = 6$ and $i = 3$ again. The vector v and the non-trivial column for each generator is given by Table 2.

From the non-trivial column of $K(\sigma_1)$, we get the relation f_1 as

$$xtq^2 + xtq^2(q-1) + (1+q+q^2)tq(q-1) = x \Rightarrow xtq^3 - tq(1-q^3) = x \Rightarrow x(tq^3-1) = tq(1-q^3),$$

which implies

$$x = \frac{tq(1-q^3)}{tq^3-1}.$$

Similarly, the non-trivial column of $K(\sigma_2)$ gives the relation f_2 as

$$xtq^3(q-1) + (1+q+q^2)tq(q-1)^2 + xtq^3 + (1+q+q^2)tq(q-1) = xq,$$

which is simplified into

$$xq(tq^3-1) = tq^2(1-q^3).$$

We get the same relation when we cancel the q 's on both sides of the equation, i.e., f_2 gives the same solution for x .

Similarly, we obtain the unique solution for y using the non-trivial columns of $K(\sigma_4)$ and $K(\sigma_5)$.

From Lemma 1, we finished the construction of the vector v . To sum up,

$$v = [\delta_1, \dots, \delta_{n-1}]$$

where

$$\delta_k = \begin{cases} [xq^{k-1}, \dots, xq^{i-2}, 1, q, \dots, q^{n-i-1}] & k < i; \\ [1, q, q^2, q^3, \dots, q^{n-i-1}] & k = i; \\ [yq^{k-i-1}, \dots, yq^{n-i-2}] & k > i \end{cases}$$

with

$$x = \frac{t(1-q^{n-i})}{tq^i-1} \text{ and } y = \frac{tq^{i+1}(1-q^{n-i})}{tq^i-1}.$$

This completes the proof. \square

TABLE 2. The corresponding v vector of $b \in B_6$ where b does not include the Artin generator σ_3 and the non-trivial columns in the Krammer representation of the Artin generators for B_6

v		σ_1	σ_2	σ_3	σ_4	σ_5
δ_1	x	tq^2	0	0	0	0
	xq	$tq(q-1)$	$tq^2(q-1)$	0	0	0
	1	$tq(q-1)$	$tq(q-1)^2$	$tq^3(q-1)$	0	0
	q	$tq(q-1)$	$tq(q-1)^2$	$tq^2(q-1)^2$	$tq^4(q-1)$	0
	q^2	$tq(q-1)$	$tq(q-1)^2$	$tq^2(q-1)^2$	$tq^3(q-1)$	$tq^5(q-1)$
δ_2	xq	0	tq^2	0	0	0
	1	0	$tq(q-1)$	$tq^2(q-1)$	0	0
	q	0	$tq(q-1)$	$tq(q-1)^2$	$tq^3(q-1)$	0
	q^2	0	$tq(q-1)$	$tq(q-1)^2$	$tq^2(q-1)^2$	$tq^4(q-1)$
δ_3	1	0	0	tq^2	0	0
	q	0	0	$tq(q-1)$	$tq^2(q-1)$	0
	q^2	0	0	$tq(q-1)$	$tq(q-1)^2$	$tq^3(q-1)$
δ_4	y	0	0	0	tq^2	0
	yq	0	0	0	$tq(q-1)$	$tq^2(q-1)$
δ_5	yq	0	0	0	0	tq^2

5. KRAMMER POLYNOMIAL OF THE n -GONAL CURVES

In this section, we compute the Krammer polynomials of two sets of the n -gonal curves. We already mentioned these results in the introduction section. Here, we will prove these computations.

First, we compute the local Krammer polynomial around a special singular fiber.

Theorem 2. The local Krammer polynomial for the monodromy around a singular fiber F of an n -gonal curve where $m < n$ components intersect is equal to zero.

Proof. Since $m < n$ components intersect over F , $n - m$ strands of the corresponding braid monodromy is fixed. In other words, the braid monodromy gives an essential braid. Hence, from Theorem 1, we can deduce that the Krammer polynomial around the fiber F is equal to zero \square

As the second result, we compute the global Krammer polynomials of the completely reducible n -gonal curves that have one singular fiber only.

Theorem 3. The global Krammer polynomial of a completely reducible n -gonal curve C that has one singular fiber only is

$$(t^{2d}q^{6d} - 1)^{\binom{n}{2}}$$

where d is maximum degree of the irreducible components of C .

Proof. Since C has only one singular fiber, it is in the form

$$(y + a_1(x - p_1)^d + p_2) \dots (y + a_n(x - p_1)^d + p_2),$$

where $a_i \neq a_j \ \forall i \neq j$, $p_1, p_2 \in \mathbb{C}$ and $d \in \mathbb{Z}$. In other words, all the irreducible components of C intersects at $x = p_1$. In this case, there are d full twists around the singular fiber F_{p_1} . Hence, the local monodromy around this fiber is $(\sigma_1 \sigma_2)^d$. Then, we have the following $m \times m$ local Libgober matrix with $m = \binom{n}{2}$:

$$L_C = \begin{pmatrix} t^{2d}q^{6d} - 1 & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & t^{2d}q^{6d} - 1 \end{pmatrix}$$

Furthermore, this matrix is the global Libgober matrix since there is only one singular fiber. Thus, the Krammer polynomial $\mathfrak{k}_C(t, q)$ is equal to the greatest common divisor of the of the order m minors in L_C which is

$$\mathfrak{k}_C(t, q) = (t^{2d}q^{6d} - 1)^m.$$

□

6. CONCLUSION

In this paper, we introduce a new polynomial invariant using Krammer representation of braid group. We compute the Krammer polynomials of essential braids and some certain n -gonal curves. We also implement an algorithm to compute the Krammer polynomial of a curve having its braid monodromies as an input in Sagemath [4] (available from <http://www.math.uco.edu/pages/maktas/Research.htm>). As a future task, it would be very interesting to find out other curves that have non-trivial Krammer polynomials. Moreover, following the same idea, we plan to use different representations of the braid group to generate different polynomial invariants. One can use, for example, the Gassner Representation of the pure braid group since the braids in our case are also pure braids.

APPENDIX A. KRAMMER REPRESENTATION OF THE BRAID GROUP

One of the popular question in late 20th century was whether the Braid group is linear, i.e., it is isomorphic to a subgroup of $GL(n, K)$ for $n \in \mathbb{N}$ and some field K . It is easy to show that the Burau representation is faithful for $n \leq 3$. Bigelow showed that it is unfaithful for $n \geq 5$ [2] and it is still unknown whether the Burau representation is faithful or not for $n = 4$.

Another representation, introduced by Lawrence [6], is studied to prove that the Braid group is linear. Krammer showed that this representation is faithful for the Braid group B_4 [5]. Then finally Bigelow showed that it is faithful for all n [3].

The representation that is mentioned above is called *Krammer representation*. It is in $GL_m \mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ where $m = \frac{n(n-1)}{2}$ [5]. Here we will present the definition of the representation.

Let $D_n = D \setminus P$ where D is the complex disk and $P = \{p_1, \dots, p_n\}$ with $p_i \in D$. Let

$$L = D_n \times D_n \setminus \Delta,$$

where Δ denotes the diagonal of $D_n \times D_n$. Let K be the set of all unoredered pairs of distinct points in D_n , i.e., $K = L/S_2$. Let $\alpha : I \rightarrow K$ be a path in K based at $k_0 = \{p_0, q_0\}$ where p_0 and q_0 are points on the boundary of D_n . Since the projection $L \rightarrow K$ is a 2-fold covering, we can lift α to L , say α' , and it is in the form $\alpha' = (\alpha'_1, \alpha'_2)$ where $\alpha'_i : I \rightarrow D_n$ for $i \in \{1, 2\}$ and $\alpha'_1(s) \neq \alpha'_2(s)$ for any $s \in I$. If α is a loop, α'_1 and α'_2 are either both loops or composition of them results a loop.

Let $\alpha \in \pi_1(K)$ and define maps a and b from $\pi_1(K)$ to \mathbb{Z} as follows:

$$a(\alpha) = \begin{cases} w(\alpha'_1) + w(\alpha'_2), & \alpha'_1 \text{ and } \alpha'_2 \text{ are both closed loops} \\ w(\alpha'_1 \alpha'_2), & \text{otherwise} \end{cases}$$

where w denotes the winding number around the puncture points p_1, \dots, p_n and

$$b(\alpha) = \beta_2 \beta_1$$

where $\beta_1 : I \rightarrow S^1$ with

$$\beta_1(s) = \frac{\alpha'_1(s) - \alpha'_2(s)}{|\alpha'_1(s) - \alpha'_2(s)|}$$

and $\beta_2 : S^1 \rightarrow H_1(\mathbb{R}P^1) \cong \mathbb{Z}$ is the induced map of the projection $S^1 \rightarrow \mathbb{R}P^1$. In other words, a counts how many times α'_1 and α'_2 winds around the puncture points and b counts how many times they wind around each other. Now, define a map $\phi : \pi_1(K) \rightarrow \langle q, t \rangle$ by

$$\phi(\alpha) = q^{a(\alpha)} t^{-b(\alpha)}.$$

Let \tilde{K} be the regular covering of K corresponding to the ker ϕ . Let h be a homomorphism of D_n to itself. Clearly, this induces a homomorphism h_K of K to itself by

$$h_K(\{d_1, d_2\}) = \{h(d_1), h(d_2)\}$$

and this can be lifted to a homeomorphism $\tilde{h}_K : \tilde{K} \rightarrow \tilde{K}$. Since the group $\langle q, t \rangle$ acts on \tilde{K} as a group of covering transformations, the homology $H_2(\tilde{K})$ is a $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ -module. Hence, \tilde{h} commutes with the covering transformations q and t , and \tilde{h} induces a $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$ -module isomorphism $\tilde{h}^* : H_2(\tilde{K}) \rightarrow H_2(\tilde{K})$. The representation $\kappa : B_n \rightarrow \text{Aut}(H_2(\tilde{K}))$ where

$$\kappa(h) = \tilde{h}^*$$

is called the *Krammer representation* of B_n .

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