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A Special Session on *Singularities and Physics* was organized for the AMS Sectional Meeting held in Knoxville, TN on March 21–23, 2014. The session focused on the theory of singularities and its interactions with different branches of theoretical physics: singularities of elliptic fibrations in string theory, renormalization issues in quantum field theory, Landau-Ginzburg models, wall-crossing phenomena, and other recent points of contact. The aim was to bring together the mathematics and physics communities, to foster further interactions. The speakers and talks at the session were as follows:

- *Lara Anderson:* Geometric Constraints in Heterotic/F-theory.
- *Jacob Bourjaily:* Scattering Amplitudes and the Positive Grassmannian
- *Mirjam Cvetic:* Elliptic fibrations with higher rank Mordell-Weil Group: F-theory compactifications with higher rank Abelian Gauge Symmetry
- *Clay Cordova:* Deformations of superconformal field theories
- *Antonella Grassi:* Deformations and Resolutions.
- *Ralph Kaufmann:* Singularities, swallowtails and topological properties in families of Hamiltonians.
- *Anatoly Libgober:* Calabi-Yau threefolds from plane singular curves.
- *Matilde Marcolli:* Rota-Baxter algebras of singular hypersurfaces and applications to quantum field theory.
- *Laurentiu Maxim:* Intersection spaces, perverse sheaves and type IIB string theory.
- *Dave Morrison:* Canonical singularities and superconformal field theories
- *Richard Rimanyi:* R-matrices acting on the cohomology of flag varieties
- *Washington Taylor:* Classifying and enumerating elliptically fibered Calabi-Yau threefolds and associated singularities.

This volume collects articles written by some of the speakers on the occasion of this meeting, expanding on the content of their talks or reporting on research originated in discussions begun at the conference. We offer the contributors to this volume and the other participants our heartfelt thanks for their work and for the relaxed yet stimulating atmosphere permeating the special session.

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Mboyo Esole

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SPECTRAL COVERS, INTEGRALITY CONDITIONS, AND HETEROTIC/F-THEORY DUALITY

LARA B. ANDERSON

Abstract. In this work we review a systematic, algorithmic construction of dual heterotic/F-theory geometries corresponding to 4-dimensional, $\mathcal{N} = 1$ supersymmetric compactifications. We look in detail at an exotic class of well-defined Calabi-Yau fourfolds for which the standard formulation of the duality map appears to fail, leading to dual heterotic geometry which appears naively incompatible with the spectral cover construction of vector bundles. In the simplest class of examples the F-theory background consists of a generically singular elliptically fibered Calabi-Yau fourfold with E_7 symmetry. The vector bundles arising in the corresponding heterotic theory appear to violate an integrality condition of an $SU(2)$ spectral cover. A possible resolution of this puzzle is explored by studying the most general form of the integrality condition. This leads to the geometric challenge of determining the Picard group of surfaces of general type. We take an important first step in this direction by computing the Hodge numbers of an explicit spectral surface and bounding the Picard number.

1. An algorithm construction of dual heterotic/F-theory geometry

Compactifications of heterotic string theory and F-theory are believed to be dual – that is to lead to the same effective low energy physics – whenever the compactification geometries take the form [9, 10, 11, 12]

$$\text{Heterotic on } \pi_h : X_n \xrightarrow{\mathbb{E}} B_{n-1} \;\Leftrightarrow\; \text{F-theory on } \pi_f : Y_{n+1} \xrightarrow{K3} B_{n-1} \tag{1.1}$$

where the $K3$ fiber of Y_{n+1} is itself elliptically fibered over a $\mathbb{P}^1$ base. The compatibility of these two fibrations leads to the observation that $\rho_f : Y_{n+1} \xrightarrow{\mathbb{E}} \mathcal{B}_n$ and $\sigma_f : \mathcal{B}_n \xrightarrow{\mathbb{P}^1} B_{n-1}$. In recent work [5] this duality was used to systematically enumerate an interesting and finite class of string backgrounds and the properties of the associated 4-dimensional effective theories. As given in (1.1), the choice of geometry in F-theory consists simply of a $K3$-fibered Calabi-Yau fourfold. For the $E_8 \times E_8$ heterotic string theory the background is determined by an elliptically fibered Calabi-Yau threefold equipped with a pair of poly-stable, holomorphic vector bundles, V_i $(i = 1, 2)$ on X_3 with structure groups, $H_i \subset E_8$.

In [5] a program was set out to systematically study the general properties and constraints of the dual effective theories and develop a general and algorithmic formalism to build consistent heterotic/F-theory backgrounds. With this goal in mind, the first step in constructing a pair of the form (1.1) is the choice of a twofold base, B_2 appearing in both the heterotic and F-theory geometry. For all smooth threefolds, X_3, the possible choices for B_2 have been classified [27] (and B_2 must be a generalized del Pezzo surface). Furthermore, to explore and test general structure there is an important dataset of such manifolds consisting of $61,539$ toric surfaces systematically constructed by Morrison and Taylor [2, 1, 8].

Key words and phrases. Heterotic string compactification, F-theory, 4-dimensional $\mathcal{N} = 1$ string dualities, algebraic geometry, surfaces of general type, Picard number.

With these results in place it iso possible to begin to build the geometry of (1.1) from the bottom up. In the Calabi-Yau fourfold geometry the next step is to choose a form for the $\mathbb{P}^1$-fibration, $\sigma_f : \mathcal{B}_3 \xrightarrow{\mathbb{P}^1} B_2$. As described in Section 2, this can be accomplished for non-degenerate fibrations by building $\mathcal{B}_3$ as a $\mathbb{P}^1$ bundle over B_2, parameterized by a "twist": a $(1,1)$-form T in B_2 (see (2.1)). In the heterotic theory this choice of twist corresponds to a piece of the heterotic vector bundle topology (more specifically, a component of the second Chern class $c_2(V)$) [12]. In [5] we established that given a twofold base B_2, the set of all possible twists is in fact bounded by the conditions imposed by 4-dimensional $N = 1$ supersymmetry. In the heterotic theory this appears through the condition of slope stability of the vector bundles V_i and in F-theory by the condition that the generically singular fourfold Y_4 admits a smooth Calabi-Yau resolution. Finally it should be noted that since we require all fibrations to admit (exactly one) section, each elliptically fibered manifold is birationally equivalent to a Weierstrass model [49] (see (2.2)). Thus, having chosen B_2 and constructed a $\mathbb{P}^1$-bundle $\mathcal{B}_3$, we have fully specified X_3 and Y_4.

With consistency conditions in place and a scheme for algorithmically constructing pairs as in (1.1), it remains to extract patterns and structure from the effective theories. Duality here provides a powerful tool to determine otherwise difficult to calculate information on both sides of the theory. While historically heterotic/F-theory duality has been used to determine the effective physics of the mysterious and non-lagrangian F-theory, in [5] we also explored ways in which the singularity structure of the F-theory fourfold could be used to determine non-trivial information about $\mathcal{M}_\omega(c(V))$ – the moduli space of sheaves that are semi-stable with respect to the Kähler form ω with fixed total Chern class $c(V)$. Such information is hard won, since very few techniques exist to determine $\mathcal{M}_\omega(c(V))$ for sheaves/bundles over Calabi-Yau threefolds (or their associated higher-rank Donaldson-Thomas invariants).

As one simple illustration of this correspondence, we note here that the presence of generic symmetries on singular Calabi-Yau fourfolds make it possible to derive correlations between the topology of a slope-stable heterotic vector bundle on a CY threefold and its structure group. Initial investigations of this nature were first undertaken in [15, 14] who constructed "lower bounds" on the second Chern class of a vector bundle with fixed structure group. In [5], we continue to explore the links between structure group and topology, exploring not only these lower bounds but also upper bounds as well (see Section 6 of [5]).

Structure Group, H	Topology	Structure Group, H	Topology
$SU(N)$	$\eta \geq N \cdot c_1(B_2)$	E_8	$\eta \geq 5 \cdot c_1(B_2)$
$SO(7)$	$\eta \geq 4 \cdot c_1(B_2)$	E_7	$\eta \geq \frac{14}{3} \cdot c_1(B_2)$
$SO(M)$	$\eta \geq \frac{M}{2} \cdot c_1(B_2)$	E_6	$\eta \geq \frac{9}{2} \cdot c_1(B_2)$
$Sp(K)$	$\eta \geq 2K \cdot c_1(B_2)$	G_2	$\eta \geq \frac{7}{2} \cdot c_1(B_2)$
		F_4	$\eta \geq \frac{7}{2} \cdot c_1(B_2)$

TABLE 1. *Constraints linking the topology, $\eta = c_2(V)|_{B_2}$, of an H-bundle V and its structure group on an elliptically fibered CY threefold, $\pi_h : X_3 \to B_2$.* [15, 14].

Systematic patterns such as those shown in Table 1 are of use in string phenomenology (for example they could simplify recent algorithmic searches for heterotic Standard Models carried such as those carried out in [20, 21, 22, 18, 17, 4]). In order to fully understand such patterns though, it is necessary to complete the geometric "dictionary" which matches heterotic/F-theory geometry. This includes the inclusion of G-flux in the F-theory background and an understanding of the zero-locus of the induced Gukov-Vafa-Witten superpotential [48]. In this context the quantization conditions on flux and the corresponding constraints in the heterotic theory become particularly important. Indeed, as described in detail [5], in our systematic search, we find many geometries which appear mysterious from the point of view of these commonly assumed integrality conditions.

In the following sections we will review the standard construction of heterotic/F-theory dual pairs. In its most explicit form, the duality map dependences on a particular method of constructing Mumford poly-stable vector bundles – namely, the *spectral cover construction* [12]. In recent work [5, 6] it has been observed that many apparently consistent F-theory fourfolds have topology which appears to be inconsistent with a naive construction of spectral cover bundles. We will explore this discrepancy further in concrete examples in the following Sections.

Out of the dataset generated in [5], we consider one of the simplest examples of such an exotic heterotic/F-theory dual pair. In particular, we explore the so-called "integrality" condition on the spectral data (see (2.16)) and set out to determine whether it is really correct/necessary as frequently implemented in the literature. In addition, we lay out the necessary geometric questions that must be addressed if this criterion is to be refined or improved. We will argue that in many cases the surface forming the $SU(2)$ spectral cover can have a larger Picard group than is generically assumed and that the heterotic bundle can in fact be described by a consistent spectral cover pair $(\mathcal{S}, \mathcal{L}_S)$, consisting of a 2-sheeted cover $\pi_{\mathcal{S}} : \mathcal{S} \to B_2$ and a line bundle over it $\mathcal{L}_S$ over it. We begin with a brief review of heterotic/F-theory duality in 4-dimensions to set the stage for these investigations.

2. Heterotic/F-theory Duality in 4-dimensions

In this section we will provide a rough outline of the geometric correspondence that arises in heterotic and F-theory dual pairs. Many excellent reviews exist in the literature and we refer the reader to classic sources such as [12, 16] and modern summaries such as [5] for a more complete treatment. In recent work, [5] a constructive algorithm was developed to consistently build and enumerate dual heterotic/F-theory geometries. As a tractable starting point for that work, heterotic backgrounds were considered consisting of a smooth elliptically fibered Calabi-Yau threefold X_3 (with a single section[1]) over a base B_2, together with two holomorphic, Mumford poly-stable vector bundles [7]. In such cases, the dual F-theory compactification geometry can be built beginning with a rationally fibered threefold base $\mathcal{B}_3$ that is a $\mathbb{P}^1$ bundle over B_2 (the same surface used to define the heterotic Calabi-Yau threefold). The F-theory compactification space is then an elliptically and K3-fibered fourfold, $\rho_f : Y_4 \to \mathcal{B}_3$. Following [12], without loss of generality, the non-degenerate $\mathbb{P}^1$-fibered base ($\mathcal{B}_3$) can be defined as a $\mathbb{P}^1$ bundle through the projectivization of a sum of two line bundles

$$\mathcal{B}_3 = \mathbb{P}(\mathcal{O} \oplus \mathcal{L})\,, \tag{2.1}$$

where $\mathcal{L}$ is a general line bundle on the base B_2. Over $\mathcal{B}_3$, the classes $R = c_1(\mathcal{O}(1)), T = c_1(\mathcal{L})$, can be defined, where $\mathcal{O}(1)$ is a bundle that restricts to the usual $\mathcal{O}(1)$ on each $\mathbb{P}^1$ fiber. The $\mathbb{P}^1$ fibration is equipped with sections Σ_- and $\Sigma_+ = \Sigma_- + T$ of $\mathcal{B}_3$ satisfying $\Sigma_- \cdot \Sigma_+ = 0$, corresponding to the relation $R(R+T) = 0$ in cohomology.

Finally, then the fourfold itself can be described in Weierstrass form as

$$y^2 = x^3 + fx + g \tag{2.2}$$

where y, x are (affine) coordinates along the elliptic fiber and $f \in H^0(\mathcal{B}_3, K_3^{-4})$, $g \in H^0(\mathcal{B}_3, K_3^{-6})$. As usual the position of singular fibers is encoded in the discriminant locus, $\Delta = 4f^3 + 27g^2$.

For this choice of an F-theory model on Y_4 and a heterotic theory on X_3, it is now possible to begin by matching topology [12, 16]. Starting with the $E_8 \times E_8$ heterotic theory, the bundle

[1] For geometries without section and some of the physics of these more general genus-1 fibrations see recent progress in [45, 46, 47].

decomposes as $V_1 \oplus V_2$, and without loss of generality, the curvatures split as

$$\frac{1}{30}\text{Tr } F_i^2 = \eta_i \wedge \omega_0 + \zeta_i\,, \qquad i = 1,2 \tag{2.3}$$

where η_i, ζ_i are (pullbacks of) 2-forms and 4-forms on B_2 and ω_0 is Poincaré dual to the zero-section of the elliptic fibration. The heterotic Bianchi identity [7] gives $\eta_1 + \eta_2 = 12c_1(B_2)$. Thus, it is possible to parameterize a solution as

$$\eta_{1,2} = 6c_1(B_2) \pm T'\,, \qquad (E_8 \times E_8) \tag{2.4}$$

where T' is a $\{1,1\}$ form on B_2. Next, returning to the F-theory geometry described above in (2.1), the canonical class of $\mathcal{B}_3$ is determined by adjunction to be

$$-K_3 = 2\Sigma_- - K_2 + T\,, \tag{2.5}$$

By studying the 4-dimensional effective theories of these dual heterotic/F-theory compactifications it is straightforward to determine that the defining $\{1,1\}$ forms T, T' in B_2 are in fact identical: $T = T'$ [12, 3]. The $\{1,1\}$-form T is referred to as the "twist" (of the $\mathbb{P}^1$-fibration) and is the crucial defining data of the simplest classes of heterotic/F-theory dual pairs.

2.1. **The spectral cover construction.** To explicitly match the degrees of freedom – including the geometric moduli – of a heterotic/F-theory dual pair, it is necessary to modify our description of the slope-stable holomorphic vector bundles arising as part of the heterotic background. A powerful tool to this end is the description of vector bundles known as the "spectral cover construction[2]" [12, 24, 25, 26]. In the simplest cases it is possible to form a $1-1$, onto map from a suitable[3] slope-stable, holomorphic, rank N vector bundle $\pi : V \to X_3$ to a pair $(\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ (referred to as the "spectral data") where $\mathcal{S}$ is a smooth divisor in X_3 (forming an N-fold cover of the base B_2 and referred to as the "spectral cover") and $\mathcal{L}_{\mathcal{S}}$ is a line bundle[4] over $\mathcal{S}$.

The spectral cover construction has been used extensively in heterotic theories to construct rank N bundles with structure group $SU(N)$ or $Sp(2N)$ that are slope-stable in some region of Kähler moduli space. As shown in [12], the class of the divisor $\mathcal{S}$ is given by

$$[\mathcal{S}] = N[\sigma] + \pi^*(\eta) \tag{2.6}$$

where σ is the zero section of $\pi : X_3 \to B_2$ and η is defined as in (2.3) and (2.4).

It is helpful to once again describe the elliptically fibered heterotic threefold in Weierstrass form:

$$\hat{Y}^2 = \hat{X}^2 + f(u)\hat{X}\hat{Z}^4 + g(u)\hat{Z}^6 \tag{2.7}$$

where $\{\hat{X}, \hat{Y}, \hat{Z}\}$ are coordinates on the elliptic fiber (described as a degree six hypersurface in $\mathbb{P}_{123}$) and $\{u\}$ are coordinates on the base B_2. Here $\hat{Z} = 0$ defines the section σ. For $SU(N)$ bundles, the *spectral cover*, $\mathcal{S}$, can be represented as the zero set of the polynomial

$$s = a_0\hat{Z}^N + a_2\hat{X}\hat{Z}^{N-2} + a_3\hat{Y}\hat{Z}^{N-3} + \dots \tag{2.8}$$

ending in $a_N\hat{X}^{\frac{N}{2}}$ for N even and $a_N\hat{X}^{\frac{N-3}{2}}\hat{Y}$ for N odd [12]. The polynomials a_i are sections of line bundles over the base B_2

$$a_i \in H^0(B_2, K_{B_2}^{\otimes i} \otimes \mathcal{O}(\eta))\,, \tag{2.9}$$

In order for the spectral cover to be an actual algebraic surface in X_3 (a necessary condition for the associated vector bundle to be Mumford slope-stability) it is necessary that $\mathcal{S}$ be an

[2]More generally, the "cameral" cover construction [25, 24].

[3]Here suitability is rigorously defined via the concept of "regularity" [13, 41].

[4]More generally, a rank 1 sheaf. For interesting physical examples where this distinction is crucial see [43, 42, 44].

effective class in $H_4(X_3, \mathbb{Z})$. There is a further condition – that the spectral cover must be *indecomposable* – that must be imposed in order for the spectral cover bundle V to be slope stable. It can be seen that $\mathcal{S}$ is indecomposable if η is base-point free (*i.e.,* has no base locus in a Zariski-type decomposition and $\eta - Nc_1(B_2)$ is effective (see [29] for example)).

All that remains to fully determine the holomorphic bundle V is the data of the rank 1 sheaf, $\mathcal{L}_{\mathcal{S}}$. As described in [12], given the projection $\pi_{\mathcal{S}} : \mathcal{S} \to B_2$, the Grothendieck-Riemann-Roch theorem [30] indicates that

$$\pi_{\mathcal{S}*}\left(e^{c_1(\mathcal{L}_{\mathcal{S}})}Td(\mathcal{S})\right) = ch(\pi_*(V))Td(B_2) \tag{2.10}$$

At the level of the first Chern class this yields

$$\pi_{\mathcal{S}*}\left(c_1(\mathcal{L}_{\mathcal{S}}) + \frac{1}{2}c_1(\mathcal{S})\right) = \frac{N}{2}c_1(B_2) + c_1(V) \tag{2.11}$$

At this point, the condition that $c_1(V) = 0$ (necessary for our choice of $SU(N)$ bundle $V \to X_3$) fixes $c_1(\mathcal{L}_{\mathcal{S}}) \in H^{1,1}(\mathcal{S}) \cap H^2(\mathcal{S}, \mathbb{Z})$ up to a class $\gamma \in \ker(\pi_{\mathcal{S}*})$. Since $\pi_{\mathcal{S}}$ is an N-sheeted cover of B_2, $\pi_{\mathcal{S}*}\pi_{\mathcal{S}}{}^*(c_1(B_2)) = Nc_1(B_2)$ and hence

$$c_1(\mathcal{L}_{\mathcal{S}}) = \frac{N\sigma + \eta + c_1(B_2)}{2} + \gamma \tag{2.12}$$

with

$$\pi_{\mathcal{S}*}(\gamma) = 0 \tag{2.13}$$

Here we are faced with the generally difficult problem of determining γ. We will return to this in the next section, but for now we simply review the observations made in [12]: $c_1(\mathcal{L}_{\mathcal{S}})$ must be an integral $(1,1)$-class on $\mathcal{S}$. For the cases of interest, such classes may be scarce since it can be verified that frequently $h^{2,0}(\mathcal{S}) \neq 0$. As a result, the only obvious $(1,1)$-classes on $\mathcal{S}$ are those inherited from X_3, namely the restriction of the zero section of the elliptic fibration, σ, and pullbacks $\pi^*_{\mathcal{S}}(\beta)$ of integral $(1,1)$ classes on B_2.

Since $\pi_{\mathcal{S}*}\sigma|_{\mathcal{S}} = \eta - Nc_1(B)$ one finds [12] that a description of $\gamma \in \ker(\pi_{\mathcal{S}*})$ in this "obvious" basis is

$$\gamma = \lambda(N\sigma|_{\mathcal{S}} - \pi^*_{\mathcal{S}}(\eta - Nc_1(B)) \tag{2.14}$$

where λ must be either integer or half integer according to

$$\lambda = \begin{cases} m + \frac{1}{2}, & \text{if } N \text{ is odd} \\ m, & \text{if } N \text{ is even} \end{cases} \tag{2.15}$$

When N is even it is clear that this **integrality condition** imposes

$$\eta = c_1(B_2) \bmod 2 \tag{2.16}$$

where "mod 2" indicates that η and $c_1(B_2)$ differ only by an even element of $H^2(B_2, \mathbb{Z})$. This leads to the form most commonly assumed in the literature [12]:

$$c_1(L_{\mathcal{S}}) = N\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\pi^*_S\eta + \left(\frac{1}{2} + N\lambda\right)\pi^*_S c_1(B_2) \tag{2.17}$$

Having fully specified the topology of the spectral cover, it is possible to infer the full topology of V itself. The Chern classes of a spectral cover bundle V, specified by η and the integers n

and λ is [12, 13, 28, 23]

$$c_1(V) = 0 \tag{2.18}$$

$$c_2(V) = \eta\sigma - \frac{N^3 - N}{24} c_1(B_2)^2 + \frac{N}{2}\left(\lambda^2 - \frac{1}{4}\right)\eta \cdot (\eta - N c_1(B_2)) \tag{2.19}$$

$$c_3(V) = 2\lambda\sigma\eta \cdot (\eta - N c_1(B_2)) \tag{2.20}$$

Note that since $c_1(V) = 0$, $\text{Ind}(V) = ch_3(V) = \frac{1}{2}c_3(V)$.

The spectral cover construction provides a powerful tool in explicitly matching the geometric moduli of heterotic/F-theory dual pairs. For the details of the duality map and the necessary stable degeneration limit, we refer the reader to the classic references [12, 16] and conclude here with only a rough hint in Table 2 of how the degrees of freedom associated to $(\mathcal{S}, \mathcal{L}_{\mathcal{S}})$ correspond to the moduli of a Calabi-Yau fourfold in F-theory. In later investigations, we will

Het/Bundle	Het/Spec. Cov.	F-theory
$H^1(End_0(V))$	$H^{2,0}(\mathcal{S}) \sim Def(\mathcal{S})$	$H^{3,1}(\tilde{Y}_4)$
	$H^{1,0}(\mathcal{S}) \sim Pic_0(\mathcal{S})$	$H^{2,1}(\tilde{Y}_4)$
	$H^{1,1}(\mathcal{S}) \sim$ Discrete data of $\mathcal{L}_{\mathcal{S}}$	$H^{2,2}(\tilde{Y}_4, \mathbb{Z})$

Table 2. A rough, schematic matching of the heterotic vector bundle moduli, encoded as spectral data $(\mathcal{S}, \mathcal{L}_{\mathcal{S}})$, and geometric moduli of the (resolved) F-theory fourfold in the stable degeneration limit [12, 16].

further compare the structure of an $SU(2)$ spectral cover with its F-theory dual consisting of a generically singular fourfold with E_7 symmetry.

3. A database of Heterotic/F-theory dual pairs

In [5] a systematic algorithm was laid out for constructing heterotic/F-theory dual pairs in which $\mathcal{B}_3$ (the base of the elliptically fibered fourfold geometry) is constructed as a $\mathbb{P}^1$ bundle over B_2. To illustrate the methods of construction, the complete dataset of Calabi-Yau fourfolds with smooth heterotic duals and *toric* twofold bases were enumerated. This consisted of 4962 Calabi-Yau fourfolds, dual to heterotic threefold/bundle geometry. Of these, 947 were found to be generically singular with an E_7 symmetry (in at least one heterotic E_8 factor, equivalently F-theory coordinate patch). In the heterotic theory the E_7 gauge symmetry is realized by the commutant structure within E_8, via an $SU(2)$ vector bundle over the dual Calabi-Yau threefold. These rank 2 vector bundles provide one of the simplest windows into the generic properties of the bundle moduli space $\mathcal{M}_\omega(c(V))$. Because of the fact that these E_7 symmetries are un-Higgsable – that is the fourfolds are generically singular for all values of the complex structure moduli, the results of Table 1 indicate that for this choice of η the moduli space of stable sheaves contains only $SU(2)$ bundles.

Since the heterotic/F-theory duality map is most clearly understood in the case that the heterotic bundles can be described via spectral covers, it is natural to ask whether we can use this formalism to explicitly match the full degrees of freedom in dual E_7 effective theories described above.

As described in [5], the three conditions on the defining topological data, η, for consistent spectral covers are

- η effective

- η base-point-free within B_2
- $\eta = c_1(B_2)$ mod 2

In [5], it was explored how these conditions compare to those arising in defining good Calabi-Yau fourfold backgrounds for F-theory. It can be shown that the first of these conditions is true for all $K3$-fibered fourfolds arising as F-theory backgrounds. Moreover, it can be shown that if the second condition is violated for a fourfold with a generic E_7 singularity, then the Calabi-Yau manifold is too singular to admit a Kähler resolution. To that point, the geometric consistency conditions on an F-theory fourfold and an $SU(2)$ heterotic spectral cover bundle are identical. However, as we will see, at the final condition, this agreement appears to end.

The condition $\eta = c_1(B_2)$ mod 2 is required for the integrality of $\mathcal{L}_S$ in (2.16). However, a direct construction of the dataset in [5] shows immediately that this is violated for most fourfolds with generic E_7 symmetries – in fact, 897 of the 947! How then are we to make sense of these dual pairs?

One obvious resolution to the puzzle could occur if none of the 897 moduli spaces of $SU(2)$ bundles could admit any bundle built via the spectral cover construction. While possible, this seems unlikely from experience of how generic spectral cover bundles appear to be in known moduli spaces [41]. Another possible answer is that the integrality condition placed on $c_1(\mathcal{L}_\mathcal{S})$ in (2.16) may be artificially restrictive. This will clearly be the case whenever the Picard number of $\mathcal{S}$ is greater than $1 + h^{1,1}(B_2)$ as assumed by [12].

One class of examples in which the Picard group of $\mathcal{S}$ is larger than the generic case was outlined in [23]. There, it was pointed out that if the matter curve $a_2 = 0$ in (4.12) (in the class $[\eta - 2c_1(B_2)]$) is reducible in B_2, its components may in fact pull back to distinct, new divisors in $\mathcal{S}$. That is, if the curve $\bar{\eta} \in [\eta - 2c_1(B_2)]$ can be written as $\bar{\eta} = D + D' \subset B_2$, then its pullback can be described as

$$\pi^*_\mathcal{S}(\bar{\eta}) = \mathcal{D} + \mathcal{D}' \tag{3.1}$$

and even if D, D' are well-understood divisors in B_2, the class $\mathcal{D}$ in $\mathcal{S}$ may not be a simple linear combination of the divisors $\sigma|_\mathcal{S}$ and $\pi^*_\mathcal{S}(\phi)$ (with ϕ an effective curve class in B_2) assumed in the generic formula (2.12). In [6] we explored whether or not this observation could alleviate the disparity of the mysterious 897 E_7 theories found in [5]. While a handful of the examples found over Hirzebruch bases could be resolved by this mechanism, the majority of them remained unexplained [6]. To really resolve this puzzle and decide whether or not these geometries consist of valid heterotic/F-theory dual pairs, it is necessary to go further and attempt to study the integrality condition in detail. We turn to this now in the context a simple example of an $SU(2)$ bundle defined over $\pi_h : X_3 \to \mathbb{P}^2$.

4. A case study: bounding the Picard number $\rho(\mathcal{S})$

To begin, it is useful to summarize the discussion of the previous sections in the context of an $SU(2)$ spectral cover. To fully specify the $SU(2)$ gauge bundle appearing in the heterotic compactification, it is not enough to choose a spectral cover of the form given in (2.6) and (2.8), we must also fully describe the line bundle, $\mathcal{L}_\mathcal{S}$ over $\mathcal{S}$. A priori, we can describe the 1st Chern class of $\mathcal{L}_\mathcal{S}$ via (2.12) as

$$c_1(\mathcal{L}_\mathcal{S}) = \frac{N\sigma + \eta + c_1(B_2)}{2} + \gamma \tag{4.1}$$

where $\pi_{\mathcal{S}*}(\gamma) = 0$. By the construction of $\mathcal{S} \subset X_3$ there are $1 + h^{1,1}(B_2)$ natural integral $(1,1)$-classes on $\mathcal{S}$ (consisting of the restriction of σ, the section of the elliptic fibration, and the pullback of classes from the base). Using these as basis (and ignoring any other possibilities for γ) the integrality condition given in (2.14) and (2.16) were obtained in [12]. However, recent work on the F-theory side of the duality [5] indicates that this integrality condition appears to be violated in the vast majority of known examples (897 of 947 generic E_7 models enumerated in [5]) we must now ask whether or not it is possible to derive a more general integrality condition for $c_1(\mathcal{L}_\mathcal{S})$? To accomplish this, $\mathcal{L}_\mathcal{S}$ must be expressed in a complete basis. We are thus led to the following question:

Question 4.1. *For a general surface $\mathcal{S} \subset X_3$ (as described above) which is a ramified, N-sheeted cover of B_2 in the class $[N\sigma + {\pi_h}^*(\eta)]$ what is the rank of the Picard group of $\mathcal{S}$?*

As illustrated in the next Section, generically $h^{1,0}(\mathcal{S}) = 0$ [12] and $\mathcal{S}$ is a surface of general type. Unfortunately, determining the Picard number of such complex surfaces is a notoriously difficult problem (see [33, 34, 35] and references therein for some recent advances). To begin, it is enough to consider ways to bound the Picard number $\rho(\mathcal{S})$ as an important first step.

Let us briefly recall a few standard definitions regarding the Picard group (see [30, 36] for example). To define divisors (and hence line bundles), one begins with the exponential sequence

$$0 \to \mathbb{Z} \xrightarrow{i} \mathcal{O} \xrightarrow{exp} \mathcal{O}^* \to 0 \tag{4.2}$$

where the map i is an inclusion and exp is the exponential map. With vanishing $Pic_0(\mathcal{S})$ (i.e. with $h^{1,0}(\mathcal{S}) = h^1(\mathcal{S}, \mathcal{O}) = 0$ there are no continuous degrees of freedom in the Picard group), the associated long exact sequence in cohomology takes the form

$$0 \to H^1(\mathcal{S}, \mathcal{O}^*) \to H^2(\mathcal{S}, \mathbb{Z}) \to H^2(\mathcal{S}, \mathcal{O}) \tag{4.3}$$

The image of $H^1(\mathcal{S}, \mathcal{O}^*)$ (modulo torsion) in $H^2(\mathcal{S}, \mathbb{Z})$ parametrizes the Neron-Severi group, $NS(\mathcal{S})$, of the surface and its rank is the Picard number (i.e. $\rho(\mathcal{S})$, the number of discrete parameters which we can use to construct $\mathcal{L}_\mathcal{S}$). The Picard group is given by the kernel of the map from $H^2(\mathcal{S}, \mathbb{Z})$ to $H^2(\mathcal{S}, \mathcal{O}) = H^{0,2}$. The Hodge decomposition and Lefschetz' theorem [30] demonstrate that it is also zero in $H^{2,0}$ and hence must be a subset of $H^{1,1}$:

$$NS(S) \simeq H^2(\mathcal{S}, \mathbb{Z}) \cap H^{1,1}(\mathcal{S}) \tag{4.4}$$

Stated simply, divisors on $\mathcal{S}$ are determined by how the complex subspace $H^{1,1}$ of $H^2(\mathcal{S}, \mathbb{C})$ intersects the discrete subgroup $H^2(\mathcal{S}, \mathbb{Z})$. For surfaces with vanishing geometric genus, i.e., when $p_g = h^{0,2} = 0$, this is a trivial identification, but few tools exist to address the general case with $p_g \neq 0$. To begin, it should be observed that there is at least a bound:

$$\rho(\mathcal{S}) \leq h^{1,1}(\mathcal{S}) \tag{4.5}$$

Since in the present work we are focused on the case of 2-sheeted spectral covers and the mysterious E_7 cases described in the previous section, here we will try to make a first step towards answering this question. We will consider a simple example appearing in [5], with $\pi_\mathcal{S} : \mathcal{S} \to \mathbb{P}^2$. As we will see, even here determining the full Neron-Severi group is a non-trivial problem in algebraic geometry and for this brief work, we content ourselves with simply bounding the Picard number, $\rho(\mathcal{S})$ as described above.

4.1. **A double cover of $\mathbb{P}^2$.** In an explicit example we can explore in detail the possible form of the spectral line bundle, $\mathcal{L}_\mathcal{S}$. We consider here a 2-sheeted spectral cover, $\mathcal{S}$, and one of the simplest examples arising in the dataset of [5]. Let $\pi : X_3 \to \mathbb{P}^2$ be a Calabi-Yau threefold described via the generic (smooth) Weierstrass model over $\mathbb{P}^2$:

$$\hat{Y}^2 = \hat{X}^2 + f(u)\hat{X}\hat{Z}^4 + g(u)\hat{Z}^6 \tag{4.6}$$

where u_i $(i = 1, 2, 3)$ are homogeneous coordinates of $\mathbb{P}^2$ and

$$f \in H^0(\mathbb{P}^2, \mathcal{O}(12H)) \ , \ g \in H^0(\mathbb{P}^2, \mathcal{O}(18H)) \tag{4.7}$$

where H is the hyperplane divisor in $\mathbb{P}^2$. This Weierstrass model can be realized as hypersurface inside a toric variety. In a language more familiar to physicists, this threefold also be written via a GLSM-style charge matrix (see [32] for example):

	$\hat{Y}$	$\hat{X}$	$\hat{Z}$	u_1	u_2	u_3
6	3	2	1	0	0	0
0	0	0	-3	1	1	1

The hodge numbers of this threefold are well-known to be $h^{1,1} = 2$, $h^{2,1} = 272$. Furthermore, a basis of divisors on X_3 is given by D_1, D_2 where $D_2 = \pi^*(H)$ is the pullback of the hyperplane in $\mathbb{P}^2$ and D_1 is related to the elliptic fiber such that the class of the zero section ($\hat{Z} = 0$) is given in this basis as $\sigma = D_1 - 3D_2$. The tangent bundle of X_3 is described via adjunction as

$$0 \to TX_3 \to T\mathcal{A}|_{X_3} \to \mathcal{O}(6D_1)|_{X_3} \to 0 \tag{4.8}$$

where $T\mathcal{A}$ denotes the tangent sheaf of the toric ambient space. This in turn is defined by an Euler sequence [30]:

$$0 \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(3D_1) \oplus \mathcal{O}(2D_1) \oplus \mathcal{O}(D_1 - 2D_2) \oplus \mathcal{O}(D_2)^{\oplus 3} \to T\mathcal{A} \to 0 \tag{4.9}$$

For this geometry we specify vector bundles and a dual F-theory geometry by making a choice of twist as in Section 2, eq.(2.4). Here we select

$$T = 10H \tag{4.10}$$

In the heterotic theory this leads to an $SU(2)$ bundle $V \to X_3$ with

$$\eta = 6c_1(\mathbb{P}^2) - T = 8H \tag{4.11}$$

In the dual F-theory geometry this corresponds to a Calabi-Yau fourfold with *g*eneric E_7 singularity [5]. From (2.6), the spectral cover is in the class $[\mathcal{S}] = [2\sigma + 8\pi^*(H)]$ which in the basis given above corresponds to a section of the line bundle $N_{\mathcal{S}} = \mathcal{O}(2D_1 + 2D_2)$. Explicitly $\mathcal{S}$ is given by (2.8) as the zero locus of

$$a_0\hat{Z}^2 + a_2\hat{X} = 0 \tag{4.12}$$

with $a_0 \in H^0(\mathbb{P}^2, \mathcal{O}(8H))$ and $a_2 \in H^0(\mathbb{P}^2, \mathcal{O}(2H))$. Let us now take a closer look at $\mathcal{S}$. The complex, Kähler surface is a ramified double cover of $\mathbb{P}^2$ and we can directly compute its three independent Hodge numbers

$$h^{2,0}(\mathcal{S}),\ h^{1,0}(\mathcal{S}),\ h^{1,1}(\mathcal{S}) \tag{4.13}$$

To explicitly determine these numbers, we can once again make use of an adjunction formula, this time for $\mathcal{S}$ itself as a hypersurface inside X_3:

$$0 \to T\mathcal{S} \to TX_3|_{\mathcal{S}} \to \mathcal{O}(2D_1 + 2D_2)|_{\mathcal{S}} \to 0 \tag{4.14}$$

Furthermore, to determine the cohomology of vector bundles restricted to $\mathcal{S}$, the Koszul sequence for hypersurfaces

$$0 \to \mathcal{O}_{X_3}(-2D_1 - 2D_2) \to \mathcal{O}_{X_3} \to \mathcal{O}_{\mathcal{S}} \to 0 \tag{4.15}$$

and its associated long exact sequence in cohomology plays a useful role (see [37] for a review). In the case at hand, all the relevant cohomology groups on X_3 can be determined by considering the defining sequences (4.14), (4.8) and (4.15) and line bundle cohomology on X_3. For this geometry we employed the techniques of [32] to compute line bundle cohomology on X_3 (as implemented in [31]).

To begin, we note that $h^{2,0}(\mathcal{S}) = H^0(\mathcal{S}, \mathcal{O}(2D_1 + 2D_2)|_{\mathcal{S}})$. Twisting (4.15) by $\mathcal{O}(2D_1 + 2D_2)$ we obtain

$$0 \to \mathcal{O}_{X_3} \to \mathcal{O}_{X_3}(2D_1 + 2D_2) \to \mathcal{O}_{\mathcal{S}}(2D_1 + 2D_2) \to 0 \tag{4.16}$$

The associated long exact sequence in cohomology leads to

$$H^0(\mathcal{S}, \mathcal{O}(2D_1 + 2D_2)|_{\mathcal{S}}) = H^0(X_3, \mathcal{O}(2D_1 + 2D_2))/\mathbb{C} \tag{4.17}$$

Which can be directly calculated to yield

$$h^0(\mathcal{S}, \mathcal{O}(2D_1 + 2D_2)|_{\mathcal{S}}) = h^0(X_3, \mathcal{O}(2D_1 + 2D_2)) - 1 = 51 - 1.$$

This provides the first of three independent hodge numbers (the geometric genus):

$$h^{2,0}(\mathcal{S}) = 50 \tag{4.18}$$

Note that this is expected via the description of $\mathcal{S}$ in (4.12). By inspection of that formula it can be noted that there are 51 degrees of freedom in the coefficients a_0, a_2 over $\mathbb{P}^2$. Subtracting 1 for the overall scale, we see that this agrees with the expectation of the embedding moduli of $\mathcal{S} \subset X_3$.

Next, note that $h^{1,0} = h^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}})$ (the "irregularity" of the surface). Here the long exact sequence in cohomology associated to (4.15) yields

$$h^{1,0}(\mathcal{S}) = 0 \tag{4.19}$$

Finally, to determine $h^{1,1}(\mathcal{S})$, consider the dual sequence

$$0 \to \mathcal{O}(-2D_1 - 2D_2)|_{\mathcal{S}} \to {TX_3}^{\vee}|_{\mathcal{S}} \to T\mathcal{S}^{\vee} \to 0 \tag{4.20}$$

To evaluate this it should first be noted that the Koszul sequence for $\mathcal{O}(-2D_1 - 2D_2)$ produces the following short exact sequence

$$0 \to \mathcal{O}_{X_3}(-4D_1 - 4D_2) \to \mathcal{O}_{X_3}(-2D_1 - 2D_2) \to \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2) \to 0 \tag{4.21}$$

and from the associated sequence in cohomology

$$\begin{aligned} &h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2)) = h^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2)) = 0 \\ &h^2(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2)) = 219 \end{aligned} \tag{4.22}$$

This gives the full cohomology of the first term bundle in (4.20). But what is $H^*(\mathcal{S}, {TX_3}^{\vee}|_{\mathcal{S}})$? The last necessary pieces can be obtained by considering (4.15) twisted by $TX_3^{\vee}$:

$$0 \to TX_3 \otimes \mathcal{O}_{X_3}(-2D_1 - 2D_2) \to TX_3^{\vee} \to TX_3^{\vee}|_{\mathcal{S}} \to 0 \tag{4.23}$$

Here the long exact sequence in cohomology produces

$$\begin{aligned} &h^0(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) = 0 \\ &h^1(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) = h^1(X_3, TX^{\vee}) + \dim(\ker(\phi)) = 2 + \dim(\ker(\phi)) \\ &h^2(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) = \dim(\mathrm{coker}(\phi)) \\ &\phi : H^2(X, TX_3 \otimes \mathcal{O}_{X_3}(-2D_1 - 2D_2)) \to H^2(X, TX_3^{\vee}) \end{aligned} \tag{4.24}$$

Since $h^2(X, TX_3 \otimes \mathcal{O}_{X_3}(-2D_1 - 2D_2)) = 393$ and $h^2(X, TX_3^{\vee}) = 272$, it follows that $\dim(\ker(\phi)) = 121 + m$ for some $m \geq 0$, and $\dim(\mathrm{coker}(\phi)) = m$ by exactness. In fact, for generic choices of spectral cover in (4.12), we expect the induced map ϕ to be surjective and $h^1(\mathcal{S}, TX_3^{\vee}|_{\mathcal{S}}) = 123$.

With this in hand, we are now in a position to put the pieces together to determine $H^1(\mathcal{S}, T\mathcal{S}^\vee)$. Using (4.22) and (4.24), and returning to the long exact sequence in cohomology associated to (4.20) gives the following long exact sequence:

$$0 \to H^2(\mathcal{S}, TX_3^\vee|_{\mathcal{S}}) \to H^{1,1}(\mathcal{S}) \to H^2(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-2D_1 - 2D_2)) \to H^2(\mathcal{S}, TX_3^\vee|_{\mathcal{S}}) \to 0 \tag{4.25}$$

It is helpful to note that $h^2(\mathcal{S}, T\mathcal{S}^\vee) = h^{1,0} = 0$, and the alternating sum of the dimensions in (4.25) leads at last to

$$h^{1,1}(\mathcal{S}) = (123 + m) + (219 - m) = 342 \tag{4.26}$$

Thus, in summary we have determined that $\mathcal{S}$ is a complex surface with $h^{1,0} = 0$, $h^{2,0} = 50$ and $h^{1,1} = 342$. It follows that the Euler number of $\mathcal{S}$ is $e = 2 + 2p_g + h^{1,1} - 4h^{1,0} = 444$ (with $e = c_2(T\mathcal{S})$) and the holomorphic Euler characteristic is $\chi = 51$ (leading to $K_{\mathcal{S}}^2 = 168$). According to Kodaira's classification, $\mathcal{S}$ is a surface of general type (Kodaira dimension 2).

Taking a step back, one can now ask what we have learned from the this example? The first observation is that in this case

$$2 \leq \rho(\mathcal{S}) \leq 342 \tag{4.27}$$

where the lower bound arises from concrete construction of divisors [12] and the upper bound is obtained from $h^{1,1}$ as described in the previous Subsection. It should be noted here that there are in principle hugely more parameters in the spectral data than are commonly assumed in the physics literature. While the full computation of $\rho(\mathcal{S})$ is beyond the scope of the present work, tools exist to analyze the intersection structure of curves in $\mathcal{S}$ and can be used to further constrain $\rho(\mathcal{S})$ in many cases. We hope to explore this in future work. For the moment, in the example above, we expect that $H^2(\mathcal{S}, \mathbb{Z}) \cap H^{1,1}(\mathcal{S})$ will generically be large. Indeed, despite the fact that $p_g = 50$, $h^{1,1}$ is sufficiently big that contrary to the expectations of [12], it may be that the Picard number $\rho(\mathcal{S})$ is considerably above its minimum value of 2. In this case, there are certainly more general choices available for the line bundle, $\mathcal{L}_{\mathcal{S}}$, and the integrality condition in (2.16) is manifestly incorrect and too restrictive.

To proceed further with this explicit example, it might be possible to consider the branch locus of the two-sheeted cover in detail. Such an analysis was undertaken in [50] for certain double covers of $\mathbb{P}^2$. There for special choices of topology, the resolution of singularities in the branch curve led to concrete descriptions of the Neron-Severi group of the double cover (which was in fact maximal in those cases). It would be interesting in the future to explore the application of these techniques to heterotic spectral covers.

Finally it should be noted that as $\mathcal{S}$ is varied within the 50-parameter family given in (4.12), the Picard number can surely change. While difficult to compute, these special, higher codimensional "Noether-Lefschetz Loci" [38, 39] may be especially significant for the underlying physics, determining for example, where the complex structure moduli of the dual F-theory geometry, Y_4 are stabilized by G-flux [40].

To conclude, the example above was provided as a simple illustration of the fact that the integrality condition for spectral cover bundles given in (2.16) may be too restrictive in many cases. Furthermore, it serves to highlight the interesting and frequently difficult geometric questions that arise in fully determining the geometry of dual heterotic/F-theory pairs. As a final comment on the mysterious 897 E_7 examples highlighted in [5], the arguments presented above indicate to us that in fact there is more to understand about integrality conditions in spectral covers and that this may provide a resolution to the seeming discrepancy in all the exotic heterotic/F-theory pairs. We hope in future work to build upon the simple examples considered here and to fully compute the Picard group of $\mathcal{S}$ systematically in the full dataset.

By addressing these remaining geometric puzzles we hope it will be possible to complete the program laid out in [5] and fully enumerate all consistent heterotic/F-theory dual pairs.

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AN INDEX FORMULA FOR SUPERSYMMETRIC QUANTUM MECHANICS

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Abstract. We derive a localization formula for the refined index of gauged quantum mechanics with four supercharges. Our answer takes the form of a residue integral on the complexified Cartan subalgebra of the gauge group. The formula captures the dependence of the index on Fayet-Iliopoulos parameters and the presence of a generic superpotential. The residue formula provides an efficient method for computing cohomology of quiver moduli spaces. Our result has broad applications to the counting of BPS states in four-dimensional $\mathcal{N}=2$ systems. In that context, the wall-crossing phenomenon appears as discontinuities in the value of the residue integral as the integration contour is varied. We present several examples illustrating the various aspects of the index formula.

1. Introduction

Supersymmetric quantum mechanics has a wide variety of applications in mathematical physics. It arises universally as the zero momentum sector of supersymmetric field theories and governs the worldline dynamics of supersymmetric particles. A basic feature of any such system is its set of supersymmetric ground states. When these states are counted with signs according to their fermion number they form the Witten index [1], perhaps the most primitive example of a quantity protected by supersymmetry.

Motivated by these general considerations, in this work we determine a general formula for the index of $\mathcal{N}=4$ quantum mechanics. We focus on the class of quantum mechanics models that have Lagrangians which arise from the dimensional reduction of four-dimensional supersymmetric gauge theories. In this context the counting of vacua may be further sharpened using R-charges. The result is a refined index

$$\Omega \equiv \mathrm{Tr}_{\mathcal{H}}\left((-1)^F \exp(-\beta H) y^{R+2J_3}\right). \tag{1.1}$$

Our main result is an integral expression for Ω derived by supersymmetric localization [2, 3].

Pragmatically speaking, our derivation of the index formula in §2 follows closely a similar calculation for the elliptic genus of two-dimensional systems with $\mathcal{N}=(2,2)$ supersymmetry. Consequently, our final answer for the index Ω takes a similar form to that uncovered in [4–6]: the index Ω can be expressed as a residue integral of a meromorphic form on a product of complex annuli $(\mathbb{C}^*)^r$.

The index Ω depends in a subtle way on two pieces of data entering the quantum-mechanical model.

- In gauge theories with abelian factors, the Lagrangian may contain Fayet-Iliopoulos parameters ζ. The index Ω depends in a piecewise constant fashion on such FI parameters. Across codimension one walls in ζ-space, supersymmetric vacua may be created or destroyed and the index Ω jumps. In our context, the FI parameters enter the index through a specification of integration contour. The jumping of the index is mapped to the change of a residue integral under large variations in the contour.

- In theories which admit non-trivial superpotentials, the refined index Ω depends on the superpotential through the R-charge assignments that the latter implies for chiral fields. We find that the residue formula accurately encodes this dependence for the case of generic superpotential.

We highlight these key features of the index in our study of examples in §4.

In §2.3 we compare the residue formula to alternative computational approaches to the index. The most straightforward technique involves two steps. First, one calculates the classical moduli space of the supersymmetric quantum mechanics. Then, one finds the desired ground state wavefunctions by quantizing the moduli space, i.e. computing its cohomology. Our residue formula bypasses the intermediate step of the classical moduli space and computes directly the refined index Ω which may be interpreted as a generating functional of the cohomology. In this way our index formula is similar in spirit to the Reineke formula [7] for the cohomology of moduli spaces of quiver representations, and to its cousin the MPS formula [8] obtained by geometric quantization of the Coulomb branch.

One of the key physical applications of the index formula occurs in the study of BPS states in four-dimensional systems with $\mathcal{N} = 2$ supersymmetry. Often, the BPS spectrum may be described via the ground states of quiver quantum mechanics. We briefly review this connection in §3. The class of physical systems to which this paradigm applies is broad and includes black holes in supergravity [9–12], dyons in four-dimensional gauge theories [13–15], and even more exotic systems decorated by external defects [16, 17].

In the context of BPS states, our result for the quantum mechanical index Ω can be interpreted as an explicit formula for the protected spin character of BPS states with an electromagnetic charge determined by the ranks of the quiver gauge groups. The jumps in Ω as the FI parameters are varied are then mapped to the ubiquitous wall-crossing phenomenon first uncovered in [18–20]. The fact that wall-crossing may be encoded by contour deformation of a residue integral is a generalization of similar ideas in systems with $\mathcal{N} = 4$ supersymmetry [21].

Wall-crossing has recently been extensively studied [22–27] due to the existence of universal formulas [28–30] encoding the discontinuities in the BPS spectrum. In the simple examples that we have investigated, the discontinuities in the residue formula for Ω agree with these universal formulas. It would be interesting to understand the relation more concretely and explain why our residue prescriptions obey wall-crossing formulas. We leave this, as well as applications of the index formula to interesting four-dimensional $\mathcal{N} = 2$ systems, as open problems for future work.

Note added: While this work was being completed the preprint [31] appeared which develops the same formula for the refined index in the context of generalized ADHM quantum mechanics. Localization formulas for the index of supersymmetric quantum mechanics have also been independently obtained in [32, 33]. See additionally [34] for related work.

2. The Index of $\mathcal{N} = 4$ Quantum Mechanics

In this section we present the residue formula for the index of $\mathcal{N} = 4$ quantum mechanics. Our derivation follows straightforwardly from the dimensional reduction of the elliptic genus formulas of [5, 6]. Our discussion is brief and we refer to those works for a more complete treatment.

2.1. Gauged Quantum Mechanics and the Refined Index.

The class of models we consider are quantum-mechanical gauge theories with four real supercharges. We assume throughout that the system is gapped so that there are a finite number of ground states which are separated in energy from the excited states. Our aim is to count (with appropriate signs), the number of ground states in such a model.

In addition to possible flavor symmetries, the systems in question have R-symmetry group $su(2)_J \times u(1)_R$. There are two classes of multiplets:

- Vector multiplets associated to gauge groups. The bosonic fields consist of a one-component gauge field A and a triplet of adjoint scalars $\vec{X}$. The gauge field is uncharged under the R-symmetry group, while the adjoint scalars transform in a $\mathbf{3}$ of $su(2)_J$ and are neutral under $u(1)_R$.
- Chiral multiplets associated to representations of the gauge groups. The bosonic fields consist of a complex scalar Φ transforming as a singlet under $su(2)_J$ and with $u(1)_R$ charge R_Φ.

In addition to the spectrum of vector and chiral multiplets the Lagrangian for our quantum mechanics depends on two additional pieces of data.

- FI parameters. Let the total gauge symmetry algebra for the quantum mechanics be $\mathfrak{g}$. We decompose $\mathfrak{g} = \tilde{\mathfrak{g}} + \mathfrak{g}_{u(1)}$, where $\tilde{\mathfrak{g}}$ is semi-simple and $\mathfrak{g}_{u(1)} = \bigoplus_i \mathfrak{u}(1)_\mathfrak{i}$ is the abelian part of the gauge algebra. We view the FI parameter ζ as an element of the dual space $\mathfrak{g}^*_{u(1)}$.
- Superpotentials. If the model admits holomorphic gauge invariant monomials in the chiral fields then we may activate them in the superpotential $\mathcal{W}$. Consider a monomial in $\mathcal{W}$ and let d_i denote the degree in this monomial of the chiral field Φ_i. The presence of such a term restricts the R-charges of the chirals as

$$R(\mathcal{W}) = 2 = \sum_i d_i R_{\Phi_i}. \tag{2.1}$$

  The above constraint must be true for each monomial term in the superpotential and restricts $\mathcal{W}$ to be quasi-homogeneous.

  In our analysis, the superpotential will enter only through the above constraints on the $u(1)_R$ charges of chiral fields. Thus our results are restricted to the case of quasi-homogeneous superpotential. Aside from the constraint (2.1), the $u(1)_R$ charges of chiral fields may be chosen arbitrarily.

  We make two additional assumptions about $\mathcal{W}$.
  - We assume that all lowest degree terms consistent with quasi-homogeneity are in fact present in $\mathcal{W}$.[1]
  - We assume that $\mathcal{W}$ is a generic polynomial of multi-degree consistent with (2.1) and the previous assumption.

  As we illustrate in the examples of §4.3, both of these assumptions are necessary for the applicability of the residue formula of §2.2.[2]

Given a fixed gauged quantum mechanics, our object of interest is the refined Witten index defined as

$$\Omega(y, \zeta) \equiv \mathrm{Tr}_\mathcal{H}\left((-1)^F \exp(-\beta H) y^{R+2J_3}\right). \tag{2.2}$$

As usual, when the system is gapped the index receives contributions only from ground states and hence is independent of β. In general, the index depends on both the FI parameter ζ and the R-charges of chiral fields.[3] The charge $R + 2J_3$ commutes with the supercharge used to form

[1]Thus, if a quadratic superpotential is possible we assume that it is present. If no quadratic superpotential is possible and a cubic potential is possible we assume the later is present. And so on.

[2]Indeed without these additional assumptions, the spectrum will generally be non-discrete and the index as studied here is incomplete.

[3]We suppress the dependence on R-charges in the notation.

the index and hence we may further grade the ground states to obtain a non-trivial function of y. It is convenient to define z as

$$y = e^{i\pi z}. \tag{2.3}$$

In the following we use z and y interchangeably.

2.2. **The Residue Formula for the Index.** The refined index $\Omega(y,\zeta)$ can be computed by a path integral on a circle with periodic boundary conditions for fermions, and background R-symmetry gauge fields. A formula $\Omega(y,\zeta)$ can be directly obtained by taking the dimensional reduction limit of the various ingredients of the localized elliptic genus formula of [5, 6] (see §3 of [6] for the derivation). Our final answer takes the form of a residue

$$\Omega(y,\zeta) = \frac{1}{|W|} \sum_{u_* \in \mathfrak{M}^*_{\text{sing}}} \underset{u=u_*}{\text{JK-Res}} \left(\mathbf{Q}(u_*), \zeta\right) \, Z_{1-\text{loop}}(z,u), \tag{2.4}$$

where $|W|$ is the order of the Weyl group and ζ is the FI parameter.

In this section we explain the elements of this formula. In the remainder of the paper we discuss its various applications.

Definition of the Space $\mathfrak{M}$.

The u variable that appears in (2.4) is valued in a space $\mathfrak{M}$ of bosonic zero modes of the vector multiplets. We restrict the gauge field and scalars to be valued in the Cartan subalgebra $\mathfrak{h}$ of the gauge algebra $\mathfrak{g}$. In the triplet of scalars in the vector multiplet, there is one real component which is neutral under the charge $R + 2J_3$ and we denote this field by X. The field X may have zero modes, while for generic y, the remaining members of the triplet do not have zero modes.

The definition of the variable u is then

$$u \equiv A^{(0)} - iX^{(0)}, \tag{2.5}$$

where $A^{(0)}$ and $X^{(0)}$ are the zero modes for the one-dimensional gauge field and the scalar X. Since A is a gauge field, large gauge transformations make the real part of u periodic. Thus the space $\mathfrak{M}$ of zero modes is a product of annuli

$$\mathfrak{M} = \mathfrak{h}_{\mathbb{C}}/Q^\vee \cong (\mathbb{C}^*)^r \tag{2.6}$$

where r is the total rank of the gauge groups and $Q^\vee$ is the coroot lattice.

Definition of the Meromorphic Form $Z_{1-\text{loop}}(z,u)$.

The quantity $Z_{1-\text{loop}}(z,u)$ is a meromorphic top form on the space $\mathfrak{M}$. It is defined by computing the one-loop determinant of the massive modes in the path integral on the circle. This one-loop determinant receives contributions from the vector multiplets and the chiral multiplets as

$$Z_{1-\text{loop}} = \prod_V Z_{V,G} \prod_\Phi Z_{\Phi,\mathbf{R}}. \tag{2.7}$$

The quantities $Z_{V,G}$ and $Z_{\Phi,\mathbf{R}}$ can be obtained from direct dimensional reduction of (2.12) and (2.8) in [5], respectively.

The contribution of a vector multiplet V with gauge group G to the one-loop determinant $Z_{1-\text{loop}}$ is

$$Z_{V,G}(z,u) = \left[-\frac{\pi}{\sin(\pi z)}\right]^{\text{rank}\, G} \prod_{\alpha \in G} \frac{\sin[\pi\alpha(u)]}{\sin[\pi\alpha(u) - \pi z]} \prod_{a=1}^{\text{rank}\, G} du_a. \tag{2.8}$$

where the product of α is over the roots of G.

The contribution of a chiral multiplet Φ in the representation $\mathbf{R}$ with $u(1)_R$ charge R is

$$Z_{\Phi,\mathbf{R}}(z,u) = \prod_{\rho\in\mathbf{R}} \frac{\sin\left[\pi\rho(u) + \pi\left(\frac{R}{2}-1\right)z\right]}{\sin\left[\pi\rho(u)+\pi\frac{R}{2}z\right]}, \tag{2.9}$$

where the product of ρ is over the weights of $\mathbf{R}$.
Definition of the Locus $\mathfrak{M}^*_{\text{sing}}$.

Next we define the locus $\mathfrak{M}^*_{\text{sing}} \subset \mathfrak{M}$. The form $Z_{1-\text{loop}}$ has poles along hyperplanes H_i in $\mathfrak{M}$ where modes, which are massive at generic u, become massless. Specifically these hyperplanes are

$$\text{vector}: \ H_i = \Big\{ -z + Q_i(u) = 0 \text{ mod } \mathbb{Z}\Big\} \qquad Q_i = \alpha, \tag{2.10}$$

$$\text{chiral}: \ H_i = \left\{ \frac{R_i}{2}z + Q_i(u) = 0 \text{ mod } \mathbb{Z}\right\} \qquad Q_i = \rho. \tag{2.11}$$

And the charge covectors $Q_i \in \mathfrak{h}^*$ can be either the roots α of the gauge algebra or weights ρ of the matter representations.

We define

$$\mathfrak{M}^*_{\text{sing}} = \left\{ u_* \in \mathfrak{M} \,\middle|\, \text{at least } r \text{ linearly independent } H_i\text{'s meet at } u_* \right\}. \tag{2.12}$$

$\mathfrak{M}^*_{\text{sing}}$ is the collection of points where the residue (2.4) is evaluated.
Definition of the Residue.

The Jeffrey-Kirwan residue operation $\underset{u=u_*}{\text{JK-Res}}\left(\mathbf{Q}(u_*),\eta\right)$ is defined abstractly in [35] and studied constructively in [36].

For notational simplicity, we shift the point where we evaluate the residue to be at $u_* = 0$. $\mathbf{Q}(u_*)$ is a collection of charge covectors $Q_i \in \mathfrak{h}^*$ with $i = 1, \cdots, n$ for some n. The collection $\mathbf{Q}(u_*)$ defines n hyperplanes meeting at $u = u_*$:

$$H_i = \left\{ u \in \mathbb{C}^r \,\middle|\, Q_i(u) = 0 \right\}. \tag{2.13}$$

In addition, the Jeffrey-Kirwan residue operation depends on a choice of covector $\eta \in \mathfrak{h}^*$.

If all the charge covectors in $\mathbf{Q}(u_*)$ are contained in a half-space of $\mathfrak{h}^*$, the hyperplane arrangement is said to be projective. For a projective arrangement, the Jeffrey-Kirwan residue is the linear functional defined by the conditions

$$\underset{u=u_*}{\text{JK-Res}}\left(\mathbf{Q}(u_*),\eta\right) \frac{du_1}{Q_{j_1}(u)} \wedge \cdots \wedge \frac{du_r}{Q_{j_r}(u)} = \begin{cases} \det|(Q_{j_1}\cdots Q_{j_r})|^{-1} & \text{if } \eta \in \text{Cone}(Q_{j_1}\cdots Q_{j_r}), \\ 0 & \text{otherwise,} \end{cases} \tag{2.14}$$

where $\text{Cone}(Q_{j_1}\cdots Q_{j_r})$ indicates the positive linear span of the covectors $Q_{j_1}, \cdots, Q_{j_r}$. In particular, if $n = r$, the hyperplane arrangement is projective. For simplicity in this paper we study examples with $n = r$.
Definition of the Contour.

Finally, we must specify the choice of the covector $\eta \in \mathfrak{h}^*$ in the definition of the Jeffrey-Kirwan residue operation (2.14). This quantity is fixed by the FI parameter ζ as

$$\eta = \zeta \in \mathfrak{g}^*_{u(1)} \subset \mathfrak{h}^*. \tag{2.15}$$

This identification may be justified by demanding that the gaussian path integral over the zero mode of the auxiliary D field in the vector multiplet passes through the point of stationary phase fixed by the FI parameter ζ.

Equation (2.15) is a key aspect of the residue formula (2.4). Because of the discontinuity in the Jeffrey-Kirwan residue operation (2.14) as ζ varies, the contour prescription (2.15) enables the index $\Omega(y, \zeta)$ to depend in a piecewise constant fashion on the FI parameter.

2.3. **Cohomology of Higgs Branch Moduli Spaces.** It is fruitful to compare the residue formula for the refined index to other methods of calculating the ground states.

The most direct approach is to calculate the moduli space of classical vacua and then quantize this moduli space to determine wavefunctions. As usual in supersymmetric gauge theory, the classical moduli space is typically separated into multiple branches: Higgs branches where matter fields are non-vanishing, and Coulomb branches where scalars from the vector multiplets are non-vanishing. We isolate one of these branches and quantize. We focus on the Higgs branch as it is typically better behaved. For Coulomb branch approaches see [8, 11, 27].

The classical Higgs branch $\mathcal{M}$ is simply the set of solutions to the F and D flatness conditions modulo the action of the gauge group.

Explicitly, let G denote the total gauge group of the gauged quantum mechanics. And let Φ_ν indicated the chiral fields transforming in representations $\mathbf{R}_\nu$ of G. We define a set in the vector space $\oplus_\nu \mathbf{R}_\nu$ as the set of Φ_ν obeying the following equations.

- For each chiral field Φ_ν, the superpotential is stationary

$$\frac{\partial \mathcal{W}}{\partial \Phi_\nu} = 0. \tag{2.16}$$

- The gauge group G has a number of abelian factors, each with an associated FI parameter ζ_i. Let q^i_ν denote the charge of Φ_ν under the i-th $U(1)$. Then for each abelian factor we demand

$$\sum_\nu q^i_\nu |\Phi_\nu|^2 = \zeta_i. \tag{2.17}$$

The Higgs branch moduli space $\mathcal{M}$ is the set of solutions to (2.16)-(2.17) quotiented by the action of the group G.

In the most widely studied class of examples, the gauge group G is a product of unitary groups and the representations $\mathbf{R}_\nu$ are chosen to be bifundamentals. In that case $\mathcal{M}$ is the moduli space of stable quiver representations [37].

In favorable circumstances, the moduli space $\mathcal{M}$ is compact and we may now extract the ground state spectrum from its cohomology. To form the refined index we must then assemble this cohomology into a generating function. Supersymmetry implies that $\mathcal{M}$ is Kähler and hence its cohomology may be bigraded into Dolbeault cohomology groups. We denote by $h^{p,q}(\mathcal{M})$ the resulting Hodge numbers, and let d denote the complex dimension of $\mathcal{M}$. Then the refined index is

$$\Omega(y, \zeta) = \sum_{p,q=0}^{d} h^{p,q}(\mathcal{M})(-1)^{p-q} y^{2p-d}. \tag{2.18}$$

Agreement between (2.18) and the residue formula (2.4) yields a direct way of extracting information about the cohomology of the moduli space $\mathcal{M}$ which is similar in spirit to [7]. Note however that the residue formula (2.4) is applicable only in the case of discrete spectrum which in the context of quiver representations implies that ranks of the gauge groups must be coprime.

3. Relation to BPS Particles of $4d$ $\mathcal{N}=2$ Systems

In this section, we briefly review the connection between supersymmetric gauged quantum mechanics and BPS states of four-dimensional $\mathcal{N}=2$ systems. See [14] for a systematic introduction and examples. This connection motivates the analysis of the refined index $\Omega(y,\zeta)$ in a broad class of quantum-mechanical models.

Fix a four-dimensional $\mathcal{N}=2$ system and a generic vacuum v on its Coulomb branch. At low energies, the physics is described by an abelian gauge theory with electromagnetic charge lattice Γ. The one-particle Hilbert space of the theory supports BPS states carrying charges $\gamma \in \Gamma$. For each occupied charge the Hilbert space in that sector is a representation of $su(2)_J \times su(2)_I$, where $su(2)_J$ is group of spatial rotations and $su(2)_I$ is the R-symmetry of the four-dimensional theory. This representation takes the general form

$$\left[(\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{2})\right]\otimes\mathcal{H}_\gamma. \tag{3.1}$$

We count BPS states by forming a protected spin character

$$\Omega(\gamma,y,v)_{4d}=\mathrm{Tr}_{\mathcal{H}_\gamma}y^{2J_3}(-y)^{2I_3}. \tag{3.2}$$

$\Omega(\gamma,y,v)_{4d}$ receives contributions only from BPS states, and is stable under small variations in the vacuum v. Under large changes in v, $\Omega(\gamma,y,v)_{4d}$ may jump according to the wall-crossing formula [28–30].

Next, let us describe an approach to the calculation of the protected spin characters $\Omega(\gamma,y,v)_{4d}$ utilizing supersymmetric quantum mechanics. The basic physical paradigm of this method is to isolate a collection of elementary BPS states, and then to view the remaining BPS particles as non-relativistic composites of the elementary states. Since the worldvolume theory of a BPS particle preserves four supercharges, the interactions governing the formation of non-relativistic bound states are controlled by $\mathcal{N}=4$ quantum mechanics. Frequently this quantum mechanics is of the gauge theory type investigated in the previous section.

In a large class of models the relevant $\mathcal{N}=4$ quantum mechanics is a quiver model with unitary gauge groups and bifundamental matter. In broad strokes, the dictionary between the two systems is as follows. Each elementary constituent BPS state is represented by a node of the quiver giving a quantum mechanical gauge group. The interactions between these nodes are encoded by the Dirac inner product of their electromagnetic charges and specify the number of arrows in the quiver. In the quantum mechanics model, these are the chiral multiplets. Finally, the central charges of the elementary BPS states map to the FI parameters ζ.[4]

The main difficulty in applying the quantum-mechanical approach outlined above is to determine an explicit basis of elementary BPS states. However in many four-dimensional theories, including for instance arbitrary gauge theories coupled to fundamental matter [14, 15], such a basis may be identified and the BPS spectrum may be investigated. When this is so we obtain a direct relationship between the four-dimensional protected spin character and the refined index of the associated gauged quantum mechanics:[5]

$$\Omega(\gamma,y,v)_{4d}=\Omega(y,\zeta), \tag{3.3}$$

where in the above we have the following explicit identification of parameters.

[4]If a superpotential is permitted by the topology of the quiver, then it must also be specified. See e.g. [38] for a class of four-dimensional gauge theories where the relevant quiver superpotential may be fixed.

[5]The identification (3.3) suggests that in models for which the correspondence holds, all the ground states of the quantum mechanics are bosonic with vanishing $u(1)_R$ charge, and that the $su(2)_I$ charge acts trivially on the spectrum of BPS particles as in the "no-exotics" conjecture of [39].

- The i-th quiver gauge group is $U(n_i)$ where the n_i are determined by expanding the charge γ as a sum of the charges of the elementary BPS states

$$\gamma = \sum_i n_i \gamma_i. \tag{3.4}$$

  Since the matter content from the chiral multiplets consists of bifundamentals, the overall $U(1) \subset \prod_i U(n_i)$ decouples and is treated as non-dynamical. Alternatively, one may freely decouple any other convenient $U(1)$ without effecting the refined index.
- The FI parameters are specified by the choice of vacuum v. Each elementary BPS state has a central charge $\mathcal{Z}_i(v)$ which depends explicitly on v. The central charge of γ is then determined from (3.4) by linearity

$$\mathcal{Z}(\gamma, v) = \sum_i n_i \mathcal{Z}_i(v) \equiv |\mathcal{Z}(\gamma, v)| \exp(i\alpha), \qquad \alpha \in \mathbb{R}. \tag{3.5}$$

  The FI parameter at the i-th node is then given by

$$\zeta_i = \Im\left(\exp(-i\alpha) n_i \mathcal{Z}_i(v)\right). \tag{3.6}$$

  Observe that by construction, the sum of the FI parameters is zero. This enables the decoupling of the overall $U(1)$ described above.

One interesting consequence of the identification (3.3) and the associated dictionary, is that the four-dimensional wall-crossing phenomenon maps to the discontinuity in the refined index $\Omega(y, \zeta)$ under large changes in ζ. Because ζ enters our residue formula (2.4) as a definition of the contour, it follows that the four-dimensional wall-crossing formulas of [28–30] must be encoded in the variations of the residue integral as the contour is deformed. This is similar the perspective on wall-crossing developed in systems with $\mathcal{N} = 4$ supersymmetry in [21].

3.1. **Toy Models.** In this section we describe simple examples of the relation between BPS particles and quiver quantum mechanics. We study these models using the residue formula in §4.

3.1.1. *Dyon Chains.* A basic example illustrating the connection between four-dimensional BPS particles and ground states of supersymmetric gauged quantum mechanics are dyon chains. These have been studied from the semiclassical soliton perspective in [40] and from the quiver quantum mechanics perspective in [11].

The relevant four-dimensional system is $SU(M)$ super-Yang-Mills. One is interested in investigating the bound states of a collection of $n + 1 \leq M$ distinct dyons. We choose the electric and magnetic charges of the dyons as

$$(e_i, m_i) = (q_i \alpha_i, \alpha_i), \tag{3.7}$$

where α_i denote simple roots of the $SU(M)$ algebra normalized such that

$$\alpha_i \cdot \alpha_j = \begin{cases} 2 & |i-j| = 0, \\ -1 & |i-j| = 1, \\ 0 & |i-j| > 1, \end{cases} \tag{3.8}$$

and q_i are integers satisfying

$$q_{n+1} > q_n > \cdots > q_3 > q_2 > q_1. \tag{3.9}$$

If we denote by k_i the quantity $q_{i+1} - q_i$, then the symplectic products of the dyon charges are

$$(e_i, m_i) \cdot (e_j, m_j) = \begin{cases} +k_i & j = i+1, \\ -k_{i-1} & j = i-1, \\ 0 & j \neq i \pm 1. \end{cases} \tag{3.10}$$

The quiver governing the bound states of these dyons is then a linear chain illustrated in Figure 1.

$$\begin{tikzcd} \boxed{1} \arrow[r, "k_1"] & \boxed{1} \arrow[r, "k_2"] & \boxed{1} \arrow[r, "k_3"] & \cdots \arrow[r, "k_{n-1}"] & \boxed{1} \arrow[r, "k_n"] & \boxed{1} \end{tikzcd}$$

FIGURE 1. The general abelian linear quiver which governs the bounds states of the specified dyons. The integers at the nodes denote the ranks of the associated gauge groups, while k_i are the number of bifundamentals (arrows).

The spectrum of bound states depends on the FI parameters ζ_i at the i-th node. When these are such that

$$\zeta_{n+1} > 0, \qquad \zeta_{n+1} + \zeta_n > 0, \qquad \cdots \qquad \zeta_{n+1} + \zeta_n + \cdots + \zeta_2 > 0, \tag{3.11}$$

there is a non-trivial classical Higgs branch, $\mathcal{M}$, of supersymmetric vacua in the quiver. By explicitly solving the F and D term equations of §2.3, one finds that the Higgs branch is a product of projective spaces

$$\mathcal{M} = \prod_{i=1}^{n} \mathbb{P}^{k_i - 1} \tag{3.12}$$

quantizing this space as in (2.18) we find that the index is

$$\Omega(y, \zeta) = \prod_{i=1}^{n} \left(y^{-k_i+1} \sum_{j=0}^{k_i - 1} y^{2j} \right), \tag{3.13}$$

which reproduces the answers obtained by quantizing monopole moduli spaces [40].

We obtain this result using the residue formula (2.4) in §4.1.

3.1.2. *Electron Halos.* Another class of interesting examples arises from studying the bound states of m identical electrons and a single monopole of magnetic charge k. In this case, the relevant quiver is shown in Figure 2.

$$\begin{tikzcd} \boxed{1} \arrow[r, "k"] & \boxed{m} \end{tikzcd}$$

FIGURE 2. The quiver relevant for studying the bound states of a monopole and a cloud of electrons. The integers at the nodes denote the ranks of the associated gauge groups, while k is the number of bifundamentals (arrows).

Let ζ indicate the FI parameter at the second node and assume $\zeta > 0$. By solving the equations of §2.3, we determine that the moduli space is the Grassmannian $Gr(m, k)$ of complex m-planes in a k-dimensional space.[6] Extracting the refined index from cohomology as in (2.18),

[6]The moduli space is empty if $m > k$.

we find that

$$\Omega(y,\zeta) = \frac{y^{m(m-k)}\prod_{i=1}^{k}(1-y^{2i})}{\prod_{i=1}^{m}(1-y^{2i})\prod_{i=1}^{k-m}(1-y^{2i})}. \tag{3.14}$$

We reproduce this result using the residue formula (2.4) in §4.2.

4. Examples

In this section we explore various examples of the residue formula (2.4) for the refined index $\Omega(y,\zeta)$. The cases we consider illustrate several interesting features of the index: wall-crossing, non-Abelian gauge groups, and superpotentials.

To achieve maximal overlap with the applications discussed in §3, we consider quantum-mechanical quiver gauge theories with unitary gauge groups. In such examples a single $U(1)$ factor of the gauge group decouples. One may choose this $U(1)$ to simplify the resulting quantum mechanics. Correspondingly, we demand that the sum of the FI parameters vanishes as in (3.6).

4.1. **Linear Abelian Quivers: Dyon Chains.** We begin with the example of linear abelian quivers. As described in §3.1.1, these quivers compute the bound states of chains of distinct dyons. We aim to reproduce the result (3.13) using the residue formula (2.4).

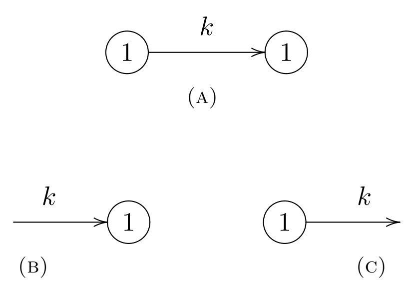


Figure 3. The two-node linear quiver. The integers at the nodes denote the ranks of the associated gauge groups, while k is the number of bifundamentals (arrows). In (b) and (c), the two ways of decoupling a $U(1)$.

4.1.1. *Two Nodes.* We start with the abelian two-node quiver with k bifundamental chiral multiplets between the two nodes. We can decouple a $U(1)$ in two different ways as shown in Figure 3a. We decouple the first node as in Figure 3b. The other alternative clearly yields the same answer.

In this case the one-loop determinant is

$$Z_{1-\mathrm{loop}}(z,u) = -\frac{\pi}{\sin(\pi z)}\left[\frac{\sin(\pi u - \pi z)}{\sin(\pi u)}\right]^k du. \tag{4.1}$$

On $\mathfrak{M}$, there is a hyperplane H (in this case, point) where $Z_{1-\mathrm{loop}}$ has a pole:

$$H: \; u = 0. \tag{4.2}$$

The corresponding charge covector Q is just 1. Let ζ_2 be the FI parameter of the second node. The Jeffrey-Kirwan residue operation satisfies

$$\underset{u=0}{\text{JK-Res}}\left(\{1\},\zeta_2\right)\frac{du}{u} = \begin{cases} 1, & \text{if } \zeta_2 > 0, \\ 0, & \text{if } \zeta_2 < 0. \end{cases} \tag{4.3}$$

In the $\zeta_2 > 0$ case, we can therefore write the Jeffrey-Kirwan residue as the usual contour integral around $H = \{ u = 0 \}$:

$$\underset{u=0}{\text{JK-Res}}\,(\{1\}, \zeta_2)\, \frac{du}{u} = \frac{1}{2\pi i} \oint_{u=0} \frac{du}{u}, \text{ if } \zeta_2 > 0. \tag{4.4}$$

The index is then given by

$$\begin{aligned}
\Omega(y, \zeta_2) &= -\frac{\pi}{\sin(\pi z)} \underset{u=0}{\text{JK-Res}}\,(\{1\}, \zeta_2) \left[\frac{\sin(\pi u - \pi z)}{\sin(\pi u)}\right]^k du \\
&= \begin{cases} -\frac{\pi}{\sin(\pi z)} \oint_{u=0} \frac{du}{2\pi i} \left[\frac{\sin(\pi u - \pi z)}{\sin(\pi u)}\right]^k & \text{if } \zeta_2 > 0, \\ 0 & \text{if } \zeta_2 < 0. \end{cases} \\
&= \begin{cases} y^{-k+1} \sum_{j=0}^{k-1} y^{2j} & \text{if } \zeta_2 > 0, \\ 0 & \text{if } \zeta_2 < 0. \end{cases}
\end{aligned} \tag{4.5}$$

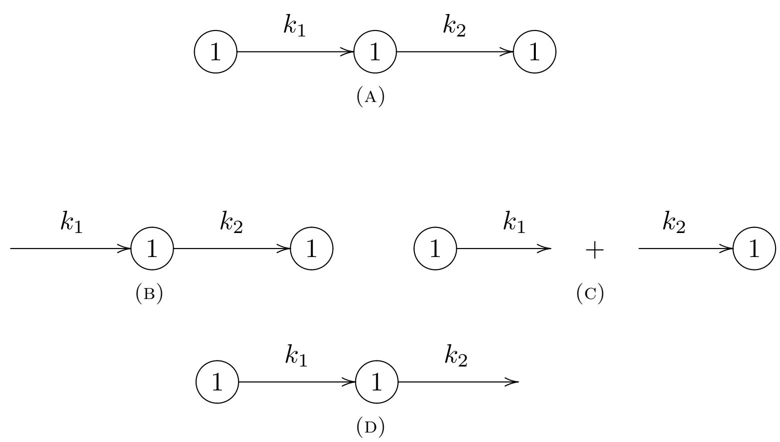


FIGURE 4. The three-node linear quiver. The integers at the nodes denote the ranks of the associated gauge groups, while k_i are the number of bifundamentals (arrows). In (b), (c), and (d), the three ways of decoupling a $U(1)$. In (b), the quiver has become disconnected and the model factorizes.

4.1.2. *Three Nodes.* Let us now move on to the three-node linear quiver with k_i bifundamental chiral multiplets between the i-th and the $(i+1)$-th nodes. There are three distinct ways to decouple a $U(1)$ from the quiver as shown in Figure 4a. For purposes of illustration we will show explicitly that all three choices yield the same answer.

The easiest choice is to decouple the second node as in Figure 4c, so that the quiver becomes two decoupled one-node quivers. The index is immediately given by the product of the answers (4.5) for the one-node quivers:

$$\Omega(y, \zeta) = \begin{cases} \left(y^{-k_1+1} \sum_{i=0}^{k_1-1} y^{2i}\right) \left(y^{-k_2+1} \sum_{j=0}^{k_2-1} y^{2j}\right), & \text{if } \zeta_1 < 0,\ \zeta_3 > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{4.6}$$

Alternatively, we can decouple the first node as in Figure 4b. The one-loop determinant is

$$Z_{1-\text{loop}}(z,u) = \left[\frac{\sin(\pi u_2 - \pi z)}{\sin(\pi u_2)}\right]^{k_1} \left[\frac{\sin(-\pi u_2 + \pi u_3 - \pi z)}{\sin(-\pi u_2 + \pi u_3)}\right]^{k_2} du_2 \wedge du_3. \tag{4.7}$$

There are two hyperplanes on the complex two-dimensional space $\mathfrak{M}$ where $Z_{1-\text{loop}}$ has poles:

$$\begin{aligned} &H_1: \; u_2 = 0, \\ &H_2: \; -u_2 + u_3 = 0. \end{aligned} \tag{4.8}$$

The corresponding charge covectors Q_i that define H_i in $\mathfrak{M}$ are

$$\begin{aligned} Q_1 &= (1,0), \\ Q_2 &= (-1,1), \end{aligned} \tag{4.9}$$

as shown in Figure 5. The intersection $H_1 \cap H_2 = \{u = 0\}$ is the point u_* at which we evaluate the residue. Since this theory is abelian, $\mathfrak{g}^*_{u(1)} = \mathfrak{h}^*$ and we can take $\eta = \zeta$ to be on any point on the $\mathfrak{h}^*$ plane in Figure 5.

From the definition of the Jeffrey-Kirwan residue operation, we have

$$\underset{u=0}{\text{JK-Res}} \left(\{Q_1, Q_2\}, \zeta\right) \frac{du_2 \wedge du_3}{u_2(-u_2+u_3)} = \begin{cases} 1, & \text{if } \zeta \in \text{Cone}(Q_1, Q_2), \\ 0, & \text{otherwise.} \end{cases} \tag{4.10}$$

If $\zeta \in \text{Cone}(Q_1, Q_2)$, we can then write the Jeffrey-Kirwan residue as

$$\underset{u=0}{\text{JK-Res}} \left(\{Q_1, Q_2\}, \zeta\right) \frac{du_2 \wedge du_3}{u_2(-u_2+u_3)} = \left(\frac{1}{2\pi i}\right)^2 \oint_{u_2=0} \oint_{u_3=u_2} \frac{du_2 du_3}{u_2(-u_2+u_3)}. \tag{4.11}$$

The index in the chamber $\zeta \in \text{Cone}(Q_1, Q_2)$ is

$$\begin{aligned} \Omega(y,\zeta) &= \left[-\frac{\pi}{\sin(\pi z)}\right]^2 \oint_{u_2=0} \frac{du_2}{2\pi i} \oint_{u_3=u_2} \frac{du_3}{2\pi i} \left[\frac{\sin(\pi u_2 - \pi z)}{\sin(\pi u_2)}\right]^{k_1} \left[\frac{\sin(-\pi u_2 + \pi u_3 - \pi z)}{\sin(-\pi u_2 + \pi u_3)}\right]^{k_2} \\ &= \left(y^{-k_1+1} \sum_{i=0}^{k_1-1} y^{2i}\right) \left(y^{-k_2+1} \sum_{j=0}^{k_2-1} y^{2j}\right). \end{aligned} \tag{4.12}$$

The condition $\zeta \in \text{Cone}(Q_1, Q_2)$ for nonzero index in components is

$$\zeta_3 > 0, \; \zeta_2 + \zeta_3 > 0, \tag{4.13}$$

which is the same as the answer obtained by decoupling the second node in (4.6). Note that we have used $\zeta_1 + \zeta_2 + \zeta_3 = 0$. Similarly one can show that the index obtained by decoupling the $U(1)$ as in Figure 4d is the same as above.

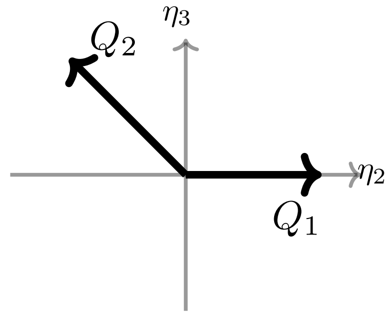


FIGURE 5. Three-node quiver with the first node decoupled. The figure shows the charge covectors Q_i in $\mathfrak{h}^* = (\mathfrak{u}(1)^2)^* \cong \mathbb{R}^2$.

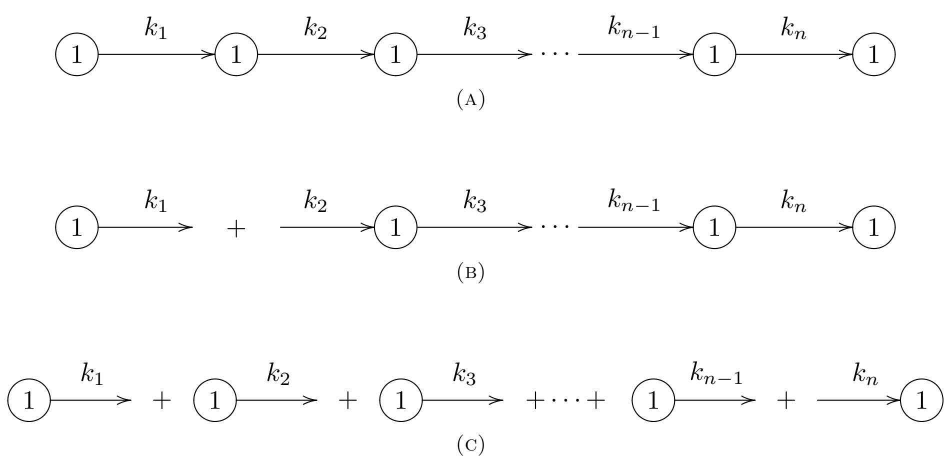


FIGURE 6. The general abelian linear quiver. The integers at the nodes denote the ranks of the associated gauge groups, while k_i are the number of bifundamentals (arrows). In (b), a convenient choice of decoupled $U(1)$. In (c), the model is reduced to a product.

4.1.3. *General Linear Abelian Quiver.* Finally, we consider the general abelian linear quiver with $n+1$ nodes and k_i bifundamental chiral multiplets between the i-th and the $(i+1)$-th nodes.

In the abelian three-node quiver case ($n = 2$), we have shown that $\Omega(y, \zeta)$ is the product of the index of the one-node quiver with k_1 chiral multiplets, and the index for the one-node quiver with k_2 chiral multiplets.

Now assume that for the n-node quiver $\Omega(y, \zeta)$ is similarly given by the product of that for $n-1$ one-node quivers with k_i chiral multiplets for the i-th decoupled node. Then for the linear quiver with $n+1$ nodes, we can decouple the second node as shown in Figure 6b and the quiver becomes the product of a one-node quiver with a n-node quiver. Inductively, we have shown that the index for the $(n+1)$-node quiver is the same as the product of indices for n one-node quivers as shown in Figure 6c.

Thus, the index of the general abelian linear quiver is:

$$\Omega(y,\zeta)=\prod_{i=1}^{n}\left(y^{-k_i+1}\sum_{j=0}^{k_i-1}y^{2j}\right), \tag{4.14}$$

if the FI parameters ζ_i satisfy the following conditions

$$\zeta_{n+1}>0, \qquad \zeta_{n+1}+\zeta_n>0, \qquad \cdots \qquad \zeta_{n+1}+\zeta_n+\cdots+\zeta_2>0, \tag{4.15}$$

as can be easily seen from Figure 6b.

This is exactly the expected result (3.13).

4.2. **Non-Abelian Phenomena: Electron Halos.** In this section we consider an example with non-abelian quiver gauge group. As described in §3.1.2 this example computes the bound states of a single monopole and m identical electrons. Our goal is to reproduce the result (3.14) using the residue formula (2.4).

(A) (B)

FIGURE 7. The two-node linear quiver with a nonabelian gauge group. The integers at the nodes denote the ranks of the associated gauge groups, while k is the number of bifundamentals (arrows). In (b), a $U(1)$ is decoupled leaving a $U(m)$ gauge theory with k fundamental chiral multiplets with $+1$ charge under the $U(1)$ of $U(m)$.

Consider the quiver in Figure 7a. We decouple the $U(1)$ node to compute the index as in Figure 7b. One can alternatively decouple the central $U(1)$ of $U(m)$ and obtain the same answer.

The one-loop determinant for a $U(m)$ vector multiplet with k chiral multiplets in the representation $\square_1$ is

(4.16)
$$Z_{1-\text{loop}}=\frac{1}{m!}\left[-\frac{\pi}{\sin(\pi z)}\right]^m\prod_{\substack{b,c=1,\\ b\neq c}}^{m}\frac{\sin(\pi u_b-\pi u_c)}{\sin(\pi u_b-\pi u_c-\pi z)}\prod_{a=1}^{m}\left[\frac{\sin(\pi u_a-\pi z)}{\sin(\pi u_a)}\right]^k du_1\wedge\cdots\wedge du_m.$$

On the complex m-dimensional space $\mathfrak{M}$, there are hyperplanes H_{ab} and H_c, with

$$a,b,c=1,\cdots,m \qquad \text{and} \qquad a\neq b,$$

where $Z_{1-\text{loop}}$ has poles:

$$\begin{aligned}&\text{vector}:\ H_{ab}:\ u_a-u_b-z=0,\ a\neq b,\\ &\text{chiral}:\ H_c:\ u_c=0.\end{aligned} \tag{4.17}$$

For the index formula, we always pick the covector η in the definition of the Jeffrey-Kirwan residue operation (2.14) to be in the $u(1)$ part of the dual Cartan subalgebra $\mathfrak{g}^*_{u(1)}$ as in (2.15). In the current example, this implies that η lies on a real one-dimensional line on the real m-dimensional space $\mathfrak{h}^*$:

$$\eta=\zeta(1,1,\cdots,1)\in\mathfrak{h}^*\cong\mathbb{R}^m, \tag{4.18}$$

where ζ is the FI parameter for $U(m)$.

For a given ζ, the index can potentially receive contribution from various intersections of H_{ab} and H_a. For example, in the $U(2)$ case shown in Figure 8, if we choose $\zeta > 0$, the Jeffrey-Kirwan residue operation receives contributions from $H_1 \cap H_2$, $H_{12} \cap H_2$, and $H_{21} \cap H_1$, while it gives zero for $\zeta < 0$. However, the contributions from $H_{12} \cap H_2$ and $H_{21} \cap H_1$ can be shown to be zero by a direct computation.

For general m in the chamber $\zeta > 0$, we therefore conjecture that the index only receives contribution from the intersection $H_1 \cap H_2 \cap \cdots \cap H_m$.

With this assumption, the index can then be computed to be

(4.19)
$$\Omega(y,\zeta) = \frac{1}{m!}\left[-\frac{\pi}{\sin(\pi z)}\right]^m \prod_{a=1}^{m}\left[\oint_{u_a=0}\frac{du_a}{2\pi i}\right]\prod_{\substack{b,c=1,\\ b\neq c}}^{m}\frac{\sin(\pi u_b - \pi u_c)}{\sin(\pi u_b - \pi u_c - \pi z)}\prod_{d=1}^{m}\left[\frac{\sin(\pi u_d - \pi z)}{\sin(\pi u_d)}\right]^k .$$

On the other hand, from the result (3.14) we know that the index is given by

$$\Omega(y,\zeta) = \begin{cases} \frac{y^{m(m-k)}\prod_{i=1}^{k}(1-y^{2i})}{\prod_{i=1}^{m}(1-y^{2i})\prod_{i=1}^{k-m}(1-y^{2i})} & \text{if } m \leq k, \\ 0 & \text{if } m > k, \end{cases} \tag{4.20}$$

when $\zeta > 0$.

We have checked by direct calculation that the two expressions (4.19) and (4.20) agree for a wide range of m and k. This provides further evidence that index only receives contribution from the intersection $H_1 \cap H_2 \cap \cdots \cap H_m$ and yields an elegant combinatorial identity for the residue integral (4.19).

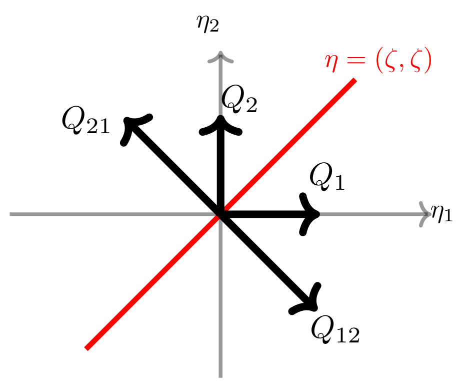


FIGURE 8. A $U(2)$ vector multiplet with k chiral multiplets in the fundamental representation with $+1$ charge under the $U(1)$ of $U(2)$. The figure shows the charge covectors Q_{ab} and Q_a on $\mathfrak{h}^* \cong \mathbb{R}^2$. In the index formula, we choose η to be on $\mathfrak{g}^*_{u(1)}$, which is the red line in the figure. As a result, we never need to consider the chamber $\mathrm{Cone}(Q_1, Q_{12})$ nor $\mathrm{Cone}(Q_2, Q_{21})$.

4.3. **Non-Trivial Superpotentials.** In the examples of §4.1 and §4.2 the quivers do not admit non-trivial superpotentials and hence the refined index is not sensitive to the choice of $u(1)_R$ charge assignments for the chiral multiplets. In this section we generalize to examples where the superpotential plays an important role. We find that as long as the superpotential satisfies the properties described in §2.1 the index formula (2.4) still accurately computes the refined index.

The examples we explore fall into the class of quivers analyzed from a representation theory perspective in [28, 41, 42].

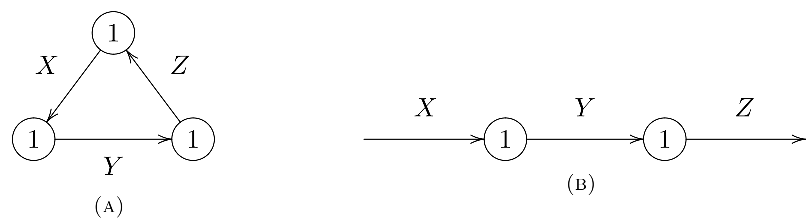


FIGURE 9. The XYZ model. The integers at the nodes denote the ranks of the associated gauge groups, while X, Y, Z label the fields. There is one arrow between each pair of nodes. In (b), a choice of $U(1)$ decoupling.

4.3.1. *The XYZ Model.* Consider a triangle quiver shown in Figure 9a with three $U(1)$ vector multiplets and three chiral multiplets X, Y, Z. We decouple the $U(1)$ node where X and Z meet as in Figure 9b.

We will assume the R-charges for the three chiral multiplets X, Y, Z to be R_X, R_Y, R_Z, respectively. Let k be a positive integer such that

$$\frac{2}{k} = R_X + R_Y + R_Z. \tag{4.21}$$

Given a k, we can allow for the following superpotential in the quantum mechanics:

$$\mathcal{W} = (XYZ)^k. \tag{4.22}$$

For $k = 1$ we have the generic cubic superpotential $\mathcal{W} = XYZ$. There are no supersymmetric ground states, so the expected answer for $\Omega(y, \zeta)$ is zero. In this case we will see that the residue formula (2.4) accurately computes the index.

When $k > 1$, the superpotential does not satisfy our hypotheses. A direct calculation in the chamber where $\zeta_1 > 0$ and $\zeta_2 > 0$ shows that the expected index from quantizing the classical moduli space is one. We will see that the residue formula does not produce this answer.

The one-loop determinant is

$$\begin{aligned} Z_{1-\text{loop}} = &\left[-\frac{\pi}{\sin(\pi z)}\right]^2 \left[\frac{\sin\left(\pi u_1 + \pi\left(\frac{R_X}{2} - 1\right)z\right)}{\sin(\pi u_1 + \pi\frac{R_X}{2}z)}\right] \left[\frac{\sin\left(-\pi u_1 + \pi u_2 + \pi\left(\frac{R_Y}{2} - 1\right)z\right)}{\sin(-\pi u_1 + \pi u_2 + \pi\frac{R_Y}{2}z)}\right] \\ &\times \left[\frac{\sin\left(-\pi u_2 + \pi\left(\frac{R_Z}{2} - 1\right)z\right)}{\sin(-\pi u_2 + \pi\frac{R_Z}{2}z)}\right] du_1 \wedge du_2 \wedge du_3. \end{aligned} \tag{4.23}$$

It has poles at the hyperplanes

$$\begin{aligned} H_X:&\ \ u_1 + \frac{R_X}{2}z = 0, \\ H_Y:&\ \ -u_1 + u_2 + \frac{R_Y}{2}z = 0, \\ H_Z:&\ \ -u_2 + \frac{R_Z}{2}z = 0. \end{aligned} \tag{4.24}$$

The corresponding charge covectors Q_X, Q_Y, Q_Z on $\mathfrak{h}^* \cong \mathbb{R}^2$ are shown in Figure 10. Since the theory is abelian, $\mathfrak{g}^*_{u(1)} = \mathfrak{h}^*$ and we can take ζ to be at any point on the plane $\mathfrak{h}^*$.

There are three chambers on the FI parameter space ζ in Figure 10. For a given ζ, the index receives contributions from one of the three intersections $H_X \cap H_Y$, $H_Y \cap H_Z$, and $H_X \cap H_Z$, depending on which chamber ζ is in. A direct computation shows that all three chambers give the same answer.

For example, if $\zeta \in \mathrm{Cone}(Q_X, Q_Y)$, the Jeffrey-Kirwan residue operation is nonzero at

$$H_X \cap H_Y = \{u_1 = -\frac{R_X}{2} z,\ u_2 = -\frac{R_X + R_Y}{2} z\}:$$

$$\begin{aligned}
&\underset{u=H_X\cap H_Y}{\text{JK-Res}} (\{Q_X, Q_Y\}, \zeta) \frac{du_1 \wedge du_2}{(u_1 + \frac{R_X}{2} z)(-u_1 + u_2 + \frac{R_Y}{2} z)} \\
&= \left(\frac{1}{2\pi i}\right)^2 \oint_{u_1 = -\frac{R_X}{2} z} \oint_{u_2 = u_1 - \frac{R_Y}{2} z} \frac{du_1 du_2}{(u_1 + \frac{R_X}{2} z)(-u_1 + u_2 + \frac{R_Y}{2} z)}
\end{aligned} \tag{4.25}$$

The index can then be computed as

$$\begin{aligned}
\Omega(y, \zeta) = &\left[-\frac{\pi}{\sin(\pi z)}\right]^2 \oint_{u_1 = -\frac{R_X}{2} z} \frac{du_1}{2\pi i} \oint_{u_2 = u_1 - \frac{R_Y}{2} z} \frac{du_2}{2\pi i} \left[\frac{\sin\left(\pi u_1 + \pi\left(\frac{R_X}{2} - 1\right) z\right)}{\sin(\pi u_1 + \pi \frac{R_X}{2} z)}\right] \\
&\times \left[\frac{\sin\left(-\pi u_1 + \pi u_2 + \pi\left(\frac{R_Y}{2} - 1\right) z\right)}{\sin(-\pi u_1 + \pi u_2 + \pi \frac{R_Y}{2} z)}\right] \left[\frac{\sin\left(-\pi u_2 + \pi\left(\frac{R_Z}{2} - 1\right) z\right)}{\sin(-\pi u_2 + \pi \frac{R_Z}{2} z)}\right] \\
&= \frac{y^{1-\frac{1}{k}} - y^{-1+\frac{1}{k}}}{y^{-\frac{1}{k}} - y^{\frac{1}{k}}}.
\end{aligned} \tag{4.26}$$

Note that the answer only depends on the sum of the R-charges $2/k$, but not on the individual assignments R_X, R_Y, R_Z. For $k = 1$, $\Omega(y, \zeta)$ vanishes as expected. For $k > 1$, however, the answer produced by the residue formula does not match that obtained by direct analysis. As explained above, this is no contradiction since in this case the superpotential does not satisfy our hypotheses.

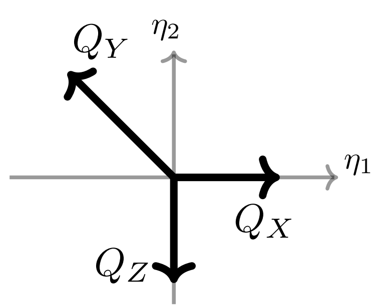


FIGURE 10. The XYZ model with one node removed. The figure shows the charge covectors Q_X, Q_Y, Q_Z on $\mathfrak{h}^* \cong \mathbb{R}^2$.

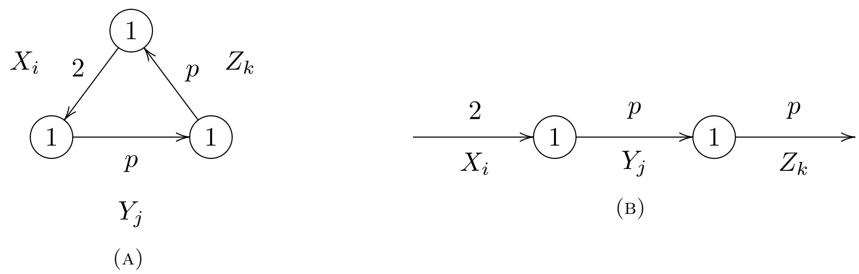


FIGURE 11. The generalized XYZ model. The integers at the nodes denote the ranks of the associated gauge groups, while the integers at the arrows denote the number of bifundamental fields. In (b), a choice of $U(1)$ decoupling.

4.3.2. *Generalized XYZ Model.* We continue our study of models with superpotential but now with a more nontrivial index and wall-crossing phenomenon.

The quiver diagram is shown in Figure 11a. We have two X_i, $i = 1, 2$, and p Y_j and Z_k, $j, k = 1, \cdots, p$, chiral multiplets. We will assume the R-charges for all the X_i are the same and will be denoted by R_X. Similarly for Y_j and Z_k.

We will also assume $2 = R_X + R_Y + R_Z$ so that the superpotential is cubic. The charge covectors are the same as the previous case shown in Figure 10. However, unlike the XYZ model in the previous subsection, the index now does depend on the choice of the chamber.

In the $\zeta \in \mathrm{Cone}(Q_X, Q_Y)$ chamber, the index is

$$\begin{aligned}\Omega(y,\zeta) &= \left[-\frac{\pi}{\sin(\pi z)}\right]^2 \oint_{u_1=-\frac{R_X}{2}z} \frac{du_1}{2\pi i} \oint_{u_2=u_1-\frac{R_Y}{2}z} \frac{du_2}{2\pi i} \left[\frac{\sin\left(\pi u_1 + \pi\left(\frac{R_X}{2}-1\right)z\right)}{\sin(\pi u_1 + \pi\frac{R_X}{2}z)}\right]^2 \\ &\quad \times \left[\frac{\sin\left(-\pi u_1 + \pi u_2 + \pi\left(\frac{R_Y}{2}-1\right)z\right)}{\sin(-\pi u_1 + \pi u_2 + \pi\frac{R_Y}{2}z)}\right]^p \left[\frac{\sin\left(-\pi u_2 + \pi\left(\frac{R_Z}{2}-1\right)z\right)}{\sin(-\pi u_2 + \pi\frac{R_Z}{2}z)}\right]^p \\ &= p. \end{aligned} \tag{4.27}$$

Similarly the chamber $\zeta \in \mathrm{Cone}(Q_X, Q_Z)$ gives the same answer as above.

In the chamber $\zeta \in \mathrm{Cone}(Q_Y, Q_Z)$, the index is

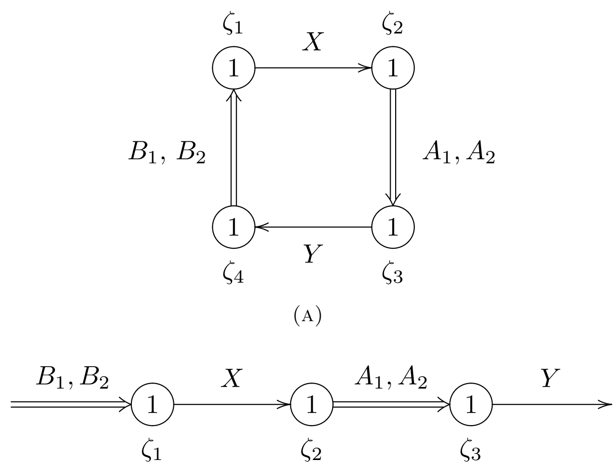
$$\begin{aligned}\Omega(y,\zeta) &= \left[-\frac{\pi}{\sin(\pi z)}\right]^2 \oint_{u_2=\frac{R_Z}{2}z} \frac{du_2}{2\pi i} \oint_{u_1=u_2+\frac{R_Y}{2}z} \frac{du_1}{2\pi i} \left[\frac{\sin\left(\pi u_1 + \pi\left(\frac{R_X}{2}-1\right)z\right)}{\sin(\pi u_1 + \pi\frac{R_X}{2}z)}\right]^2 \\ &\quad \times \left[\frac{\sin(-\pi u_1 + \pi u_2 + \pi\left(\frac{R_Y}{2}-1\right)z)}{\sin(-\pi u_1 + \pi u_2 + \pi\frac{R_Y}{2}z)}\right]^p \left[\frac{\sin(-\pi u_2 + \pi\left(\frac{R_Z}{2}-1\right)z)}{\sin(-\pi u_2 + \pi\frac{R_Z}{2}z)}\right]^p \\ &= \begin{cases} \sum_{j=2}^{p}(j-1)\left(y^{2(p-j)} + y^{-2(p-j)}\right) & \text{if } p > 1, \\ 0 & \text{if } p = 1. \end{cases} \end{aligned} \tag{4.28}$$

Note that the index again does not depend on the individual R-charges R_X, R_Y, R_Z but only on their sum. We assume $R_X + R_Y + R_Z = 2$ to allow for the generic cubic superpotential.

These results match those obtained by directly quantizing the quiver moduli space in [17].

4.3.3. *4d $\mathcal{N} = 2$ $SU(3)$ Yang-Mills Theory.* As a final example we consider the quiver quantum mechanics which governs the BPS states of four-dimensional $\mathcal{N} = 2$ $SU(3)$ Yang-Mills theory [13, 14] shown in Figure 12a. We study an example where the ranks of the quiver gauge groups are all one. The corresponding BPS particle is a W-boson. We expect that this particle is stable, and hence ground states of the quantum-mechanics exist, in the weak coupling region of the four-dimensional moduli space. This is the region in ζ-space where

$$\zeta_2 < \zeta_3, \qquad \zeta_4 < \zeta_1. \tag{4.29}$$



(A) (B)

FIGURE 12. The BPS quiver for $4d$ $\mathcal{N}=2$ $SU(3)$ Yang-Mills theory. The integers at the nodes denote the ranks of the associated gauge groups in the quantum mechanics, while the integers at the arrows denote the number of bifundamental fields. The ζ_i indicate our convention for FI parameters. The corresponding BPS particle is a W-boson in the $4d$ theory. In (b), a choice of $U(1)$ decoupling.

The superpotential is

$$\mathcal{W} = B_1 X A_1 Y - B_2 X A_2 Y. \tag{4.30}$$

From the symmetry of the quiver, we assume the R-charges for X and Y are the same and denote them by R. From the superpotential and the symmetry we deduce that the R-charges for A_1, A_2, B_1, B_2 are all $1-R$.

We decouple the $U(1)$ as in Figure 12b. The meromorphic top form is

$$\begin{aligned} Z_{1-\text{loop}} = & \left[-\frac{\pi}{\sin(\pi z)}\right]^3 \left[\frac{\sin\left(\pi u_1 + \pi\left(\frac{1-R}{2}-1\right)z\right)}{\sin\left(\pi u_1 + \pi\frac{1-R}{2}z\right)}\right]^2 \left[\frac{\sin\left(-\pi u_1 + \pi u_2 + \pi\left(\frac{R}{2}-1\right)z\right)}{\sin\left(-\pi u_1 + \pi u_2 + \pi\frac{R}{2}z\right)}\right] \\ & \times \left[\frac{\sin\left(-\pi u_2 + \pi u_3 + \pi\left(\frac{1-R}{2}-1\right)z\right)}{\sin\left(-\pi u_2 + \pi u_3 + \pi\frac{1-R}{2}z\right)}\right]^2 \left[\frac{\sin\left(-\pi u_3 + \pi\left(\frac{R}{2}-1\right)z\right)}{\sin\left(-\pi u_3 + \pi\frac{R}{2}z\right)}\right] du_1 \wedge du_2 \wedge du_3 \end{aligned} \tag{4.31}$$

There are four hyperplanes in $\mathfrak{M}$ where $Z_{1-\text{loop}}$ has poles:

$$
\begin{aligned}
H_1 :\ & u_1 + \frac{1-R}{2} z = 0, \\
H_2 :\ & -u_1 + u_2 + \frac{R}{2} z = 0, \\
H_3 :\ & -u_2 + u_3 + \frac{1-R}{2} z = 0, \\
H_4 :\ & -u_3 + \frac{R}{2} z = 0.
\end{aligned}
\tag{4.32}
$$

The charge covectors in $\mathfrak{h}^*$ that define these hyperplanes in $\mathfrak{M}$ are

$$
\begin{aligned}
Q_1 &= (1,0,0), \\
Q_2 &= (-1,1,0), \\
Q_3 &= (0,-1,1), \\
Q_4 &= (0,0,-1).
\end{aligned}
\tag{4.33}
$$

$\mathfrak{M}^*_{\text{sing}}$ contains the following four points from the intersections of any three of the four hyperplanes H_i:

$$
\begin{aligned}
u_*^{(1)} &= H_2 \cap H_3 \cap H_4 = \left(\frac{1+R}{2} z,\ \frac{1}{2} z,\ \frac{R}{2} z \right), \\
u_*^{(2)} &= H_1 \cap H_3 \cap H_4 = \left(-\frac{1-R}{2} z,\ \frac{1}{2} z,\ \frac{R}{2} z \right), \\
u_*^{(3)} &= H_1 \cap H_2 \cap H_4 = \left(-\frac{1-R}{2} z,\ -\frac{1}{2} z,\ \frac{R}{2} z \right), \\
u_*^{(4)} &= H_1 \cap H_2 \cap H_3 = \left(-\frac{1-R}{2} z,\ -\frac{1}{2} z,\ -\frac{2-R}{2} z \right).
\end{aligned}
\tag{4.34}
$$

Given the FI parameter $\zeta \in \mathfrak{h}^*$, it belongs to a cone generated by three of the four Q_i's. The Jeffrey-Kirwan residue only receives contribution from the intersection of the three corresponding hyperplanes H_i. For example, we can write the Jeffrey-Kirwan residue operation in the $\text{Cone}(Q_2, Q_3, Q_4)$ chamber as

$$
\begin{aligned}
&\underset{u=u_*^{(1)}}{\text{JK-Res}} \left(\{Q_2,Q_3,Q_4\}, \zeta\right) \frac{du_1 \wedge du_2 \wedge du_3}{\left(-u_1+u_2+\frac{R}{2}z\right)\left(-u_2+u_3+\frac{1-R}{2}z\right)\left(-u_3+\frac{R}{2}z\right)} \\
=&(-1)^3 \left(\frac{1}{2\pi i}\right)^3 \oint_{u_3=\frac{R}{2}z} du_3 \oint_{u_2=u_3+\frac{1-R}{2}z} du_2 \oint_{u_1=u_2+\frac{R}{2}z} du_1 \\
&\times \frac{1}{\left(-u_1+u_2+\frac{R}{2}z\right)\left(-u_2+u_3+\frac{1-R}{2}z\right)\left(-u_3+\frac{R}{2}z\right)}.
\end{aligned}
\tag{4.35}
$$

The index in the four chambers are

$$
\Omega(y,\zeta) = \begin{cases} 0 & \text{if } \zeta \in \text{Cone}(Q_2,Q_3,Q_4), \\ y+y^{-1} & \text{if } \zeta \in \text{Cone}(Q_1,Q_3,Q_4), \\ 0 & \text{if } \zeta \in \text{Cone}(Q_1,Q_2,Q_4), \\ y+y^{-1} & \text{if } \zeta \in \text{Cone}(Q_1,Q_2,Q_3). \end{cases}
\tag{4.36}
$$

In terms of the components of the FI parameters, the chambers can be described as

$$\Omega(y,\zeta)=\begin{cases}0 & \text{if } \zeta_4>0,\ \zeta_1<0,\ \zeta_1+\zeta_2<0,\\ y+y^{-1} & \text{if } \zeta_1>0,\ \zeta_2<0,\ \zeta_2+\zeta_3<0,\\ 0 & \text{if } \zeta_2>0,\ \zeta_3<0,\ \zeta_1+\zeta_2>0,\\ y+y^{-1} & \text{if } \zeta_3>0,\ \zeta_4<0,\ \zeta_2+\zeta_3>0.\end{cases} \tag{4.37}$$

This agrees with the expectation (4.29) and provides a complete picture of the walls of marginal stability where the W-boson decays.

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GEOMETRY AND TOPOLOGY OF STRING JUNCTIONS

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Abstract. We study elliptic fibrations by analyzing suitable deformations of the fibrations and vanishing cycles. We introduce geometric string junctions and describe some of their properties. We show how the geometric string junctions manifest the structure of the Lie algebra of the Dynkin diagrams associated to the singularities of the elliptic fibration. One application in physics is in F-theory, where our novel approach connecting deformations and Lie algebras describes the structure of generalized type IIB seven-branes and string junction states which end on them.

1. Introduction

An elliptic fibration is a morphism $\pi : X \to B$ such that $\pi^{-1}(p) = E_p$ for a general point $p \in B$ is a smooth elliptic elliptic curve (a torus T^2); the discriminant locus is

$$\Sigma = \{q \in B \text{ such that } \pi^{-1}(q) \neq T^2\}.$$

In this paper we take X and B to be smooth; if π is also assumed to have a section σ, X is the (smooth) resolution of the Weierstrass model W of the fibration [24] with Gorenstein singularities. W is defined by the Weierstrass equation: $y^2z-(x^3+fxz^2+gz^3)=0$, where f, g are sections of appropriate bundles on B. If $\dim W = 2$, the singularities are the rational double points. It was noted by Du Val and Coxeter [8, 5] that rational double points are classified by the Dynkin diagrams of the simply laced Lie algebras $\mathfrak{g}$ of type $\mathfrak{a}_n, \mathfrak{d}_n, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$: in fact, the dual diagram of the exceptional divisors in the minimal resolution is one of the above Dynkin diagrams. In the case of higher dimensional elliptic fibrations also non simply-laced Dynkin diagram can occur.

We study elliptic fibrations by analyzing suitable deformations of the fibrations; we introduce "geometric string junctions". String junctions were defined in the physics literature by DeWolfe, Gaberdiel and Zwiebach [11, 6] as equivalence relations of closed paths in a punctured disc $\mathbb{C} \setminus \Sigma$; the disc is the base of an elliptic fibration with discriminant locus Σ. The authors assign composition rules and show that the junctions reflect the structure of exceptional gauge algebras of the elliptic fibration. The gauge algebras which arise in this construction are the simply laced ones $\mathfrak{a}, \mathfrak{d}, \mathfrak{e}$. Bonora and Savelli [3] later derived some non-simply laced algebras from junctions; their construction is algebraic and not expressed in terms of the geometry of higher dimensional elliptic fibrations. We will do this later in the paper. The techniques presented in this paper have a number of applications in physics; for example, in describing BPS states of $d = 4$ $\mathcal{N} = 2$ supersymmetric gauge theories, or in F-theory where they provide a direct approach to the analysis of generalized type IIB seven-branes and the string junction states which end on them [29, 22, 23, 7].

In Section 2 we consider a smooth elliptic surface on the open unit disc U with nodal singular fibers over a collection of points $\{q_j\}$. We then consider suitable, disjoint, paths in U from a base point to each q_j (a junction, in the physics language) and the corresponding vanishing cycles and construct thimbles, the prongs in the physics language, in the relative homology. A general *geometric junction* J defines then a chain with boundary in the elliptic fiber over the base point p. Following the physics literature we define the asymptotic charge $a(J) \in H_1(E_p, \mathbb{Z})$, $a(J) = \partial[J]_p \in H_1(E_p)$. We show that the junctions with asymptotic charge zero are the images of spherical classes in $[J] \in H_2(X)$ (Theorems 2.5, 2.7). We then define a self-intersection product in the space of junctions, and we show that if $a(J) = 0$ the topological intersection is equal to the the self-intersection product (Theorem 2.9). We also define an intersection pairing $\langle J, K\rangle$, which we show it coincides with the topological pairing for junctions of zero asymptotic charge. We provide an explicit formula in terms of the J_i and the intersection numbers of the vanishing cycles. If $X = W$ is a Weierstrass model we also provide an explicit alternative descriptions of the class of junctions, which is implemented in a computer program in [28], [25]. In Section 3 we consider a smooth elliptic surface in Weierstrass equation over a disc, with a unique singular fiber over the origin, an ADE singularity. Klein showed that resolutions and deformation of ADE surface singularities (also known as kleinan singularities) are diffeomorphic. We deform the elliptic fibration to a smooth fibration with nodal fibers, namely we perform a complete Higgsing of the Weierstrass model. We study the junctions in the deformed model and we prove that the lattice structure found in the previous Section 2 provides the weight structure of the $\mathfrak{a}, \mathfrak{d}, \mathfrak{e}$ algebras associated to the Dynkin diagram of the original singularities. As a particular case, we obtain the roots of the $\mathfrak{a}, \mathfrak{d}, \mathfrak{e}$ central singularity and the associated Cartan matrix from the junctions with asymptotic charge zero. Our deformation analysis of the surface case show that the junctions with a fixed non-zero asymptotic charge are associated to weights of other representations and all the possible weights occur; we show that in higher dimensional variety these weights can become associated to junctions of non-zero asymptotic charge and assume geometric meaning, they become massless in the physics language [16, 13]. In contrast analysis of the resolved surface provides only the root structure.

In addition we show that the deformation analysis distinguish the Kodaira type III (two tangent rational curves) from I_2 (a cycle of two rational curves), which are associated to the $\mathfrak{su}(2)$ gauge algebra, and the Kodaira type IV (three rational curves meeting at point) from I_3 (a cycle of three rational curves, which are associated to the $\mathfrak{su}(3)$ gauge algebra. This reflects in physics the different brane structure of the two singularities. These results were first presented in our previous papers [13, 16], and were obtained with the aid of a computer package especially developed [25]. In [13] we show also that the local deformation techniques of the string junction analysis can be adapted in compact cases, even in cases when global deformation do not exist and also in higher dimension. In Section 4 we revisit the example of the $\mathfrak{g}_2$ algebra first presented in [16], and elliptic fibration of threefolds.

The techniques developed in Section 2 do not assume the existence of a section of the fibration, and in principle can be applied also in that case. Resolutions to a smooth model with trivial canonical class and an equidimensional fibration may not be available in higher dimension, minimal models can have terminal singularities, however the deformation analysis can still be performed. We will address such situations in a upcoming paper [14]. Our techniques can also be extended to other type of fibrations between varieties which are not necessarily algebraic, for example on varieties with G_2 holonomy,

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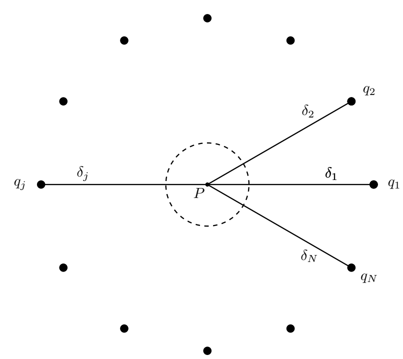
2. Geometric String Junctions

Consider the smooth elliptic fibration $\pi : X \longrightarrow U$ over an open set $U \subset \mathbb{C}$ with I_1 singular fibers above $\Sigma = \{q_1 \ldots, q_N\}$. Fix a base point $p \in B \setminus \Sigma$ with E_p the elliptic fiber $\pi^{-1}(p)$. Choose a set of continuous embedded paths $\delta_1, ..., \delta_N$, assumed disjoint except for the common starting point $\delta_j(0) = p$, ending at the points $\delta_j(1) = q_j$. Also assume the order is such that the points in which the paths meet a small circle around p go around counter clockwise; for example for a small r, $\delta_j(r) = re^{2\pi i \frac{(j-1)}{N}}$ in suitable local coordinates around q as the origin. We also assume that for ϵ a small real number, there is also the formula $\delta_j(1-\epsilon) = \epsilon\, e^{2\pi i \theta_j}$, θ_j an "angle" in suitable local coordinates around q_j as the origin. (The local co-ordinates around q_j could of course be chosen so $\theta_j = 0$, but below we will use other paths coming in to q_j at different angles.)



Any smooth fiber bundle $Y \to S^1$ over a circle is given by a monodromy of the fiber F. This means there is a diffeomorphism [27]. In our case, over the circle of radius ϵ around q_j, , the corresponding diffeomorphism $\psi_{j,1-\epsilon} : E_{\delta_j(1-\epsilon)} \to E_{\delta_j(1-\epsilon)}$ will be referred to as the *local monodromy* around q_j .

The assumption that the singular fibers are of type I_1 means that there is a real curve $C_{j,1-\epsilon}$ on the fiber $E_{\delta_j(1-\epsilon)}$, ϵ small, which the local monodromy $\psi_{j,1-\epsilon}$ fixes, where:

$$\psi_{j,1-\epsilon} : E_{\delta_j(1-\epsilon)} \to E_{\delta_j(1-\epsilon)}.$$

This curve collapses to a point $\hat{q}_j$ as $\epsilon \to 0$; $\hat{q}_j$ is the (nodal) singular point in the singular fiber $\pi^{-1}(q_j)$. With any choice of orientation for this curve, the map induced on first homology, $(\psi_{j,1-\epsilon})_* : H_1(E_{\delta_j(1-\epsilon)}) \to H_1(E_{\delta_j(1-\epsilon)})$, satisfies [1], [2].

$$(\psi_{j,1-\epsilon})_*(x) = x - (x \cdot [C_{j,1-\epsilon}])\,[C_{j,1-\epsilon}] \qquad , \qquad x \in H_1(E_{\delta_j(1-\epsilon)})\,. \tag{1}$$

The equation is a special case of the Picard-Lefshetz formula for this situation. Here $x \cdot [C_{j,1-\epsilon}]$ is the intersection number of x with the homology class $[C_{j,1-\epsilon}]$ of the curve $C_{j,1-\epsilon}$.

Fix a small ϵ_0 . The fibration π is trivial over the contractible set $\delta_j([0, 1-\epsilon_0])$; let

$$\Psi_j : [0, 1-\epsilon_0] \times E_{\delta_j(1-\epsilon_0)} \to \pi^{-1}(\delta_j([0, 1-\epsilon_0])) \tag{2}$$

be a trivialization with $\Psi_j(1-\epsilon_0, z) = z$. Then we define the vanishing cycle $\gamma_j \in H_1(E_p)$ as

$$\gamma_j = (\Psi_j | \{0\} \times E_{\delta_j(1-\epsilon_0)})_* [C_{j,1-\epsilon_0}]\,. \tag{3}$$

This is the same as the homology class $[C_j]$ of the curve $C_j = \Psi_j|\{0\} \times E_{\delta_j(1-\epsilon_0)}(C_{j,1-\epsilon_0}) \subset E_p$ that is is the image of the curve $C_{j,1-\epsilon_0} \subset E_{\delta_j(1-\epsilon_0)}$, and we also set

$$C_{j,t} = \Psi_j|\{t\} \times E_{\delta_j(1-\epsilon_0)}(C_{j,1-\epsilon_0}), \tag{4}$$

so $C_j = C_{j,0}$. The class γ_j is only defined up to sign, but we will systematically suppress this ambiguity (however see Corollary 2.2 below).

Finally, we can use the diffeomorphism $\Psi_j|\{0\} \times E_{\delta_j(1-\epsilon_0)}$ of $E_{\delta_j(1-\epsilon_0)}$ to E_p to tranfer the local monodromy at q_j to a *global monodromy* $\psi_i : E_p \to E_p$ that fixes C_j and also satisfies the Picard-Lefshetz formula 1. The topological type of the fibration is determined by the isotopy classes of the global monodromies [15].

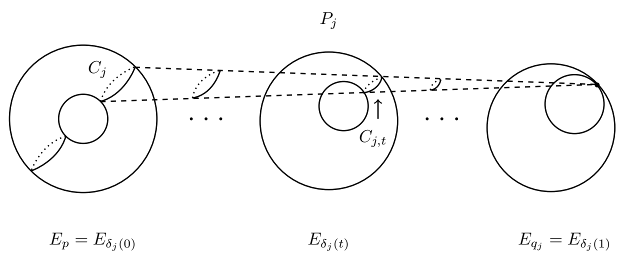
We define the "prong" (physics terminology) or "thimble" (symplectic geometry terminology)

$$\Gamma_j = \Psi_j([0, 1-\epsilon_0] \times C_{j,1-\epsilon_0}) \cup \bigcup_{0<\epsilon\leq\epsilon_0} C_{j,1-\epsilon} \cup \{\hat{q}_j\}, \tag{5}$$

and more generally we will need to use

$$\Gamma_{j,t} = \Psi_j([t, 1-\epsilon_0] \times C_{j,1-\epsilon_0}) \cup \bigcup_{0<\epsilon\leq\epsilon_0} C_{j,1-\epsilon} \cup \{\hat{q}_j\}. \tag{6}$$

The prong is a disk and represents a class $[\Gamma_j] \in H_2(X, E_p)$ with $\partial[\Gamma_j] = \gamma_j$.



We have the following alternate description when there is a Weierstrass equation:

Proposition 2.1. [15] *Let $X = W$ have the Weierstrass equation*

$$zy^2 = x^3 + f\,xz^2 + g\,z^3 \tag{7}$$

with section σ. Then $E_q - \sigma(q)$ is the two-fold branched cover of $\mathbb{C}$ branched at the roots of $x^3 + f(q)x + g(q) = 0$. For $0 \leq t \leq 1$, let $\rho_{j,1}(t)$ and $\rho_{j,2}(t)$ be continuous paths of two of the roots at $\delta_j(t)$, with the property $\rho_{j,1}(1) = \rho_{j,2}(1)$ at the singular point q_j Let $\rho_{j,3}(t)$ be the path of the remaining root, and assume that for all $0 \leq s, t \leq 1$, $\rho_{j,3}(s) \neq \rho_{j,1}(t)$ and $\rho_{j,3}(s) \neq \rho_{j,2}(t)$. Let $\bar{C}_{j,t}$ be the closed loop (not necessarily embedded) in $E_{\delta_j(t)}$ that lies over the path $\rho_{j,t} = \{\rho_{j,1}(s), \rho_{j,2}(s) | t \leq s \leq 1\}$. Then these loops have a consistent orientations so that if $\bar{\Gamma}_j = \bigcup_{0\leq t\leq 1} \bar{C}_{j,t}$, then $[\bar{\Gamma}_j] = [\Gamma_j]$; in particular $\partial[\bar{\Gamma}_j] = \gamma_j$.

The proof of the above proposition, presented in [15], also provides the following algorithm for determining vanishing cycles:

Corollary 2.2. *Assume, in addition to the hypotheses of Proposition 2.1 (for simplicity) the roots ρ_1, ρ_2, ρ_3 of $x^3 + f(p)x + g(p) = 0$, the first two being the ones that merge at q_j, are not on a common (real) line. Let m_1 be the number of times $\bar{\rho}_{j,0}$ crosses from one side of the interior of the straight line joining $\rho_1 = \rho_{j,1}(0)$ and $\rho_2 = \rho_{j,1}(0)$ to the other. Let m_2 be one half the sum of the intersection numbers of the path $\rho_{j,0}$ (with either orientation) and this straight line* *at the endpoints. (If an intersection at an endpoint is not transverse, make it so by a small perturbation and count any additional crossings of the line that this produces in m_1 as well.) Let $Z_1, Z_2, Z_3 \in H_1(E_p)$ be represented by loops that are inverse images in E_p of straight lines joining ρ_1 and ρ_2, ρ_2 and ρ_3, and ρ_3 and ρ_1. If $m_1 + m_2$ is even then $\gamma_j = \pm Z_1$. If $m_1 + m_2$ is odd, then $\gamma_j = \pm Z_2 \pm Z_3$, with any choice of signs so that $\gamma_j \cdot Z_1 \neq 0$.*

In [13, 16] and in the examples in this paper we apply Proposition 2.1 and the algorithm in simple special cases.

For the rest of this section we do not assume the existence of a Weierstrass model. Nevertheless, it can be shown [15] that there exists a *topological* section $\sigma : U \to X$. We will not give a proof here, but the reason is that the global monodromy maps, which determine the topological type of the fibration, are isotopic to maps with a common fixed point.

Definition 2.3. *As above, consider the smooth elliptic fibration $\pi : X \longrightarrow U$ over an open set $U \subset \mathbb{C}$ with I_1 singular fibers above $\Sigma = \{q_1 \ldots, q_N\}$ and corresponding vanishing cycles $\{\gamma_1, \ldots, \gamma_N\}$. Fix a base point $p \in B \setminus \Sigma$ with $E_p = \pi^{-1}(p)$.*
$J = (J_1, \ldots, J_N) \in \mathbb{Z}^N$ is a **junction** *and the cycle $a_p(J) = a(J) = \sum_1^N J_i \gamma_i \in H_1(E_p, \mathbb{Z})$ is its* **asymptotic charge**.

Remark 2.4. A junction defines a chain (actually the image of a union of 2-disks) $\sum_1^N J_j \Gamma_j$ or $\sum_1^N J_j \bar{\Gamma}_j$ in X, and hence a homology class

$$[J]_p = \sum_1^N J_j [\Gamma_j] \in H_2(X, E_p)\,, \tag{8}$$

(This homology class actually depends on the order of singular points, up to a cylcic permutation of order N ..)

Clearly $a(J) = \partial [J]_p \in H_1(E_p)$; hence if $a(J) = 0$, $[J]_p$ will be the image of a class in $[J] \in H_2(X)$; it is only well defined up to a multiple of the image of orientation class $[E_p]$ of E_p in $H_2(X)$. It will be unique subject to the extra condition that $[J] \cdot \sigma(U) = 0$; this intersection number is well defined because the image of the section is a proper submanifold. If $a(J) = 0$, then from the explicit construction one can see that $[J]$ is spherical, i.e. represented by a map $S^2 \to X$. The class $[J]$ also depends on the basepoint p and the choice of paths.

Theorem 2.5. *Let $\mathbf{J}$ denote the abelian group of junctions. Assume U is the interior of a region bounded by a closed embedded smooth curve. Then $J \mapsto [J]_p = \sum_1^N J_j [\Gamma_j] \in H_2(X, E_p)$ induces an isomorphism*

$$\mathbf{J} \cong H_2(X, E_p) \tag{9}$$

and $J \mapsto [J]$ induces an isomorphism

$$\{J \in \mathbf{J} | a(J) = 0\} \cong H_2(X)/\mathbb{Z}[E_p] \cong \{x \in H_2(X) \,|\, x \cdot \sigma(U) = 0\}\,. \tag{10}$$

Remark 2.6. The hypothesis on U can be weakened considerably at the cost of added complications in the proof below.

Proof. Recall from above that there exists a topological section $\sigma : U \to X$, which gives a splitting of the first map of the long exact homology sequence of a pair [9] :

$$.... \to H_2(E_p) \to H_2(X) \to H_2(X, E_p) \to H_1(E_p) \to ... \tag{11}$$

since $[E_p] \cdot \sigma(U) = 1$. The statement in (9) follows then from (10) and the long exact sequence above.

The hypothesis on U implies that there is a topological extension of π to $\overline{\pi} : \overline{X} \to \overline{U} \subset \mathbb{C}$, also a fiber bundle outside the points $\{q_1, ..., q_N\}$. There is a well defined topological intersection pairing [26][4][18]

$$H_2(X, E_p) \times H_2(\overline{X} - E_p, \overline{X} - X) \to \mathbb{Z}\,. \tag{12}$$

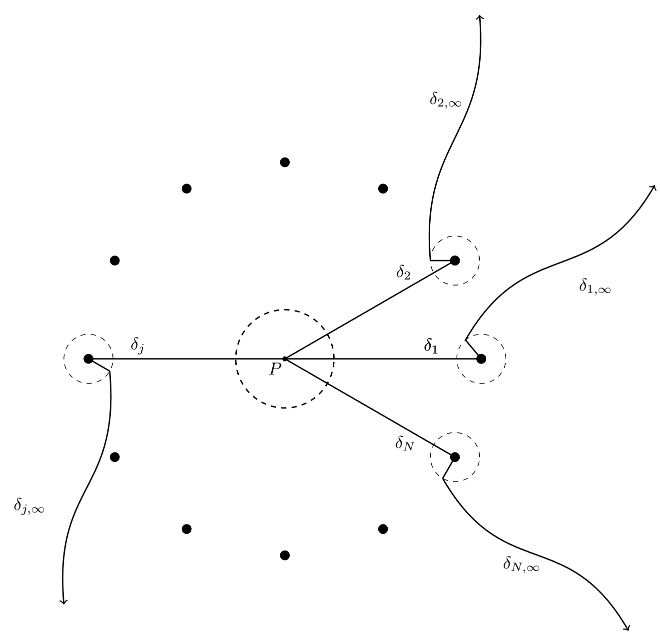
(Here is an intuitive geometric definition: Let $(A, \partial A)$ and $(B, \partial B)$ be oriented relative chains representing α and β in these groups. Since $\partial A \cap \partial B = \emptyset$, after an arbitrarily small deformation fixing the boundaries, it can be assumed A and B are in "general position", meaning that they intersect transversely in points in the interior of simplices. The intersection number $\alpha \cdot \beta$ will then be the number of these points, counted sign determined by the orientations.)

Possibly choosing ϵ_0 above smaller, let $\hat{\delta}_j : [1 - \epsilon_0] \to U$ be a small path, disjoint from δ_j except at the endpoint $\delta_j(1) = q_j = \hat{\delta}_j(1)$, where the two paths meet in one point. For example, in local coordinates as above around q_j, take $\hat{\delta}_j(1 - \epsilon) = \epsilon e^{2\pi i \hat{\theta}_j}$ for some angle $\hat{\theta}_j \neq \theta_j$.

Let $\hat{C}_{j,1-\epsilon}$, $0 < \epsilon \leq 1$, be the corresponding vanishing cycle. Let

$$\hat{\Gamma}_{j,\epsilon_0} = \bigcup_{0<\epsilon\leq\epsilon_0} \hat{C}_{j,1-\epsilon} \cup \{\hat{q}_j\}\,. \tag{13}$$

be the corresponding local prong or thimble. Let $\delta_{j,\infty}$ be a path from $\hat{\delta}_j(1 - \epsilon_0) = \delta_{j,\infty}(1 - \epsilon_0)$ to a point $\delta_{j,\infty}(0) \in \overline{X} - X$, disjoint from the paths δ_k.



Let

$$\Gamma_{j,\infty} = \Psi_{j,\infty}([0, 1 - \epsilon] \times \hat{C}_{j,1-\epsilon_0}) \cup \bigcup_{0<\epsilon\leq\epsilon_0} \hat{C}_{j,1-\epsilon} \cup \{\hat{q}_j\} = \Psi_{j,\infty}([0, 1 - \epsilon] \times \hat{C}_{j,1-\epsilon_0}) \cup \hat{\Gamma}_{j,\epsilon_0}\,, \tag{14}$$

be the corresponding prong, $\Psi_{j,\infty}$ a trivialization of $\overline{\pi}|\overline{\pi}^{-1}\delta_{j,\infty}([0, 1 - \epsilon_0])$.

Then

$$\Gamma_{j,\infty} \cap \Gamma_j = \{q_j\} \tag{15}$$

transversely and

$$\Gamma_{j,\infty} \cap \Gamma_k = \emptyset \tag{16}$$

for $k \neq j$. Therefore the classes $[\Gamma_{j,\infty}] \in H_2(\overline{X} - E_p, \overline{X} - X)$ and $[\Gamma_k] \in H_2(X, E_p)$ satisfy

$$[\Gamma_j] \cdot [\Gamma_{j,\infty}] = \pm 1 \qquad ; \qquad [\Gamma_k] \cdot [\Gamma_{j,\infty}] = 0 \qquad k \neq j\,. \tag{17}$$

(The non-zero intersection number is actually -1; this will be discussed in more detail in the proof of Theorem 2.9.) Therefore the map $J \mapsto [J]_p$ induces an isomorphism of $\mathbf{J}$ onto a summand of $H_2(X, E_p)$ of rank N .

If $Z \to S^1$ is a smooth fiber bundle over a circle with fiber F , then there is a monodromy $\phi : F \to F$ such that Z is diffeomorphic to the mapping cylinder $F \times [0,1]/(x,0) \sim (\phi(x),1)$ via a diffeomorphism that carries the bundle projection to the map to map $(f,t) \mapsto e^{2\pi i t}$. It follows from the Mayer Vietoris sequence [9](or a spectral sequence argument [19]) that there is long exact sequence

$$... \to H_k(F) \overset{\phi_*-1}{\to} H_k(F) \to H_k(E) \to H_{k-1}(F) \to ... \tag{18}$$

Further, the fiber bundle has a section $\sigma : S^1 \to Z$ if and only if ϕ has a fixed point.

We apply this to $\pi^{-1}B_j(\epsilon_0)$, the ball of radius ϵ_0 in the local coordinates around q_j . The boundary of this manifold is a bundle over the circle $\partial B_j(\epsilon_0)$, with torus fiber. The monodromy is the the map $\psi_{i,1-\epsilon}$. It follows that $H_1(\pi^{-1}\partial B_j(\epsilon_0)) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by the orientation class of a fiber and $[C_{j,1-\epsilon_0} \times \sigma(\partial B_j(\epsilon_0))]$, and $H_2(\pi^{-1}\partial B_j(\epsilon_0)) \cong \mathbb{Z} \oplus \mathbb{Z}$ generated by any class τ with $\tau \cdot \gamma_j = 1$. and the class $[\sigma(\partial B_j(\epsilon_0))$. However $\pi^{-1}B_j(\epsilon_0)$ collapses homotopically equivalently to the singular fiber over E_{q_j} , whose second homology is generated by the image of the orientation class under the collapse, and whose first homology by the image of τ , i.e. the collapse of a class represented by a curve D_1 that meets $C_{j,1-\epsilon}$ in one point. Also, $\sigma(\partial B_j(\epsilon_0)) = \partial\sigma(B_j(\epsilon_0))$. Hence $H_2(\pi^{-1}B_j(\epsilon_0), \pi^{-1}\partial B_j(\epsilon_0)) \cong \mathbb{Z}$, generated by the homology class of $\sigma(B_j(\epsilon_0), \partial B_j(\epsilon_0))$.

Let $X_0 = \pi^{-1}U_0$, $U_0 = U - \bigcup_1^N B_j(\epsilon_0)^\circ$. We now claim the map

$$H_2(X, E_p) \to H_2(X, X_0) \cong \bigoplus_1^N H_2(\pi^{-1}B_j(\epsilon_0), \pi^{-1}\partial B_j(\epsilon_0)) \tag{19}$$

is trivial. In fact, the image of the connecting homomorphism in the long exact homology sequence of the triple (X, X_0, E_p) is generated by the classes

$$[\sigma(\partial B_j(\epsilon_0))] \in H_1(X_0, E_p), \quad \pi_*[\sigma(\partial B_j(\epsilon_0))] = [\partial B_j(\epsilon_0))] \in H_1(U_0, p)\,,$$

and for $j = 1, ..., N$ these classes form a basis of this group. Therefore the connecting homomorphism is one-to-one.

More precisely, the inclusion

$$\bigcup_1^N \delta_j([0, 1-\epsilon_0]) \cup B_j(\epsilon_0)) \hookrightarrow U_0 \tag{20}$$

is a homotopy equivalence, and hence so is

$$\pi^{-1}\bigcup_{1}^{N}\delta_j([0,1-\epsilon_0])\cup B_j(\epsilon_0))\hookrightarrow X_0 \tag{21}$$

It follows by excision that

$$H_2(X_0,E_p)\cong H_2(X_0,\pi^{-1}\bigcup_{1}^{N}\delta_j([0,1-\epsilon_0])\cong\bigoplus_{1}^{N}H_2(\pi^{-1}\partial B_j(\epsilon_0)),E_{\delta_j(1-\epsilon_0)})\,. \tag{22}$$

Also by excision, for any bundle $Z\to S^1$ as discussed above, the quotient map induces and isomorphsim $H_i(F\times[0,1],F\times\{0,1\})\to H_i(Z,F)$. (This is part of one proof of the above long exact sequence.) Therefore, in our case $H_2(\pi^{-1}\partial B_j(\epsilon_0)),E_{\delta_j(1-\epsilon_0)})\cong\mathbb{Z}\oplus\mathbb{Z}$ and one of the generators is represented by the closed class (i.e. in the image of the absolute homology) represented by the torus $[C_{j,1-\epsilon_0}\times\sigma(\partial B_j(\epsilon_0))]$. This class obviously maps to zero in $H_2(X,E_p)$; therefore this group is a quotient of a free abelian group of rank N. Therefore, $J\mapsto[J]_p$, already shown to be a one-to-one map onto a summand, is an isomorphism. □

We now outline how our construction also provides some topological information about the representation of homology classes in $H_2(X)$ by embedded spheres:

Theorem 2.7. *(See also [16]) Let $J=(J_1,...,J_N)\in\mathbf{J}$ with $a(J)=0$ and $|J_i|\le 2$. Then $[J]\in H_2(X)$ is represented by a smoothly embedded 2-sphere $S^2\subset X$.*

Proof. (Outline) As mentioned above, the construction of prongs and the proof of 2.5 can be used to provide an explicit geometric cycle representing $[J]$, J a junction with $a(J)=0$. Suppose first $|J_i|\le 1$, $1\le i\le N$. The cycle representing $[J]$ can be then described as a union of some of the (oriented) prongs $\pm\Gamma_{j,t}$ for a small value of t, together with a punctured sphere (a sphere with some disks removed), in a product neighborhood $E_0\times D^2$ of the smooth fiber, that bounds the union of the bounding curves $\pm C_{j,t}$. The construction of the punctured disk is indicated [16], and, as explained there, the union of the prongs and the punctured disk is a smoothly embedded S^2. This proves the case with $|J_i|\le 1$. For $|J_j|=\pm 2$ we need to consider $\Gamma_{j,t}$ and a parallel copy $\hat{\Gamma}_{j,t}$ of the corresponding prong, using a slight deformation of the path δ_j in the base. This construction is described in more detail below in the proof of the next theorem. The prong and the deformed prong can then be added to one another in a way that eliminates the intersection point at the singularity of E_j and provides an annulus bounding $\pm C_{j,t}\cup\pm\hat{C}_{j,t}$. These annuli, the oriented prongs with coefficients ±1, and the punctured sphere bounding the union of the vanishing cycles in the different nearby smooth fibers again provide a smoothly embedded S^2 representing $[J]\in H_2(X)$. □

This result is similar in spirit to the (simpler) topological fact that in $H_2(\mathbb{P}^2)$ a generator or twice a generator can be represented by a smoothly embedded S^2 [17]; however, in X these representatives are not algebraic. As is the case with $\mathbb{P}^2$ [10], it appears that every element of $H_2(X)$ can be represented by an topologically embedded S^2 with a non-locally smoothable point. It would be interesting the determine the minimal genus of a smoothly embedded closed (oriented) real 2-manifold representing a given general element of $H_2(X)$.

Definition 2.8. *Consider a junction J. Define a self-intersection*

$$\langle J,J\rangle=-\sum_{k>j\ge 2}J_kJ_j\,\gamma_k\cdot\gamma_j-\sum_{j=1}^{N}J_j^2 \tag{23}$$

Theorem 2.9. *Let Jbe a junction with $a(J) = 0$. Then the topological intersection number satisfies*

$$[J] \cdot [J] = \langle J, J \rangle\,. \tag{24}$$

Remark 2.10. Different formulae are obtained by cyclically permuting the indices, i.e. rotating the small circle around p. All these satisfy the conclusion of the Theorem. In fact, let $\hat{p}$ be a point near p that lies between the paths δ_1 and δ_2. Then there will be defined a small "deformation" of the paths δ_j, $j = 1, ..., N$, to paths $\hat{\delta}_j$ from $\hat{p}$ to the points q_j, which will be described in the proof. The corresponding prongs will actually represent classes $[J]_p \in H_2(X - E_{\hat{p}}, E_p)$ and $[J]_{\hat{p}} \in H_2(X - E_p, E_{\hat{p}})$. There is also a well defined topological intersection pairing on these groups, and it will be shown that

$$[J]_p \cdot [J]_{\hat{p}} = \langle J, J \rangle\,. \tag{25}$$

Thus the above formula for classes which do not have asymptotic charge zero has a topological interpretation in terms of a deformation of the prongs, and all the possible formulas correspond to the different possible small deformations, up to a suitable notion of homotopy.

Remark 2.11. The missing index $j = 1$ in the first term on the right side of 23 corresponds to the fact that in the deformation used below (see the figure in the proof), since the deformed basepoint $\hat{p}$ lies between between δ_1 and δ_2, we can reach q_1 with a path near δ_1 that does not intersect δ_1 except at q_1. We could also reach q_2 without crossing δ_2 but on the opposite side of δ_2 from that indicated in the figure, leading to a change in the formula in the corresponding place in the second term.

Remark 2.12. An intersection pairing is defined by

$$\langle J, K \rangle = \frac{1}{2}(\langle J + K, J + K \rangle - \langle J, J \rangle - \langle K, K \rangle)\,. \tag{26}$$

It follows that

$$[J] \cdot [K] = \langle J, K \rangle\,. \tag{27}$$

if $a(J) = a(K) = 0$.

Proof. (of Theorem 2.9) We start by determining the sign in equation (17). We keep the same notation. It was shown in the proof of Theorem 2.5 that $H_2(\pi^{-1}B_j(\epsilon_0), E_{\delta_j(1-\epsilon_0)})$ is infinite cyclic and the connecting homomorphism to $H_1(E_{\delta_j(1-\epsilon_0})$ is injective (with image generated by the class of the local vanishing cycle.) In the general fiber bundle $Z \to S^1$ with monodromy ϕ as in the previous proof, let w be a $k-1$ cycle of F. Then the image of the relative $F \times \{0, 1\}$ cycle $w \times [0, 1]$ in the quotient Z represents an element $S(w) \in H_k(Z, F)$ with

$$\partial S(w) = (\phi_*[w] - [w]) \in H_1(F)\,.$$

In our case, take for w a curve with

$[w] \cdot [C_{j,1-\epsilon_0}] = -1$. Then by 1, $\partial S(w) = [C_{j,1-\epsilon_0}] \in H_1(E_{\delta_j(1-\epsilon_0)})$. Let

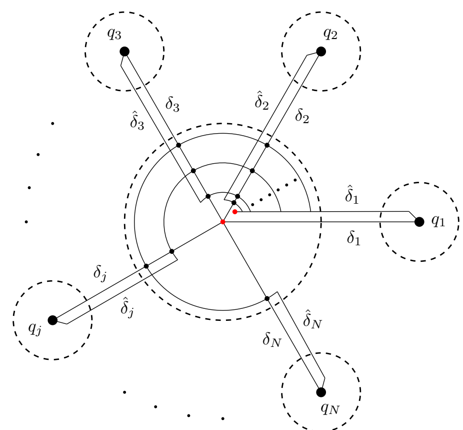
$$i_* : H_2(\pi^{-1}\partial B_j(\epsilon_0), E_{\delta_j(1-\epsilon_0)}) \to H_2(\pi^{-1}B_j(\epsilon_0), E_{\delta_j(1-\epsilon_0)}) \tag{28}$$

be the map induced by inclusion. Then $\partial i_* S(w) = [C_{j,1-\epsilon_0}]$ also, by naturality. Therefore $i_* S(w) = [\Gamma_{j,\epsilon_0}]$, the class of the local "prong" corresponding to the restriction of δ_j to $[1-\epsilon_0, 1]$. The intersection pairing is also defined between elements of $H_2(X, E_{\delta_j(1-\epsilon_0)})$, to which the target of i_* maps by another inclusion induced map, and elements of $H_1(\overline{X} - E_{\delta_j(1-\epsilon_0)}, \overline{X} - X)$, in which $\Gamma_{j,\infty}$ represents an element. Therefore

$$[\Gamma_{j,\epsilon_0}] \cdot [\Gamma_{j,\infty}] = S(w) \cdot [\hat{\Gamma}_{j,\infty}] = [w \times \{\hat{\theta}_j\}] \cdot [\hat{C}_{j,1-\epsilon_0}] = [w] \cdot [C_{j,1-\epsilon_0}] = -1 \tag{29}$$

It follows from the local nature of intersection numbers that the local prongs Γ_{j,ϵ_0} and $\hat{\Gamma}_{j,\epsilon_0}$ meet with intersection number -1.

Let $\hat{p}$ be near p, lying between the paths δ_1 and δ_2,; for example, take $\hat{p} = \hat{r}e^{2\pi\eta_1}$ in local the local coordinates near p, $\hat{r} < r_0$ and $0 < \eta_1 < N^{-1}$ both small; we assume for $0 \le r \le r_0$, $\delta_j(r) = re^{\frac{2\pi i}{N}}$ in the local coordinates around p, as above. Define paths $\hat{\delta}_j$ from $\hat{p}$ to q_j in four parts as follows: First take a straight line from $\hat{q}$ to $r_j\, e^{2\pi i\eta}$, $\hat{r} < r_1 < r_2 < .... < r_j < r_0$. Follow this with a circular arc $r_j e^{2\pi i t}$, $\eta \le t \le \frac{j-1}{N} + \eta$. Parameterize what has been done so far so that $\hat{\delta}_j(0) = \hat{p}$ and $\hat{\delta}_j(r_j) = r_j e^{2\pi i(\frac{j-1}{N}+\eta_j)}$. Then follow with paths parallel to δ_j, until we reach $B_j(\epsilon_0)$ at a point $\epsilon_0 e^{2\pi i\hat{\theta}_j}$ in local coordinates around q_j, and then follow the straight line to q_j.



Let $\hat{\Gamma}_j$ be the corresponding prongs. For $r \le r_0$,

$$i_r : H_2(X, E_{\delta_j(r)}) \to H_2(X, \pi^{-1}B_r(p)) \qquad \hat{i}_r : H_2(X, E_{\hat{\delta}_j(r)}) \to H_2(X, \pi^{-1}B_r(p)) \tag{30}$$

be the maps induced by inclusion; these are isomorphisms (as the fiber bundle is trivial over $B_r(p)$. Then it follows by letting $\eta, \hat{r} \to 0$ that

$$\hat{i}_r[\hat{\Gamma}_{j,r}] = i_r[\Gamma_{j,r}] \tag{31}$$

and that under the inclusion induced isomorphisms,

$$H_1(E_{\delta_j(r)}) \cong H_1(\pi^{-1}B_r(p)) \quad \text{and} \quad H_1(E_{\hat{\delta}_j(r)}) \cong H_1(\pi^{-1}B_r(p)),$$

the vanishing cycles $[C_{j,r}]$ and $[\hat{C}_{j,r}]$ have the same image. The orientation classes of the fibers also have the same image in H_2. In particular if $\hat{\delta}_k(s) = \delta_j(t)$, then

$$[\hat{C}_{k,s}] \cdot [C_{j,t}] = [C_k] \cdot [C_j] = [\hat{C}_k] \cdot [\hat{C}_j]. \tag{32}$$

Now we compute the intersection numbers of the two sets of prongs. If $k < j$, then $\hat{\Gamma}_k \cap \Gamma_j = \emptyset$; hence $[\hat{\Gamma}_k] \cdot [\Gamma_j] = 0$.

$$\hat{\Gamma}_1 \cap \Gamma_1 = \{q_1\}. \tag{33}$$

It follows from the first paragraph that $[\hat{\Gamma}_1]\cdot[\Gamma_1] = -1$. For $2 \le j \le N$,

$$\hat{\Gamma}_j \cap \Gamma_j = \{q_j\} \cup \Big[\hat{C}_{j,\hat{\delta}_j(s_j)} \cap C_{j,\hat{\delta}_j(r_j)}\Big]\,, \tag{34}$$

s_j the unique value with $\hat{\delta}_j(s_j) = \delta_j(r_j) = r_j e^{\frac{2\pi i(j-1)}{N}}$. At the point of intersection the path $\hat{\delta}$ meets δ with sign -1 ; hence

$$[\hat{\Gamma}_j]\cdot[\Gamma_j] = -1 - [\hat{C}_{j,s_j}]\cdot[C_{j,r_j}] = -1 - [C_j]\cdot[C_j] = -1\,. \tag{35}$$

Finally, if $k > j$,

$$\hat{\Gamma}_k \cap \Gamma_j = \hat{C}_{k,\hat{\delta}_j(s_{k,j})} \cap C_{j,\hat{\delta}_j(r_j)}\,, \tag{36}$$

$s_{k,j}$ the unique value with $\hat{\delta}_j(s_{k,j}) = \delta_j(r_j) = r_j e^{\frac{2\pi i(j-1)}{N}}$. Therefore

$$[\hat{\Gamma}_k]\cdot[\Gamma_j] = [\hat{C}_{k,s_{k,j}}]\cdot[C_{j,r_j}] = [C_k]\cdot[C_j] \tag{37}$$

Hence by bilinearity,

$$\Big(\sum_1^N J_j[\hat{\Gamma}_j]\Big)\cdot\Big(\sum_1^N J_j[\Gamma_j]\Big) = -\sum_{k>j\ge 2} J_k J_j\,\gamma_k\cdot\gamma_j - \sum_{j=1}^N J_j^2\,. \tag{38}$$

In other words, in the junction notation, if $J = (J_1, ..., J_N)$

$$[J]_{\hat{p}}\cdot[J]_p = \langle J, J\rangle\,. \tag{39}$$

Clearly, from the preceding paragraph, $\partial[J]_p = 0$ if and only if $\partial[J]_{\hat{p}} = 0$. In this case it is clear that any two closed classes $A, B \in H_2(X)$ with images $[J]_p$ and $[J]_{\hat{p}}$ satisfy $B\cdot A = \langle J, J\rangle$ and have the same image in $H_2(X, \pi^{-1}B_r(p)) \cong H_2(X, E_p)$. Therefore A and B are equal up to a multiple of the orientation class of E_p ; imposing the condition $A\cdot\sigma(U) = B\cdot\sigma(U) = 0$ then implies $A = B = [J]$, so $[J]\cdot[J] = \langle J, J\rangle$. □

3. Deformations, String Junctions, Lie Algebras and more

In this section we consider deformations of elliptic surfaces and the appearance of string junctions in the deformed geometry. Let $\pi : X \longrightarrow U$ be an elliptic fibration over an open set $U \subset \mathbb{C}$ with section σ, with $\pi_W : W \longrightarrow U$ its associated Weierstrass model; suppose that the ramification divisor Σ consists of the origin, namely $\Sigma = \{0\} \subset U$. W has local equation $y^2 = x^3 + fx + g$, then the possible singular fibers are described in the following table (see [20]):

Kodaira Fiber Type	*ord(f)*	*ord (g)*	*ord(Δ)*	Singularity Type
smooth	≥ 0	≥ 0	0	–
I_n	0	0	n	$\mathfrak{a}_{n-1}$
II	≥ 1	1	2	–
III	1	≥ 2	3	$\mathfrak{a}_1$
IV	≥ 2	2	4	$\mathfrak{a}_2$
I_n^*	2	≥ 3	$n+6$	$\mathfrak{d}_{n+4}$
I_n^*	≥ 2	3	$n+6$	$\mathfrak{d}_{n+4}$
IV^*	≥ 3	4	8	$\mathfrak{e}_6$
III^*	3	≥ 5	9	$\mathfrak{e}_7$
II^*	≥ 4	5	10	$\mathfrak{e}_8$

Theorem 3.1. (i) *There exists a deformation W_0 of the Weierstrass equation, such that $\pi_0 : W_0 \to U$ is an elliptic fibration with $\pi_0^{-1}(q)$ an I_1 singular fiber $\forall q \in \Sigma$.*

(ii) *Let $J^{(-2)}$ be the set of junctions J of W_0 such that $a(J)=0$ and $\langle J,J\rangle=-2$. Equivalently, let $J^{(-2)}\subset H_2(W_0)$ consists of those elements x with $x\cdot x=-2$ and $x\cdot\sigma(U)=0$. Then $\sharp\, J^{(-2)}$ is the number of roots of the Lie algebra associated to the singularity of π.*

(iii) *If $J=(J_1,...,J_N)\in J^{(-2)}$, then $|J_i|\leq 1$. In particular, all elements of $J^{(-2)}$ are represented by smoothly embedded S^2's.*

(iv) *There exist subsets $\{\alpha_1,...,\alpha_r\}\subset J^{(-2)}$ with r elements such that $\langle\alpha_i,\alpha_j\rangle$ is the negative Cartan matrix associated to the singularity of π.*

Proof. We will illustrate the proof in three of the cases from the table. All the singularities on the table can be handled in the same way, see [13, 16]. For the first case, assume that $\pi^{-1}(0)$ is of type I_{r+1}, in other words, an $\mathfrak{a}_r$ singularity. Then by [20] the defining equation in the complement of the image of the section σ can be written near $\pi^{-1}(0)$ as

$$y^2=x^3-3a^2x+2a^3+s^{r+1} \tag{40}$$

The derivative on the right hand side vanishes for $x=\pm a$. Therefore, assuming U was small enough to exclude the $r+1$-st roots of $-2a^3$, the singular locus is $\Sigma=\{0\}$.

In this case, take for W_0 be the deformation of W defined by

$$y^2=x^3-3a^2x+2a^3+s^{r+1}+\epsilon \tag{41}$$

for $\epsilon\in\mathbb{C}$. For $|\epsilon|$ small enough, the new discriminant locus of this equation will intersect U in the singular set of the fibration

$$\Sigma_\epsilon=\{e^{\frac{2\pi ij}{r+1}}\epsilon_0\,|\,j=0,...,r\} \tag{42}$$

with $\epsilon_0^{r+1}=\epsilon$ a specific $r+1$-st root. The fiber $\pi^{-1}(s)-\sigma(s)$ is the two fold branched cover of $\mathbb{C}$ branched along the roots of $x^3-3a^2x+2a^3+s^{r+1}+\epsilon=0$; in particular at each point of Σ_ϵ there is a multiple root corresponding to an I_1-singularity. Let $\delta_j(t)=te^{\frac{2\pi ij}{r+1}}\epsilon_0$, $0\leq t\leq 1$,be the straight line path from the origin to the jth point in Σ_ϵ. Then the equation of $\pi^{-1}(\delta_j(t))-\sigma(s)$,

$$y^2=x^3-3a^2x+2a^3+t^{r+1}\epsilon_0+\epsilon\,, \tag{43}$$

is independent of j. For example, if ϵ is real and positive and we also take ϵ_0 to be real and positive, then as we move along each path from zero to one, the two imaginary roots converge to a common real real of multiplicity two at the end point and the real root remains always real. In any case, whether we set it up this way or not, it follows from the preceding section that the vanishing cycles γ_j are all equal; $\gamma_1=....=\gamma_{r+1}$. From this it then also follows that $\{J\in\mathbf{J}|a(J)=0\}$ has the basis $\alpha_1,...,\alpha_r$, with $\alpha_1=(1,-1,0,...,0)$, $\alpha_2=(0,1,-1,0,...,0)$, etc. Since $\gamma_j\cdot\gamma_k=0$ because all these classes are equal, $\langle\alpha_j,\alpha_j\rangle=-2$, $\langle\alpha_j,\alpha_k\rangle=1$ for $|j-k|=1$ and zero for $|j-k|\geq 2$. This clearly implies the result for this case, we get the roots of the Dynkin diagram and Cartan matrix of $\mathbf{su}(r)$.

Next consider the case of singular fiber $\pi^{-1}(0)$ of type III, defined by

$$y^2=x^3+sx+s^2\,. \tag{44}$$

In this case for U small enough the the discriminant locus is only the origin. A deformation can be defined by

$$y^2=x^3+(s+\epsilon)x+(s^2+\epsilon) \tag{45}$$

Then it is not hard to see that, for U small enough, $\Sigma_\epsilon=\{q_1,q_2,q_3\}$ consists of three points. We take the basepoint to the origin $s=0$, which is now a smooth fiber $E_0=\pi_\epsilon^{-1}(0)$ of the deformed fibration π_ϵ. The fiber $E_0-\sigma(0)$ is the double branched cover of $\mathbb{C}$ along the roots of $x^3+\epsilon x+\epsilon=0$. The inverse image in the branched over of the three lines joining these roots determine curves representing elements $Z_i\in H_1(E_0)$; choose the orientations so that

$Z_1+Z_2+Z_3=0$. In this case the algorithm in the previous section yields, for the three vanishing cycles, $\gamma_i = Z_i$. In [13] this is obtained for a specific choice of small ϵ using a computer program solving cubics, but the result can also be obtained (tediously) by hand. Therefore there are exactly two junctions, $J = (1, 1, 1)$ and its negative, with $\langle J, J\rangle = -2$ and $a(J) = 0$, and we obtain the roots and Cartan matrix of $\mathbf{su}(2)$.

The third case, of type IV, is given by the equation

$$y^2 = x^3 + s^2x + s^2 \,. \tag{46}$$

$$y^2 = x^3 + (s^2 + 2\epsilon)x + s^2 + \epsilon \,. \tag{47}$$

provides a deformation of the local model. Near the origin there are now four points in the deformed discriminant. Using the same notation of the preceding paragraph for the homology classes determined by the roots, this time we will get the vanishing cycles $\gamma_1 = \gamma_3 = Z_1$ and $\gamma_2 = \gamma_4 = Z_3$ for the set of ordered vanishing cycles, with the signs chosen so that $Z_1 \cdot Z_3 = 1$. Again, this is done with a computer program in [13] for a specific choice of ϵ , but it can also be worked out by hand. The set of junctions with $a(J) = 0$ therefore has dimension two with basis, for example, $J_1, J_2 = \{(1, 0, -1, 0), (-, 1, 0, -1)\}$, $\langle J_i, J_i\rangle = -2$, $\langle J_1, J_2\rangle = 1$, there are six elements in $J^{(-2)}$, and we get the roots and negative Cartan matrix of $\mathbf{su}(3)$.

□

The example of the fiber of type IV actually arises from restricting a general Weierstrass model $\pi : W_g \to \mathbb{F}_3$ for an elliptic Calabi-Yau threefold over the Hirzebruch surface $\mathbb{F}_3$. This is an example of a "non-Higgsable cluster" (in physics language) with a type IV fiber; for this fibration, there there exists no smoothing deformation of the global model to a fibration with only I_1 singularities [21]. Nevertheless

$$y^2 = x^3 + (s^2 + 2\epsilon)x + s^2 + \epsilon \,. \tag{48}$$

provides a deformation of the local model.

In the papers [13, 16] we actually derive the entire representation structure of the Lie algebra associated to the singularity using geometric string junctions. However, perhaps the main advantage of this method is its usefulness in considering higher dimensional elliptic fibrations (see also [12]).

4. Higher Dimension, higher codimension

In physics, matter can appear when there is a codimension two stratum in the discriminant locus, arising as the intersection of two codimension one strata. Resolutions may be hard to use or may not even be available in all cases; we will conclude this paper with two illustrative examples of the deformation technique.

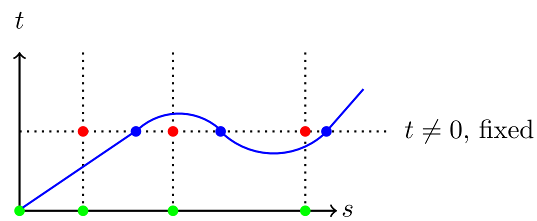
The example will be an elliptic fibration with section over an open set in $\mathbb{C}^2$ whose discriminant locus is the union of two curves meeting transversely in a point. Over each general point in the complement of the intersection, on one component we have an I_0^* singularity, on the other an I_1 singularity. Then without loss of generality it can be assumed there is a local equation in Weierstrass form [20]:

$$y^2 = x^3 - 3c^2s^2x + 2c^3s^3 + ats^3 \,. \tag{49}$$

Here s, t are coordinates in the base. The fibration over the line obtained by fixing a non-zero value of t has an I_0^* singularity at $s = 0$. For fixed $s \neq 0$ the fibration has an I_1 singularity. The deformation we consider varies with t:

$$y^2 = x^3 - 3c^2s^2x + 2c^3s^3 + ats^3 + t\epsilon. \tag{50}$$

For a fixed $t \neq 0$ and fixed s there are two possible singular points, $y = 0, x = \pm cs$. For $x = cs$, the corresponding points on the singular locus are the three cube roots of $-\frac{\epsilon}{a}$ denoted by the red dots in the picture below; for $x = -cs$ the points are the cube roots of $-\frac{\epsilon t}{at+4c^3}$. denoted by the blue dots in the picture below.



Each of these is an I_1 singularity and we have completely split the I_0^* singularity along $s = 0$. The smooth fiber $E_{0,t}$ for a fixed t and $s = 0$, minus the point at infinity i.e. minus $\sigma(0, t)$, is the double cover of $\mathbb{C}$ branched along the roots of $x^3 + t\epsilon = 0$.

Consider first the three cube roots of $-\frac{\epsilon}{a}$ and let us assume that ϵ, a are real, $\epsilon > 0$, $a < 0$. Fix $t \neq 0$ and consider the (real) plane $\mathbb{C}$ in the variable s. Let $\delta(r)$, $0 \leq r \leq 1$, be the straight line path from the origin to the real cube root. Let $\zeta = e^{\frac{2\pi i}{3}}$. Then $\zeta\delta(r)$ and $\zeta^2\delta(r)$ will the the paths to the other two roots. Then, if $\rho_1(r), \rho_2(r), \rho_3(r)$ are the roots of

$$x^3 - 3c^2(\delta(r))^2x + 2c^3(\delta(r))^3 + at(\delta(r))^3 + t\epsilon = 0, \tag{51}$$

the roots of

$$x^3 - 3c^2(\zeta^k\delta(r))^2x + 2c^3(\zeta^k\delta(r))^3 + at(\zeta^k\delta(r))^3 + t\epsilon = 0, \tag{52}$$

will be $\zeta^k\rho_1(r), \zeta^k\rho_2(r), \zeta^k\rho_3(r)$. For example, if t is a real positive number, then all along the path $\delta(r)$ the real root, say it is $\rho_1(0)$, remains real for $0 \leq r \leq 1$ whereas the complex roots $\rho_2(0)$ and $\rho_3(0) = \overline{\rho_2(0)}$ remain complex until $r = 1$, at which point they merge into a real root of multiplicity two. Then, along the path $\zeta\delta(r)$, the roots $\zeta\rho_2(0) = \rho_3(0)$ and $\zeta\rho_3(0) = \rho_1(0)$ will merge while $\rho_2(r)$ remains disjoint from the paths taken by the other roots. Similarly along $\zeta^2\delta(r)$, the roots $\rho_1(0)$ and $\rho_2(0)$ will merge. The paths of the merging roots of equation (52) will be the path of the merging roots of equation (51) multiplied by ζ or ζ^2. If we assume $2c^3 > -at$, then it is not hard to see that the real root $\rho_1(s)$ decreases with s. Therefore, since $\rho_1(s) + \rho_2(s) + \rho_3(s) = 3c^2(\delta(r))^2$ increases with s, so does the real part of the complex roots. Therefore the path of the merging roots never crosses the line joining the complex roots of $x^3 + t\epsilon = 0$. It follows from the algorithm above or a simple direct argument that the vanishing cycle corresponding to the real cube root of $-\frac{\epsilon}{a}$ will be a homology class $Z \in H_1(E_0)$ represented by the curve lying over this straight line.

The resulting determination of the vanishing cycles can be formulated as follows: Multiplication by ζ, i.e. rotation through $\frac{2\pi}{3,}$ induces a homeomorphism of the smooth fiber E_0, viewed as the double branched cover of $\mathbf{P}^1$ along the roots of $x^3 + t\epsilon = 0$ and "infinity", and hence an isomorphism $\zeta_* : H_1(E_0) \to H_1(E_0)$. The vanishing cycles corresponding the cube roots of $-\frac{\epsilon}{a}$ will be $\{Z, \zeta_*Z, \zeta_*^2Z\}$.

Now suppose that we choose c real so that $at + 4c^3 > 0$. The same argument shows that we get the same vanishing cycles for these points also. When we order all six points by increasing argument in the complex plane, we therefore obtain as our ordered set of vanishing cycles $\{Z, \zeta_*^2 Z, \zeta_* Z, Z, \zeta_*^2 Z, \zeta_* Z\}$. Note that, in the usual counterclockwise orientation, $Z \cdot \zeta Z = \zeta Z \cdot \zeta^2 Z = \zeta^2 Z \cdot Z = 1$. From the relation $1 + \zeta + \zeta^2 = 0$, it follows that the set $\{J \,|\, a(J) = 0\}$ of junctions with vanishing asymptotic charge has dimension four. The elements

(53)
$$\alpha_1 = (0,0,0,-1,-1,-1) \quad \alpha_2 = (0,0,-1,0,0,1) \quad \alpha_3 = (0,-1,0,0,1,0) \quad \alpha_4 = (-1,0,1,-1,0,1)$$

are a basis and satisfy

$$\begin{aligned} &\langle \alpha_1, \alpha_1 \rangle = \langle \alpha_2, \alpha_2 \rangle = \langle \alpha_3, \alpha_3 \rangle = \langle \alpha_4, \alpha_4 \rangle = -2 \\ &\langle \alpha_1, \alpha_2 \rangle = \langle \alpha_1, \alpha_3 \rangle = \langle \alpha_1, \alpha_4 \rangle = 1 \\ &\langle \alpha_2, \alpha_3 \rangle = \langle \alpha_2, \alpha_4 \rangle = \langle \alpha_3, \alpha_4 \rangle = 0\,. \end{aligned} \tag{54}$$

Thus we get the $\mathfrak{d}_4$ Dynkin diagram, and in fact this set of junctions gives the root lattice and weight structure of the $\mathfrak{d}_4$ Lie algebra; see [16] for more on the Lie algebra details given the geometric junctions.

Finally, we determine the monodromy around the component $t = 0$ of the singular locus to exhibit the collapse of the $\mathfrak{d}_4$ algebra (in physics language the "gauge algebra" $\mathfrak{d}_4$) to a $\mathfrak{g}_2$ algebra ("gauge algebra" in physics language). Instead of only t real, take $t(\theta) = te^{i\theta}$. For t small enough, the cube roots of $-\frac{\epsilon t(\theta)}{at+4c^3}$ will be closer to the origin than those of $-\frac{\epsilon}{a}$ and will rotate clockwise in an almost circular motion around as θ goes from zero to 2π. When θ gets to 2π the roots will have been permuted by multiplication by ζ. For each θ, let $E_0(\theta)$ be the singular fiber over the origin, the branched cover of $\mathbf{C}$ along the roots of $x^3 + t(\theta)\epsilon = 0$ and infinity . Then multiplication by $e^{i\theta}$ induces $(e^{i\theta})_* : H_1(E_0) \to H_1(E_0(\theta))$. Since the lines from the origin to these points on the discriminant will also rotate around with θ, only changing length slightly, it is clear that the vanishing cycles of the cube roots of $-\frac{\epsilon t(\theta)}{at+4c^3}$ will be $\{(e^{\frac{i\theta}{3}})_* Z,\ (e^{\frac{i\theta}{3}})_* \zeta_* Z,\ (e^{\frac{i\theta}{3}})_* \zeta_*^2 Z\}$. Therefore the effect of θ going from 0 to 2π is that the vanishing cycles do not change, but the three points on the discriminant locus undergo a rotation though $\frac{2\pi}{3}$ i.e. the order of all six vanishing cycles will have changed as the other three do not move.

In fact, as the three cube roots of $-\frac{\epsilon}{a}$ do not move as θ changes, the ordered sets of pairs of roots that coalesce as we move out from zero to any of these cube roots will have be permuted cyclically each time θ goes around through 2π. (For each discriminant point and one value of θ, a third root will cross one of these at a value of r less than one.) Thus, when we get to $\theta = 2\pi$, these vanishing cycles will have moved to $\{\zeta_* Z, \zeta_*^2 Z, Z\}$. In other words, the effect of θ going from zero to 2π will be, since these vanishing cycles moved and the other discriminant points rotated,

$$\{Z, \zeta_*^2 Z, \zeta_* Z, Z, \zeta_*^2 Z, \zeta_* Z\} \mapsto \{\zeta_* Z, Z, \zeta_*^2 Z, \zeta_* Z, Z, \zeta_*^2 Z\}\,. \tag{55}$$

In other words, on the junction we have the "outer monodromy" map

$$\lambda(a,b,c,a,b,c) = (c,a,b,c,a,b)\,.$$

(Note: As θ rotates around, there will be three points where pairs of points on the discriminant locus are on the same line segments from the origin. This is the reason the "obvious" continuity argument gives the wrong result.)

Now we replace α_4 with $\alpha_4' = (-1,0,0,1,0,0)$. The Dynkin diagram of the intersection form on the junctions $\alpha_1, \alpha_2, \alpha_3, \alpha_4'$ is also that of $\mathfrak{d}_4$, and α_4' is also a root. Clearly $\lambda(\alpha_1) = \alpha_1$, $\lambda(\alpha_2) = \alpha_4'$, and $\lambda(\alpha_4') = \alpha_3$ and $\lambda(\alpha_3) = \alpha_1$. Therefore, this monodromy map fixes the root corresponding the the central node of the $\mathfrak{d}_4$ Dynkin diagram and permutes the other three nodes, and when we divide by this action, we get the $\mathfrak{g}_2$ Dynkin diagram. However, α_4' is not a simple root in the $\mathfrak{d}_4$ algebra. Or, to put it other way, if we take these to be simple roots, we do not get the simple root lattice of $\mathfrak{d}_4$.

The monodromy does not preserve the Weyl chamber spanned by $\alpha_1, ..., \alpha_4$ but moves it to a different one. However, since it is of order three and must fix the central node because of invariance of intersection numbers, there are only two possibilities for what the monodromy induces on the Dynkin diagram, either a rotation of the extremal nodes or the identity. The calculation with the non-simple root eliminates the trivial case, and therefore it can be concluded from this argument using a non-simple root that the monodromy reduces the $\mathfrak{d}_4$ root lattice and weight structure, i.e. the Lie algebra, to that of $\mathfrak{g}_2$.

The result can also be establishes by considering the full set $J^{(-2)}$. In [16] we showed that there are precisely 192 four element subsets of $J^{(-2)}$ which can serve as simple roots; this number matches the dimension of the Weyl group of $\mathfrak{d}_4$, as it should. Thus J^{-2} contains the full data of the Lie algebra within it. It is straightforward to show that the outer monodromy map preserves $J^{(-2)}$ and is not trivial. Thus it gives an automorphism of the algebra, and from the junction description it is of order three; it cannot be trivial on the Dynkin diagram without being so on all of $J^{(-2)}$. Therefore it must induces an action on the Dynkin diagram which reduces $\mathfrak{d}_4$ to $\mathfrak{g}_2$.

For further details on the root structure and the full representation theory see [16].

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SINGULAR GEOMETRY OF THE MOMENTUM SPACE: FROM WIRE NETWORKS TO QUIVERS AND MONOPOLES

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Abstract. A new nano–material in the form of a double gyroid has motivated us to study (non)–commutative C^* geometry of periodic wire networks and the associated graph Hamiltonians.

Here we present a general more abstract framework, which is given by certain quiver representations, with special attention to the original case of the gyroid as well as related cases, such as graphene. The resulting effective C^*–geometry is that of the momentum space, which parameterizes the quasi–momenta.

This geometry is usually singular, where the singularities describe so–called band intersections in physics. We give geometric and algebraic methods to study these intersections; their origin being singularity theory and representation theory. A technique we newly apply to this situation is the use of topological invariants, which we formalize and explain in the paper. This uses K–theory and Chern classes as well as "slicing methods" for their computation. In this method the invariants can be computed using Berry's connection in the momentum space. This brings monopole charges and issues of topological stability into the picture.

Adding a constant magnetic field or allowing projective representations makes the C^* geometry non–commutative. In this case, we can also use K–theory, albeit in a different way, to make statements about the band structure using gap labeling.

Introduction

Recently, a new nano–material in the form of a double gyroid has been synthesized [38]. It is based on a thickened triply-periodic minimal surface, whose complement consists of two non-intersecting channels. These can be filled with conducting or semiconducting materials [38] to function as nanowire networks with potentially useful electronic properties [26]. The nontrivial topology of such a network has motivated our study of its commutative and non–commutative geometry [21]. Following Bellissard and Connes [4, 9, 30], we proceed by identifying the relevant C^*–algebra, which in our case is spanned by the symmetries and the tight-binding (Harper) Hamiltonian of the skeletal graph obtained as a deformation retract of the channel. This approach leads to an effective geometry described by a family of finite dimensional Hamiltonians and their spectra; the latter determine the band structure of the original nanostructured solid in the tight-binding approximation.

In this paper, we analyze this effective geometry, which in condensed matter physics is called the momentum space and a cover of it, using methods from singularity theory and topology. The singularities of the geometry are of particular interest as they determine physical properties of the material. The most prominent example of this are the so–called Dirac points of graphene which lead to its amazing properties including room temperature quantum Hall effect [8]. In our setup, we show that these Dirac points can be thought of as pull backs of A_1 singularities for the particular case of the honeycomb lattice.

Singularities can also be forced by symmetries. One way to obtain such symmetries is via symmetries of the underlying graph/quiver. This construction is not direct and proceeds through several steps. The starting point is a re–gauging groupoid for matrix representations of the

Hamiltonian. These are then "transferred" to actions on the base spaces of the effective geometry. The outcome is given by projective representations of subgroups of the symmetries of the graph on bundles over the subsets of the momentum space stabilized by the respective subgroups.

In this paper, we generalize the condensed matter setup to quiver representations stemming from finite graphs, thus making the theory more applicable to other contexts. In the process we adapt the techniques of [23] and [24] to this more general situation. In particular, we get a classification of singularities in the spectrum—the band intersections. The simplest of these is a conical intersection of two bands — these are the Dirac points mentioned above. We give analytic tools to compute locations and properties of all the singular points.

In the general framework above, we also give a formulation and application of the Berry phase phenomenon [7] in terms of K–theory and Chern–classes generalizing the observations of Thouless *et al.* (TKNN) [37] and Simon [34]. These concepts include topological charges in various guises: scalar, K–theoretic and cohomological. When the parameter space is three-dimensional, isolated conical degeneracies are magnetic monopoles in the parameter space [7]. In the present case, the parameters are components of the crystal momentum $\mathbf{k}$; their number equals the dimensionality of the original periodic structure. Thus, in three spatial dimensions—the case of the gyroid—Dirac points are monopoles in the momentum space and, as we will see, are stable with respect to small deformations of the graph Hamiltonian. Furthermore, using foliations, we consider a slicing technique which leads to an effective numerical tool for finding singular points in the spectrum, generalizing the method used for this purpose in [40]. This technique has been implemented in [25] and corroborates the topological stability of the gyroid's Dirac points. This stability is not a common characteristic of all Dirac points: those of graphene, which is described by the honeycomb lattice, do not exhibit this property for general deformations.. There of course might be deformations which do preserve them see e.g. [12].

This fact has an elegant and short explanation in our approach. We expect that this analysis will contribute to the understanding of potential applications of gyroid-based nanomaterials, as well as to the theory of three-dimensional generalizations of the quantum Hall effect, along the lines of [5]. In two dimensions, the TKNN equations for generalized Dirac–Harper operators have been worked out in [29]. Analyses of higher-dimensional situations are contained in [11, 14, 6, 27, 15].

Even without going to complete generality provided by quiver representations, our approach to studying wire networks is not restricted to the gyroid system and applies to any embedded periodic wire network in $\mathbb{R}^n$. We have already used it to study more examples, namely, Bravais lattices, the honeycomb lattice and two other triply periodic surfaces and their wire networks, the primitive cubic (P surface) and the diamond (D surface). We refer to these as the geometric examples. We recall some results here and include a new consideration of the topological charges.

The effective C^* geometry becomes non–commutative if we add an external magnetic field or more generally allow projective representations for the quivers. In the embedded wire network cases the noncommutative geometry is given by a subalgebra of a matrix algebra with coefficients in the noncommutative torus. Here the parameters of the torus correspond to the coefficients of the constant B–field that the material is subjected to.

One surprising fact is that some properties of the non–commutative situation are similar to the situation without a magnetic field, and there is evidence for duality between these two situations. The duality concerns the degenerate subspaces of the torus that appears as the relevant moduli space in both cases. In the commutative case, i.e. in the absence of a magnetic field, the torus is the base for the family of Hamiltonians and the requisite subspace is where the spectrum of the Hamiltonian has degeneracies. In the noncommutative case, the same torus parameterizes the B–field and the locus of degeneracy is that of those values of B where the C^*–algebra is not the full matrix algebra.

The paper is organized as follows: In Chapter 1, we start with a description of the material that motivated this study and its underlying geometry. This is independent of the rest of the paper and may be skipped by the reader interested in the more general setup. The first chapter also discusses how the gyroid surface geometry is reduced to that of the skeletal graph—the deformation retract of a channel component of the complement to the triply periodic surface. We additionally introduce other related geometries which we consider in parallel. These are the honeycomb lattice underlying graphene, and the P and D surfaces, which are the other triply periodic self–symmetric surfaces.

Chapter 2 describes the mathematical model we work with. This includes the Harper Hamiltonian and the relevant Hilbert space and C^* algebra for the commutative and non–commutative cases. We first consider the case of a periodic lattice embedded in $\mathbb{R}^n$ and then give the generalization to groupoid quiver representations in section 2.2. The reader more interested in the general mathematical framework can use this section as a starting point for reading the paper.

We discuss the resulting C^* geometry in Chapter 3. This includes the general setup identifying the singular locus as a pull–back from a miniversal unfolding, the representation theory using the re–gauging groupoid and our analysis of the Berry connection, topological charges and stability of the singular points as well as a slicing method to detect singular points or monopoles. Chapter 4 contains all results for the specific examples of the triply periodic wire systems P, D and G, as well as the two–dimensional honeycomb system and Bravais lattices in any dimension. This includes the new results about the topological charges. Using the methods of Chapter 3, we give the singular locus that is the degeneracies in the spectrum of the Harper Hamiltonian. As a second set of results, we review the classification results for the non–commutative geometries for the cases above. Here the parameter space is given by the background magnetic field 2–forms. In Chapter 5 we give a brief outlook including an observation of an almost duality.

1. The Double Gyroid (DG) and Related Geometries and Material

1.1. **The Geometry.** The gyroid is a triply periodic constant mean curvature surface that is embedded in $\mathbb{R}^3$ [16]. Figure 1 shows a picture of the gyroid. It was discovered in 1970 by Alan Schoen [33]. A single gyroid has symmetry group $I4_132$ in Hermann-Maguin notation. Here the letter I stands for bcc. The gyroid surface can be visualized by using the level surface approximation [28]

$$L_t : \sin x \cos y + \sin y \cos z + \sin z \cos x = t \tag{1}$$

In nature the single gyroid was observed as an interface for di–block co–polymers [18]. The **double gyroid** consists of two mutually non–intersecting embedded gyroids. Its symmetry group is $Ia\bar{3}d$ where the extra symmetry comes from interchanging the two gyroids. It also has a level surface approximation which is given by the above expression (1) with L_w and L_{-w} for $0 \leq w < \sqrt{2}$. The picture on the left-hand side of Figure 1 is actually a double gyroid or a "thick" surface.

Let us fix some notation. We will denote by $S = S_1 \amalg S_2$ the double gyroid surface. Its complement $C = \mathbb{R}^3 \setminus S$ has three connected components, which we will call C_+, C_- and W. W can be thought of as a "thickened" (fat) surface which we will refer to as DG wall. There is a deformation retract of W onto a single gyroid.

There are also two channel systems C_+ and C_-, shown in Figure 1. These channels form Y-junctions where three channels meet under a 120 degree angle. Each of these channel systems can

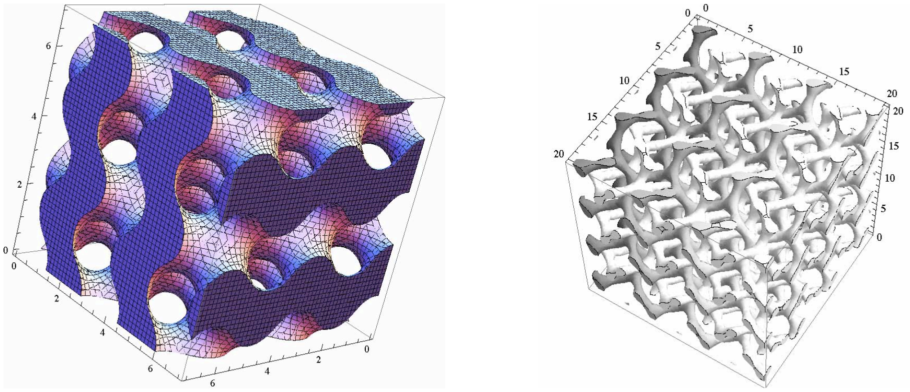


FIGURE 1. The fat gyroid surface W (left) and the two channel systems C_+ and C_- (right)

be deformation retracted to a skeletal graph $\Gamma_\pm$. We will concentrate on one of these channels and its skeletal graph Γ_+, shown in Figure 2.

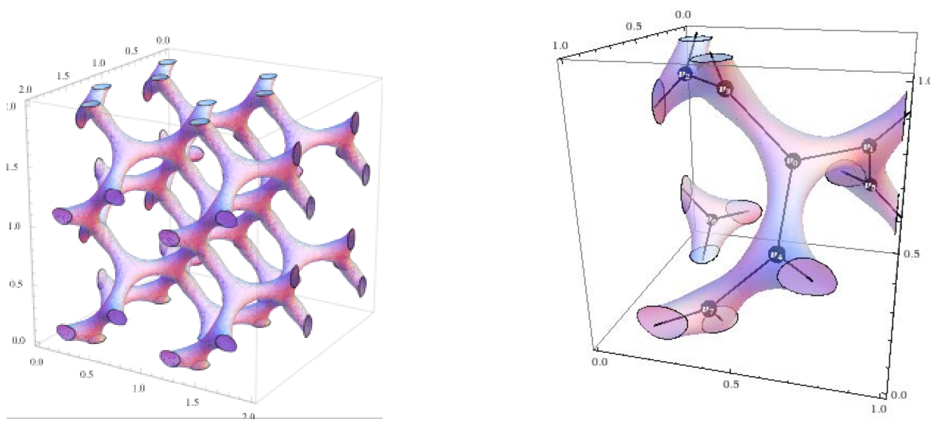


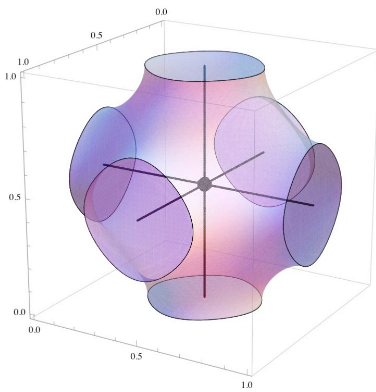
FIGURE 2. One of the two channels (left) and its skeletal graph in the unit cell (right)

1.2. **The Material and Production.** A solid-state double gyroid can be synthesized by self-assembly at the nanoscale, as demonstrated by Urade et al. [38]. The first step is production of a nanoporous silica film with the structure of unidirectionally cotracted double gyroid (DG) with lattice constant of about 18 nm. The pores in the structure can then be filled with other materials to form nanowires. Fabrication of platinum DG nanowires by electrodeposition has been demonstrated in [38], where it has also been mentioned that the process can be used for other metals or semiconductors.

1.3. **Related Geometries: the P and D surfaces.** There are two other triply periodic self–dual and symmetric CMC surfaces- the cubic (P) and the diamond (D) network. They are shown in Figure 3 together with their wire networks obtained in the same way as for the gyroid. Here we summarize the results from [22].

The P surface has a complement which has two connected components each of which can be retracted to the simple cubical graph whose vertices are the integer lattice $\mathbb{Z}^3 \subset \mathbb{R}^3$. The translational group is $\mathbb{Z}^3$ in this embedding, so it reduces to the case of a Bravais lattice.

The D surface has a complement consisting of two channels each of which can be retracted to the diamond lattice $\Gamma_\diamond$. The diamond lattice is given by two copies of the fcc lattice, where the second fcc is the shift by $\frac{1}{4}(1, 1, 1)$ of the standard fcc lattice, see Figure 3. The edges are nearest neighbor edges. The symmetry group is $Fd\bar{3}m$.



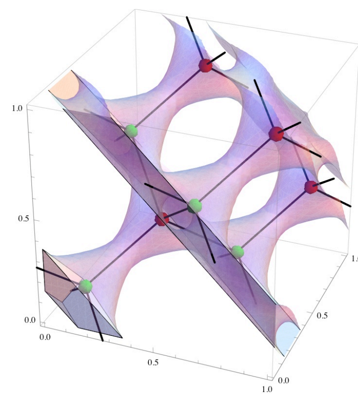


FIGURE 3. The cubic (P) (left) and the diamond (D) wire network (right)

1.4. **Graphene.** Graphene consists of one-atom thick planar sheets of carbon atoms that are densely packed in a honeycomb crystal lattice. This two–dimensional material has attracted much interest recently, partially because of the existence of Dirac points where excitations show a linear dispersion relation. Its electronic properties are described by a Harper Hamiltonian: see the review [8] and references therein. Here we will reproduce some of the known facts, such as the Dirac points using our non–commutative geometry machine.

2. Mathematical Model and Generalization: Graphs and Groupoid Representation

2.1. **Discrete model and Harper Hamiltonian.** We will now describe how to obtain the Harper Hamiltonian for any given graph $\Gamma \in \mathbb{R}^n$ with a given maximal translation group $L \simeq \mathbb{Z}^n$ [19]. We will start with the commutative case without an external field, and then progress to the non–commutative case where the graph is placed in a constant external magnetic field. The mathematical set–up we will describe below can be understood in terms of Weyl quantization and Peierls substitution in physics [32]. Without the magnetic field the Harper Hamiltonian is given by translations, but in the presence of a magnetic field all translations turn into magnetic translations or Wannier operators, which cease to commute with each other.

Mathematically the discretization by the above process yields the Hilbert space $\mathscr{H} = \ell^2(V(\Gamma))$, where $V(\Gamma)$ are the vertices of Γ, and a projective representation of the translation group L as well as an operator H, the Harper Hamiltonian. Concretely, the elements l of L act on the functions Ψ via the usual translations $T_l : T_l\Psi(l') = \Psi(l - l')$.

2.1.1. *Quotient Graph and Harper Hamiltonian.* In general, given an embedded graph $\Gamma \in \mathbb{R}^n$, with a given maximal translation group $L \simeq \mathbb{Z}^n$, we consider the quotient graph $\bar{\Gamma} := \Gamma/L$ and the projection $\pi : \Gamma \to \bar{\Gamma}$. The quotient graphs for our four main examples are given in Figure 4.

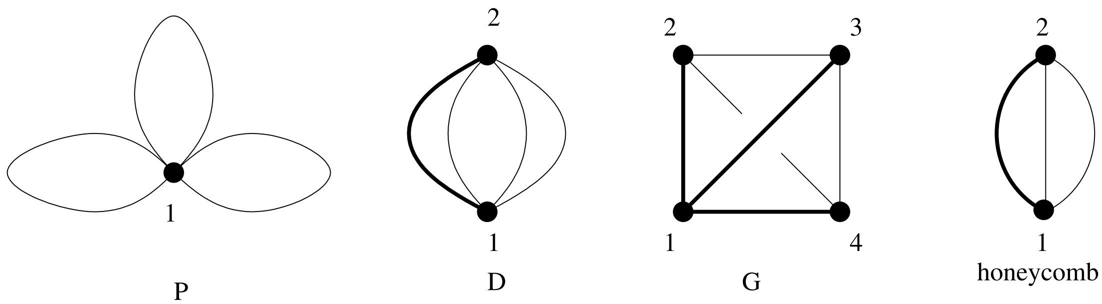


FIGURE 4. The quotient graphs of the P,D,G surfaces and the honeycomb lattice, together with a spanning tree and an order of the vertices.

The vertices of this graph are in 1–1 correspondence with vertices or sites of Γ in a fundamental cell. We can think of the graph $\bar{\Gamma}$ as embedded into $T^n = \mathbb{R}^n/\mathbb{Z}^n$. Each edge e of $\bar{\Gamma}$ lifts to a pair of edge vectors $\vec{e}, \overleftarrow{e} = -\vec{e}$ where the underlying line segment is any lift of e to Γ. This is well defined since any two lifts differ by a translation.

To each vertex $v \in \bar{\Gamma}$ we can associate the Hilbert space $\mathscr{H}_v := \ell^2(\pi^{-1}(v))$. Then the whole Hilbert space $\mathscr{H}$ decomposes as

$$\mathscr{H} = \bigoplus_{v \text{ vertex of } \bar{\Gamma}} \mathscr{H}_v \tag{2}$$

Since all the $\mathscr{H}_v$ are separable Hilbert spaces, they are all isomorphic.

The Harper Hamiltonian is then given as follows. For each edge e between two vertices v and w of $\bar{\Gamma}$ let $T_{\vec{e}}$ be the translation operator from $\mathscr{H}_w \to \mathscr{H}_v$. This extends to an operator $\hat{T}_{\vec{e}}$ on $\mathscr{H}$ via $\hat{T}_{\vec{e}} = i_{\bar{v}} T_{\vec{e}} P_{\bar{w}}$ where $i_{\bar{v}} : \mathscr{H}_{\bar{v}} \to \mathscr{H}$ is the inclusion and $P_{\bar{w}} : \mathscr{H} \to \mathscr{H}_{\bar{w}}$ is the projection.

The Harper Hamiltonian is

$$H = \sum_{e \text{ edges of } \bar{\Gamma}} \hat{T}_{\vec{e}} + \hat{T}_{-\vec{e}} \tag{3}$$

2.1.2. *Harper Hamiltonian in the presence of a magnetic field.* Adding a constant magnetic field requires a slightly different definition of the Harper Hamiltonian. We will use projective translation operators whose commutators include the fluxes of the magnetic field as follows: We define a 2–cocycle $\alpha_B \in Z^2(L, U(1))$ by a two–form $\hat{\Theta}$ on the ambient $\mathbb{R}^n$. Such a two–form is given by a skew symmetric matrix Θ with $\hat{\Theta} = \Theta_{ij} dx_i \wedge dx_j$. We let $B = 2\pi\hat{\Theta}$ and interpret it as a quadratic form[1]. In this way we obtain a two–cocycle

$$\alpha_B \in Z^2(\mathbb{R}^n, U(1)) : \ \alpha_B(u,v) = \exp(\frac{i}{2} B(u,v))$$

which we then restrict to L.

We define magnetic translations by starting from A, which is a potential for B (on $\mathbb{R}^n$). The magnetic translation partial isometry is now acting on a wave function as

$$U_{l'}\psi(l) = e^{-i\int_l^{(l-l')} A}\,\psi(l-l')$$

[1]This is the quadratic form on constant vector fields, which can be identified with a quadratic form on $\mathbb{R}^n$. The matrix Θ is also the matrix for this quadratic form.

The magnetic Harper operator is defined as

$$H = \sum_{e \text{ edges of } \bar{\Gamma}} U_{\vec{e}} + U_{\overleftarrow{e}} \tag{4}$$

2.2. **Generalization: Groupoid and quiver representations.** In the setting above, which we call the geometric examples, we have distilled the following data: a finite graph $\bar{\Gamma}$, the translational groups L and a projective representation of it on $\mathscr{H} = \bigoplus \mathscr{H}_v$ and finally the Hamiltonian H.

We will now explore the possibility of obtaining such data from a more general setup. There are two ways to do this: in terms of groupoids or in terms of quivers.

2.2.1. *Groupoid representation.* Recall that a groupoid is a category whose morphisms are all invertible. A representation of a groupoid is a functor ρ from this category into a linear category. In our case this will be the category of separable Hilbert spaces which is the full subcategory of the category of vector spaces whose objects are separable Hilbert spaces.

A graph $\bar{\Gamma}$ (here $\bar{\Gamma}$ need not be finite) determines a groupoid $\mathcal{G}$ as follows. The objects are the vertices of Γ. The morphisms are *generated* by the edges. That is for each oriented edge between v and w there is one generator $\phi_{\vec{e}}$ in $Hom(v, w)$. The morphisms in this category are then the composable words in the $\phi_{\vec{e}}$ where composable means that the source of a latter is the target of the predecessor, with the relations that

$$\phi_{\vec{e}}\phi_{\overleftarrow{e}} = id_v \in Hom(v, v), \text{ the identity element} \tag{5}$$

What this means is that the morphisms are the paths on Γ up to homotopy, with the constant path yielding the identity.

A groupoid representation of $\mathcal{G}$ in separable Hilbert spaces then assigns to each vertex v of $\bar{\Gamma}$ a separable Hilbert space $\rho(v) = \mathscr{H}_v$ and to each oriented edge $\vec{e}$ from v to w a morphism $\rho(\phi_{\vec{e}}) = \Phi_{\vec{e}} \in Hom(\mathscr{H}_v, \mathscr{H}_w)$ with the relation that $\Phi_{\vec{e}}\Phi_{\overleftarrow{e}} = id_{\mathscr{H}_v}$. We will abbriviate $\rho(\phi_{\vec{e}})$ by $\rho(\vec{e})$.

The groupoid representation is unitary if all the $\Phi_{\vec{e}}$ are.

Remark 2.1. Notice that there is an involution $*$ on the morphisms, by transposing the word and reversing the orientation of each letter. So we can only look at involutive functors, that is functors which send $*$ to $\dagger$, that is the Hermitian adjoint. This guarantees that the representation is unitary.

2.2.2. *Quiver representation.* There is a way to formulate this in quiver language. Given a graph $\bar{\Gamma}$ and an arbitrary choice of directions for the edges determines a quiver. Now one can construct the double of the quiver, where each oriented edge is doubled with reverse orientation. If we started from a graph, this means that each unoriented edge e is replaced by the two oriented edges $\vec{e}$ and $\overleftarrow{e}$. Now the double of the quiver is independent of the original choice of orientation. As above, there is an involution $*$ on the set of its oriented edges which is given by reversing the orientation. We will restrict the quiver representations we consider to those where the involution $*$ goes to $\dagger$.

2.2.3. *Hamiltonian of the representation.* Just as above we set $\mathscr{H} := \bigoplus_{v \text{ vertex of } \bar{\Gamma}} \mathscr{H}_v$ and define

$$H := \sum_{e \text{ edges of } \bar{\Gamma}} \rho(\vec{e}) + \rho(\overleftarrow{e}) : \mathscr{H} \to \mathscr{H}$$

2.2.4. *Representation of* $\pi_1(\bar{\Gamma})$. If we fix a vertex v_0 of $\bar{\Gamma}$, the groupoid representation naturally gives a representation of $\pi_1(\bar{\Gamma}, v_0)$ as follows. We fix a set of symmetric generators of $\pi_1(\bar{\Gamma}, v_0)$ which is isomorphic to the free group in $b_1 = 1-\chi$ generators $\mathbb{F}_{b_1}$, where $\chi = \#\text{vertices} - \#\text{edges}$ is the Euler characteristic and b_1 is the first Betti number. Each such generator g_i is a directed simple loop on the graph which is given by a sequence of directed edges $\vec{e}_{1i}, \dots, \vec{e}_{n_i i}$. Then $\rho(g_i) = \rho(\vec{e}_{1i}) \circ \cdots \circ \rho(\vec{e}_{n_i i})$ gives a representation of $\pi_1(\bar{\Gamma}, v_0)$ on $\mathscr{H}_{v_0}$.

Definition 2.2. We will denote the algebra generated by $\rho(\pi_1)$ by $\mathscr{T}$. We say ρ is maximal if the generators of π_1 map to linearly independent operators and we say that ρ is of torus type if $\mathscr{T} = \mathbb{T}^n_\Theta$, the non–commutative n-torus with parameters given by the skew-symmetric matrix Θ. Here necessarily $n = 1 - \chi(\Gamma)$.

If ρ is of torus type then representation ρ of $\pi_1(\bar{\Gamma}, v_0) = \mathbb{F}_{b_1}$ as a projective representation factors through $H_1(\bar{\Gamma}) = \mathbb{Z}^{b_1} = \mathbb{F}_{b_1}/[\mathbb{F}_{b_1}, \mathbb{F}_{b_1}]$, the Abelianization of π_1. In the geometric setup of nano-wire networks, these are given by a constant background B field and the parameter Θ is the matrix corresponding to that field as discussed above; see [21] for additional details.

In the geometric situation of Chapter 1, maximality is equivalent to the fact that the translational symmetry group is maximal.

2.2.5. *Spanning trees.* If we pick a rooted spanning tree of $\bar{\Gamma}$ then we get isomorphisms $\phi_{0v} : \mathscr{H}_{v_0} \simeq \mathscr{H}_v$ by using ρ and concatenation along the unique shortest path of oriented edges from v_0 to v in the spanning tree. Let $\Phi = \bigoplus_v \phi_{0v} : \mathscr{H}_{v_0}^{|V|} \to \mathscr{H}$. Then this isomorphism yields a representation $\tilde{\rho}$ on $\mathscr{H}_{v_0}^{|V|}$ via pullback.

Likewise ϕ_{v0} induces an isomorphism of $\pi_1(\bar{\Gamma}, v)$ and $\pi_1(\bar{\Gamma}, v_0)$. Using this identification, we get a representation $\hat{\rho}$ of $\mathscr{T}$ on $\mathscr{H}$ and via pull-back with Φ on $\mathscr{H}_{v_0}^{|V|}$.

A rooted spanning tree (τ, v_0) also gives rise to one more bijection. This is between a set of (symmetric) generators of π_1 and the edges *not* in the spanning tree. The bijection is as follows.

If $\vec{e}$ is a directed edge from v to w then there is a generator $g_{\vec{e}}$ which is given by the following path of ordered edges: (1) the unique shortest path in τ from v_0 to v (2) the directed edge $\vec{e}$ and (3) the unique shortest path in τ from w to v_0. It is clear that $g_{\vec{e}} = g^{-1}_{\overleftarrow{e}}$. By contracting the spanning tree, we see that this is indeed a set of symmetric but otherwise independent generators.

For convenience, we set $g_{\vec{e}} = 1$ if $e \in \tau$.

2.2.6. *Quiver C^*–Geometry, the algebra $\mathscr{B}$.* Given a groupoid representation in separable Hilbert spaces of a finite graph $\bar{\Gamma}$ we call the C^* algebra generated by the operators H and $\mathscr{T}$ via $\hat{\rho}$ on $\mathscr{H}$ the Bellissard–Harper algebra of the pair $(\bar{\Gamma}, \rho)$ and denote it by $\mathscr{B}$.

This general set gives the generalization of one of the results of [21].

Theorem 2.3. *Any choice of spanning tree together with an order on the vertices gives rise to a faithful matrix representation of $\mathscr{B}$ in $M_{|V|}(\mathscr{T})$.*

Proof. This follows from the fact that under Φ, $\rho(\vec{e})$ gets transformed to the matrix entry $\rho(g_{\vec{e}})$ between the copies of $\mathscr{H}_{v_0}$ corresponding to $\mathscr{H}_v$ and $\mathscr{H}_w$ under Φ. Enumerating these vertices yields a matrix. $\square$

In the following given a rooted spanning tree τ we will only choose orders $<$ such that the root is the first element. The resulting matrix Hamiltonian will be denoted by $H_{\tau,<}$.

3. C^*–GEOMETRY

3.1. **Structure theorems for the C^*–geometry.** The non–commutative C^*–geometry of such a quiver representation in general and the one stemming from the geometric situation in particular is that of $\mathscr{B}$.

Just like in [21, 22] one can now ask the question whether or not $\mathscr{B}$ is isomorphic to the full matrix algebra and hence Morita equivalent to $\mathscr{T}$ itself. In the geometric case, we obtain a family of algebras depending on a choice of magnetic field. To take this into account, we will denote by $\mathscr{B}_\Theta$ the resulting matrix algebra for the choice of magnetic field determined by Θ. In this notation, $\mathscr{T}_\Theta = \mathbb{T}_\Theta$ is generically simple and led to the expectation —which we proved in [21]— that generically $\mathscr{B}_\Theta = M_{|V|}(\mathbb{T}_\Theta)$. This of course need not be the case in general.

It stands to reason that other more complicated physical phenomena could be described by such algebras.

It actually turns out that in the geometric examples not only is the algebra indeed the full matrix algebra at generic irrational parameter values, but that there are even only finitely many or at most a co–dimension–2 subset of matrix parameters Θ, where $\mathscr{B}_\Theta \subsetneq M_k(\mathbb{T}_\Theta)$.

Theorem 3.1. [21, 22] *For the geometric cases of the G surface and the honeycomb lattice the algebra $\mathscr{B}_\Theta$ is isomorphic to the full matrix algebra except at finitely many values of Θ given in Chapter 4. For the P surface and all Bravais lattices $\mathscr{B}_\Theta = \mathscr{T}_\Theta = M_1(\mathbb{T}_\Theta)$. For the D surface, the set of values of Θ for which $\mathscr{B}_\Theta \subsetneq M_2(\mathbb{T}_\Theta)$ is given by 6 one dimensional families and finitely many special points (also listed in Chapter 4). If $\mathscr{B}_\Theta$ is the full matrix algebra then it is Morita equivalent to $\mathbb{T}_\Theta$.*

Remark 3.2. Note that except for the P and general Bravais case, these families above give examples of continuos variations of algebras whose K–theory does not vary continuously. In those cases the K–theory for the commutative case $\Theta = 0$, which corresponds to a nontrivial ramified cover of the torus [21, 22], is different from the generic situation, which has the K-theory of the torus. There are also certain special points where the algebra and hence the K–theory is isomorphic to the commutative case, although the magnetic field is not 0. This happens if the magnetic flux through the relevant cells is integer. We also expect the K–theory to drop at the other special points, due to the presence of additional symmetries.

3.1.1. *Inspecting the spectrum via K–theory labeling.* One application of the non–commutative approach is gap labeling by K–theory. If the Hamiltonian H has a spectrum bounded from below, then each gap in the spectrum gives rise to a projector $P_{<E}$ onto the Eigenspaces with Eigenvalues less than any fixed value E in the gap, see e.g. [4, 30]. The gap labeling then associates the K–theory class of $P_{<E}$ to the gap.

By the above result, via the inclusion $\mathscr{B} \hookrightarrow M_k(\mathscr{T})$ the projector $P_{<E}$ also gives rise to a K–theory class in $K(M_k(\mathscr{T})) \simeq K(\mathscr{T})$. Using this embedding, one can deduce analogues of the famous Hofstadter Butterfly.

Theorem 3.3. *If $(\bar{\Gamma}, \rho)$ is toric non–degenerate, then the Hamiltonian H as an operator on $\mathscr{H}$ has only finitely many gaps if the magnetic field is rational in the sense that the matrix Θ is rational.*

3.2. **Effective geometry in the commutative case: the momentum space and the Eigenvalue cover.** If $\mathscr{B}$ is commutative, for instance if $\Theta = 0$ in the geometric situation, then by the Gel'fand–Naimark theorem, there is a compact[2] Hausdorff space X, such that $\mathscr{B} \simeq C_0(X)$. The points of X can be thought of as characters, i.e. C^*–homomorphisms $\chi : \mathscr{B} \to \mathbb{C}$. More

[2] $\mathscr{B}$ is unital.

precisely these characters are in bijection with the maximal ideals of $\mathscr{B}$ which are the points. If we wish to make this distinction, we write p_χ for the point of X corresponding to the character χ and vice–versa χ_p for the character corresponding to p. Likewise there is a space T which corresponds to the C^*–algebra $\mathscr{T}$. In the geometric case $T = T^n = \mathbb{R}^n/L$.

As usual the correspondence between the algebra of functions and the spaces is contravariant. This means that the inclusion $\hat{\rho} : \mathscr{T} \to \mathscr{B}$ gives rise to a morphism $\pi : X \to T$. If (ρ, Γ) is maximal, then $\mathscr{T} \to \mathscr{B}$ is injective and hence $\pi : X \to T$ is surjective.

Furthermore let us consider the algebra $\mathscr{T}^{\oplus k}$ given by the direct sum of k copies of $\mathscr{T}$. The space corresponding to this algebra is simply $T \amalg \cdots \amalg T$ k–times.

Since after choosing an order and a rooted spanning tree $\mathscr{B} \subset M_{|V|}(\mathscr{T})$, we can lift any character χ of $\mathscr{T}$ to a C^*-homomorphism: $\hat{\chi} : M_{|V|}(\mathscr{T}) \to M_{|V|}(\mathbb{C})$ of $\mathscr{B}$ by applying χ to each entry.

Definition 3.4. We call a point χ of T degenerate if $\hat{\chi}(H)$ has less than $|V|$ distinct Eigenvalues and we will denote this locus as T_{deg}.

We also set $X_{deg} := \pi^{-1}(T_{deg})$. These are the singular points of X.

Repeating the proof of [21] we arrive at the following

Theorem 3.5. *If $(\rho, \bar{\Gamma})$ is maximal the map $\pi : X \to T$ is ramified over the degenerate points and furthermore X is the quotient of the trivial k–fold cover of T where the identifications are made in the fibers over degenerate points. Moreover these correspond to the degeneracies of H over these points.*

In other words, X can be thought of as the spectrum of the family of Hamiltonians $H(p) = \chi_p(H)$ parameterized over T.

The key ingredient is the image of H under the map $\mathscr{B} \to \mathscr{T}^{\oplus k}$ dual to the map $\amalg_{i=1}^k T \to X$

$$H \mapsto \sum_i \lambda_i e_i \tag{6}$$

where e_i are the idempotents corresponding to the i–th component and λ_i is the i–th Eigenvalue.

3.3. Singular geometry of the momentum space and the Eigenvalue cover.

3.3.1. *Characteristic map and Swallowtails.* In the commutative case, the locus X_{deg} has a nice characterization in terms of singularity theory, [23]. First, there is an embedding of X into $T \times \mathbb{R}$, where X is identified with the pairs (t, λ_i) for which λ_i is an Eigenvalue of $H(t)$. Here $H(t) = \hat{\chi}_t(H)$, i.e. the point t corresponds to the character χ_t under the Gel'fand representation.

The key ingredient is a newly defined characteristic map: for this let

$$P(z,t) = \det(zId - H(t)) = z^k + b_{k-1}(t)z^{k-1} + \cdots + b_0(t),$$

let

$$P(z - \frac{b_{k-1}}{k}, z) = z^k + a_{k-2}(t)z^{k-2} + \cdots + a_0(t)$$

and let g be the isomorphism on $T \times \mathbb{R}$ which sends (t, z) to $(t, z - \frac{b_{k-1}}{k})$.

The coefficients $a_{k-2}(t), \ldots, a_0(t)$ define a map $\Xi : T \to \mathbb{C}^{k-1}$ called the characteristic map. We recall that the miniversal unfolding of the A_{k-1} singularity z^k is given by the family of functions $f_{a_0,\ldots,a_{k-2}} = z^k + a_{k-2}z^{k-2} + \ldots a_1 z + a_0$, with the parameters $(a_0, \ldots, a_{k-2}) \in \mathbb{C}^{k-1}$ giving the base of this variation, see e.g. [2]. This is the base of the covers whose fiber over a point $(a_0, \ldots, a_{k-2})$ are the roots of $f_{a_0,\ldots,a_{k-2}}$. The terms $z^i : 0 \le i \le k-2$ correspond to a basis of the Milnor or Jacobian ring $\mathbb{C}[z]/(z^{k-1})$. It is miniversal in the sense that any other variation of resolving the singularity is a pull-back of a variation equivalent by diffeomorphism to this one. Notice that the term with z^{k-1} is missing, which is why we used the map g to reparameterize the characteristic equation.

Identifying $\mathbb{C}^{k-1}$ with the base of the miniversal unfolding of the A_{k-1} singularity, we obtain the following (cf. [23]):

Theorem 3.6. *The branched cover $X \to T$ is equivalent via g to the pull back of the miniversal unfolding of the A_{k-1} singularity along the characteristic map Ξ. Explicitly, if $\hat{P} = P \circ g$ then the pull back along Ξ of the cover of the miniversal unfolding is the cover corresponding to the roots of $\hat{P}$. Via g this cover is equivalent to the one of P and hence to the cover X. Moreover if the family of Hamiltonians is traceless the cover is the pull–back on the nose.*

The family is traceless if $\bar{\Gamma}$ has no small loops —that is edges which are a loop at one vertex— and if, additionally, the graph is also simply laced, then $a_{k-2} \equiv |E(\bar{\Gamma})|$. In other words, the image of T under Ξ is contained in the corresponding slice $a_{k-2} = |E(\bar{\Gamma})|$ of the base of the miniversal unfolding.

This means that if $\Sigma \subset \mathbb{C}^{k-1}$ is the discriminant locus or swallowtail, then $T_{\text{deg}} = g^{-1}(\Xi^{-1}(\Sigma))$ and the fiber of π over a point t is exactly $g^{-1}\pi_A^{-1}(\Xi(t))$ where π_A is the projection of the miniversal unfolding.

Here the swallowtail Σ is the set of points $(a_0, \dots, a_{k-2}) \subset \mathbb{C}^{k-1}$ where $f_{a_0,\dots,a_{k-2}}$ has roots of higher multiplicity – see Figure 5 for the A_2 and A_3 cases.

In other words the fibers over degenerate points are identified with the corresponding fibers over their image points in the swallowtail.

Using Grothendieck's characterization [10] of the swallowtail as stratified by lower order singularities obtained by deleting edges in the corresponding Dynkin diagram, and pulling this back via Ξ, we obtain:

Corollary 3.7. *Consider a variation of Hamiltonians given by $(\bar{\Gamma}, \rho)$ and assume that the Hamiltonians are traceless. The only possible types of singularities in the spectrum of this variation are $(A_{r_1}, \dots, A_{r_s})$ with $1 \le s \le \lfloor k/2 \rfloor$, and $\sum_{i=1}^{s} r_i \le k - s$.*

In the simply laced case with no loops, $r_i < k$

Remark 3.8. Notice that our approach is "orthogonal" to the considerations of [13] where the projection $T \times \mathbb{C} \to \mathbb{C}$ was used instead of the projection $T \times \mathbb{C} \to T$ which we use. Also in their context, T needs to be complex one–dimensional and hence their arguments do not generalize to arbitrary (odd and even) dimensions. This is because their theory relies on deep theorems which are special to the algebraic geometry of curves.

Remark 3.9. Theorem 3.6 and the corollary above can be viewed as a generalization/refinement of what is commonly referred to as the "von Neumann–Wigner theorem". This is not a theorem *per se*, but the expectation that for a "generic" variation the degenerate locus is of codimension 3. This goes back to the result of [39] that for the full family of all Hermitian Hamiltonians $Herm(k)$ (the space of all $k \times k$ Hermitian matrices) the degenerate locus is indeed of codimension 3.

The most prominent results about the geometry of $Herm(k)$ were already obtained in [39]. Here one can find the co–dimensions of the strata of degenerate Eigenvalues, basically by a dimension count. This was carried further in [3], where a fibration was introduced. For this and other discussions it is sometimes convenient to mod out the k^2-real-dimensional vector space $Herm(k)$ by translations and dilatations. Indeed shifting the spectrum or scaling it does not change the topology of the situation. The translations are done by adding scalar matrices and the dilatation, as usual, by multiplying by non–zero constants. Modding out by the translations

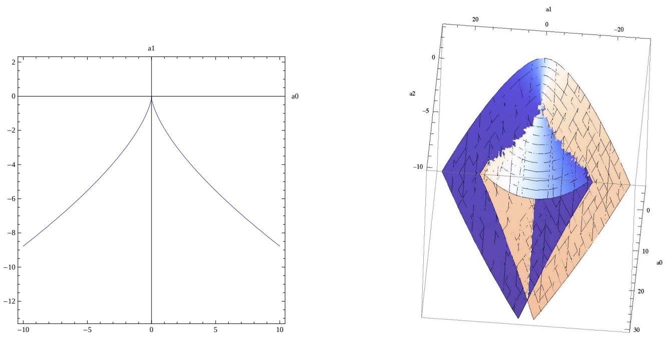


FIGURE 5. The swallowtail for the A_2 (left) and A_3 (right) singularities

means that we can restrict to traceless matrices and modding out by dilatations means that we can take the norm to be 1, unless we are dealing with the 0 matrix. The quotient space of the space of *non–scalar* Hermitian matrices under the simultaneous action, which is naturally identified with the co-invariants, is then a $k^2 - 2$–dimensional sphere which we denote by $\mathbb{S}(k)$. This sphere then has a filtration by pieces F_p for which the first p Eigenvalues are equal. Arnold [3] studied this filtration and that study has been continued in [1].

Our main focus is the geometry of a given (not necessarily generic) family $H : T \to Herm(k)$ dictated by a quiver representation. In this setting the exact codimension depends on the whole family T and is given precisely as the preimage of Ξ. To be more precise locally it is the dimension of the intersection of the image under Ξ with the swallowtail and the dimension of the fiber.

Proposition 3.10. *In the maximal toric case increasing the number of links to arbitrarily high values, the dimension of the degenerate locus T_{deg} generically becomes $-\chi(\Gamma)$, so that the stable expected codimension of the degenerate locus is* 1*.*

Proof. Since the domain of Ξ is compact, so is the image. Its size is limited by the coefficients of the Hamiltonian. The value of the i, j–th entry under a lifted character $\hat{\chi}$ is sharply bounded by l where l is the number of edges between v_i and v_j. This is follows from to the definition of the Hamiltonian as translations along edges and is a generalization to many edges of [23] [Section 2 (equation (6)]. As the number of edges grows, this bound increases. This implies that the sharp bound on the coefficients a_i also increases. If this is large enough, the image of Ξ will fill out a bounded region of the complement of the swallowtail Σ over which the discriminant is positive. Then the boundary of the image given by a part of the swallowtail Σ will be of codimension 1 and of dimension $|V_\Gamma| - 2$. The generic dimension of the fiber will be $dim(T) - (|V_\Gamma| - 1)$. In total this gives the dimension of the critical locus as $1 - \chi(\Gamma) - |V_\Gamma| + 1 + |V_\Gamma| - 2 = -\chi(\Gamma)$. $\square$

The test case of the triangular graph with possibly multiple edges has been calculated in [22] which gives an example of the phenomenon described above. This is illustrated in Figure 6 and Figure 7.

3.3.2. *Characterizing Dirac points.* Physically very interesting singularities of X are conical singularities, which are also called Dirac points. In order to find these singularities, we consider the ambient space $T \times \mathbb{R}$ and the function $P : T \times \mathbb{R} \to \mathbb{R}$ as given in §3.3.1. As we argued in [23], Dirac points in the spectrum are isolated Morse singularities of P with signature $(+, -, \ldots, -)$. That argument did not need the specifics of the geometric situation and hence generalizes.

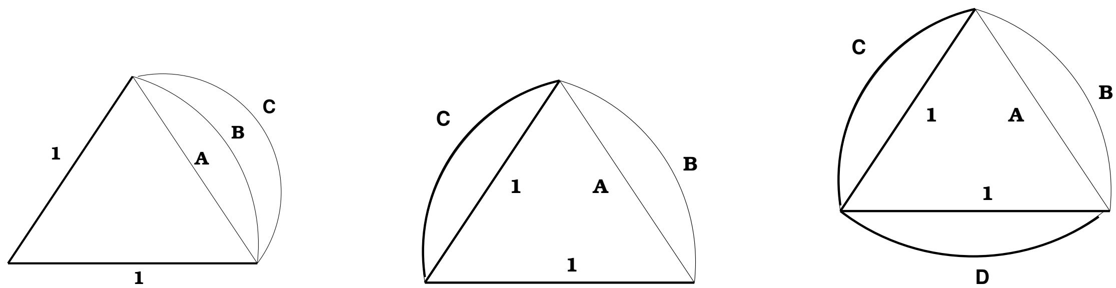


FIGURE 6. Triangle graphs with possibly multiple edges

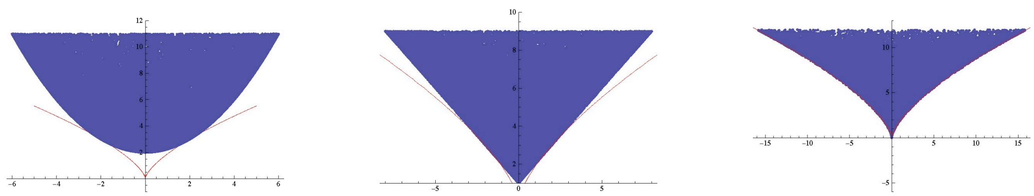


FIGURE 7. Spanning tree and characteristic region for the triangle graphs

As Morse singularities are of type A_1 it is a necessary condition from the above is that there is an A_1 singularity in the fiber, e.g. via the methods given above. For a Dirac point, one in addition needs to check the signature.

3.4. **Forced degeneracies by Symmetries.** One reason that singular points have to be present is given by symmetries. If the momentum space geometry stems from a graph, such symmetries can be induced by symmetries of the underlying graph. The procedure for this is not straightforward though and proceeds via re–gauging groupoid and a "lift" of its action to the momentum space [24]. The result of this rather elaborate process is the existence of *projective* representations of subgroups of the symmetry group of the graph that appear as stabilizers in the geometric action on the momentum space. We describe this construction below.

In the geometric examples of wire networks, we showed in [24] that all the singularities of the Eigenvalue cover are forced by these enhanced re–gauging symmetries.

3.4.1. *General setup.* Going back to the embedding of $\mathscr{B}_\Theta$ into $M_k(\mathbb{T}^n_\Theta)$ the relevant matrix representation depended on the choice of a rooted spanning tree (τ, v_0) and an order $<$ on the vertices. We will now fix that the first element in that order is given by the root. In [24] we showed that the re–gauging from $(\tau, <)$ to $(\tau', <')$ is given by conjugation by a unitary matrix $U^{\tau',<'}_{\tau,<}$. These matrices are more complicated than just the permutation group and incorporate local gaugings. These are given by diagonal matrices with invertible elements in $\mathbb{T}_\Theta$ indexed by the vertices of the graph.

Moreover in this way, the automorphism group of Γ acts by re–gaugings. Namely, if $\phi \in Aut(\Gamma)$ then given $(\tau, <)$, the image of τ, $\phi(\tau)$, and the push forward of the order, $\phi_*(<)$, give rise to a re–gauging by $U^{\phi(\tau),\phi_*(<)}_{\tau,<}$. Usually this action on a given Hamiltonian is not trivial, due to the fact that ρ need not be trivial.

All these observations directly generalize to the more general case of a groupoid representation $(\bar{\Gamma}, \rho)$. In this case $\mathbb{T}_\Theta$ is replaced by $\mathscr{T}$. The arguments of [24] are not sensitive to the particular structure of $\mathbb{T}_\Theta$ and hence carry over to the more general situation. We summarize the logical steps here.

3.4.2. *Re–gauging groupoid.* The re–gaugings form a secondary groupoid, the re–gauging groupoid. Its objects are given by tuples $(\tau, <)$ and between any two objects there is a unique morphism $((\tau, <), (\tau', <'))$. There is a morphism λ to matrices with coefficients in $\mathscr{T}$ by sending $((\tau, <), (\tau', <'))$ to $U_{\tau,<}^{\tau';<'}$. This morphism need not be a representation, however, since we are only guaranteed that $\lambda(g_1)\lambda(g_2)\lambda(g_1g_2)^{-1}$ is a non–commutative 2–cocycle with values in $U(\mathscr{T})$, the unitary elements of $\mathscr{T}$. The reason for this is that under the identification given in §2.2.5 the re–gauging basically corresponds to an isomorphism of $\pi_1(\bar{\Gamma}, v_0)$ with $\pi_1(\bar{\Gamma}, v_0')$ along a path, v_0 and v_0' being the roots of τ and τ' respectively. Concatenating the isomorphisms along these paths as above, we end up with an isomorphism under a loop; but this is precisely conjugation with an element of $\pi_1(\bar{\Gamma}, v_0)$. In the representation, this element becomes an element in $U(\mathscr{T})$.

3.4.3. *Projective Groupoid Representations.* In the commutative case the cocycle above gives rise to a central extension by $U(\mathscr{T})$ and the matrices $U_{\tau,<}^{\tau';<'}$ give a representation in $M_k(\mathscr{T})$ of the central extension.

Evaluating with a character $\hat{\chi}$, the extension becomes an extension by $U(1)$ and the matrices $\hat{\chi}(U_{\tau,<}^{\tau';<})$ form a projective representation of the groupoid in $M_k(\mathbb{C})$.

3.4.4. *Stabilizer Groups, Lifts, Projective Actions and Group Extensions.* If we have a fixed point, that is a Hamiltonian that is invariant under the action of non–trivial groupoid elements, then these elements form a *group* of re–gaugings. Technically the representation of stabilizer subgroupoid factors through the group given by identification of all objects in that groupoid to one point.

In order to find such a stabilizer group, we look for an automorphism of T which compensates the re–gauging by automorphisms of $\bar{\Gamma}$. That is, given an automorphism ϕ of $\bar{\Gamma}$, let $\Phi_{\tau',<'}^{\tau;<}$ be the associated re–gauging. We then look for an automorphism $\Psi_{\tau',<'}^{\tau;<}$ of T such that

$$\hat{\chi}_t(\Phi_{\tau',<'}^{\tau;<}(H_{\tau,<})) = \hat{\chi}_{\Psi_{\tau',<'}^{\tau;<}(t)}(H_{\tau,<}) \tag{7}$$

This is done for one orbit of $(\tau, <)$ under $Aut(\bar{\Gamma})$. This tool is most effective if the graphs are completely symmetric, like the cases we considered.

If we find such a lift of the automorphism group $Aut(\bar{\Gamma}) \to Aut(T)$, then we can look for points of enhanced symmetry. If $t \in T$ has a non–trivial stabilizer group under this action of $Aut(\bar{\Gamma})$ then the matrix $\hat{\chi}_t(H_{\tau,<})$ has a non–trivial re–gauging fixed group. This action by conjugation yields a projective representation of the stabilizer group.

Given such a projective representation, we know that it is a representation of a central $U(1)$ extension of the stabilizer group. If the stabilizer group is finite, we would furthermore like to find a smaller if possible finite group which already carries the representation. That is an extension of the stabilizer group by a finite group. For this one uses the theory of Schur multipliers.

The upshot is that the isotypical decomposition of the representation has to be commensurate with the Eigenspace decomposition of the Hamiltonian – for that particular value $t \in T$. Practically this means that on the one hand if in the given representation there are irreps of dimension bigger than one, one can infer that there are degeneracies in the spectrum of at least these dimensions. On the other hand, the one dimensional isotypical components fix Eigenvectors and hence make it easy to find the Eigenvalues. In general of course one only has to diagonalize the Hamiltonian inside the isotypical summands.

3.5. **Invariants of the momentum space geometry.** In the commutative case, a last way to characterize the singularities in the momentum space geometry is to use topological invariants. These come from the fact that although the cover $X \to T$ is unramified and trivial outside of T_{deg} the line bundles defined by each non-degenerate Eigenvalue carry non–trivial topology.

3.5.1. *Basic bundles, and their K-theoretic and Cohomology Valued Charges.* More precisely, let $X_{\text{deg}} = \pi^{-1}(T_{\text{deg}})$ be the closed singular locus of X. Then the restriction

$$\pi : X_0 := X \setminus X_{\text{deg}} \to T_0 := T \setminus T_{\text{deg}}$$

is the trivial k–fold cover, since the Eigenvalues are real. A trivialization is given by choosing the order in each fiber according to the order in $\mathbb{R}$ of the Eigenvalues $\lambda_1 < \cdots < \lambda_k$.

On this restriction the C^*–algebra $C_0(X \setminus X_{\text{deg}})$ contains pairwise orthogonal projections P_i such that $H = \sum_i \lambda_i P_i$. Each of these P_i defines a rank 1 sub–bundle L_i of the trivial bundle $X_0 \times \mathbb{C}$ which is the Eigenbundle corresponding to the Eigenvalue λ_i. The projector or equivalently the bundle L_i defines an element in K–theory $[L_i] \in K(T_0)$. We will continue with the geometric interpretation of line bundles and K–theory here, although in a forthcoming analysis we will concentrate on the C^* version of K–theory in oder to move to a non–commutative setup.

We call the classes $[L_i]$ the K–theoretic charges and the associated Chern classes

$$\beta_i := c_1(L_i) \in H^2(T_0)$$

the cohomological charges. We also let $C = \bigoplus_i L_i$, and $[C] \in K(T_0)$ be its class in K–theory. Finally we define the polynomial invariant $Q_c(t_i) = \prod_i (1 + t_i\beta_i) \in H^{\text{ev}}(T_0)[t_i]$. This class contains all the cohomological information of the L_i and C.

Remark 3.11. One can generalize most of the arguments to non–Hermitian variations: $H : T \to GL(k, \mathbb{C})$, but then one should impose that T_0 is simply connected and $\pi_1(T_0) = 1$ in order for the characteristic polynomial to be irreducible over $C_0(T_0)$ which is necessary to define the P_i, see e.g. [17].

Remark 3.12. We assumed that the Hamiltonians are generically non–degenerate. It is sufficient to assume that the ranks of the Eigenbundles are generically constant. In this case, we have vector bundles V_i and total Chern classes $c(V_i)$.

3.5.2. *Numerical Invariants/Charges.* One can try to get numerical information about Q_c and the β_i by pairing them with appropriate homology classes. For this it is easier to assume that we are dealing with oriented manifolds. If we furthermore have a differentiable structure, we know that we can evaluate Chern classes by using Chern–Weil theory.

The paring then corresponds to the integral of the curvature form for any connection over a submanifold of the correct even degree. The set of *all* such numbers on a set of generators of homology of T_0 then determines the cohomological charges as functions on homology. By the classification theorem for line bundles, see e.g. [20] the first Chern class fixes the isomorphism class of the line bundle. Furthermore, if we use at least $\mathbb{Q}$ coefficients, usually in physics we take $\mathbb{R}$ of $\mathbb{C}$, cohomology and K-theory are isomorphic via the Chern character and we can represent homology by using submanifolds [20, 35]. Notice that by the results of Thom [35] all second homology classes are of this type even over $\mathbb{Z}$.

It then follows that the charges are trivial if T_0 has vanishing second cohomology, which is where the first Chern classes live, (e.g. if T_0 is 2–connected). In that case the Chern classes β_i vanish and the line bundles $[L_i]$ are trivializable. This is the case in some examples, notably the honeycomb. The effect of the line bundles being trivializable is that the associated points of degeneracy are not topologically stable, see §3.6.

The two–torus or the two–sphere do however have non–vanishing H^2 and thus are prime candidates to detect first Chern classes if we embed them into the momentum space.

In this particular case, we can evaluate the first Chern class of a line bundle with a connection on a 2–dimensional submanifold by pulling back, i.e. restricting, the line bundle to the surface and integrating the curvature form of the connection. Here if A is a connection form for the line bundle, and the first Chern–class is represented by the curvature form $\Omega = dA + \frac{1}{2}A \wedge A$.

Explicitly, if Σ is an oriented compact surface and $i : \Sigma \to T$ is an embedding, then

$$Q_{\Sigma,i} := \int_{\Sigma} i^* c_1(L_i) = \langle c_1(L_i), i_*([\Sigma])\rangle \tag{8}$$

where $\langle\ ,\ \rangle$ is the standard pairing between cohomology and homology.

3.5.3. *Berry connection.* It was Berry's [7] great insight in this context, that adiabatic transport provides such a connection and that this connection is indeed not always trivial and produces the so–called Berry phase as a possible monodromy.

Simon [34] realized that Berry's formula is just the calculation of the first Chern class of $[L_i]$ using a connection and Chern-Weil theory. Thus one can use any other connection, for instance the so–called canonical connection used by Simon. By general theory the first Chern class is the only obstruction for the line bundle, and hence the monodromy, to be trivial.

3.5.4. *Standard Setup.* Let us first fix the concrete setup which is usually present. Assume T is compact orientable potentially with boundary and that $T_{\deg}$ is in codimension at least 1; i.e. T is generically non–degenerate. We furthermore assume that $T_{\deg} \cap \partial T = \emptyset$. Then T_0 is an orientable manifold with boundary. Let N be a tubular neighborhood of $T_{\deg}$ in T and $\bar{N}$ its closure.[3] Then $B = T \setminus N$ is a compact sub–manifold with boundary $\partial B = \partial T \amalg \partial \bar{N}$ where $\partial \bar{N} = \bar{N} \setminus N$.

E.g. If we assume that $T_{\deg}$ is a manifold with singularities and the smooth part of $T_{\deg}$ is of codimension r then $\partial \bar{N}$ is an S^{r-1} bundle over the smooth part of $T_{\deg}$. In particular, if $T_{\deg}$ is a discrete set of points p_i we can take N to be the union of small open balls centered at each point and $\bar{N}$ will be the union of the closed balls, while ∂N will be the union of the corresponding spheres.

3.5.5. *Even dimensional B.* If B is even dimensional, we can pair with B and consider $\int_B Q_c(t_i)$.

If in particular $T_0 = T$ and T is two–dimensional then $B = T$ and we obtain all the individual charges by using $B = \Sigma$ as $Q_i := \int_B \beta_i$.

Following Simon [34] this if for instance the case for the quantum Hall effect. Here $T = T^2$ has no degenerate locus and we have that $B = T$ can carry non–trivial line bundles. Indeed the arguments of TKNN [37] establish the non–triviality of the corresponding line bundle.

3.5.6. *Odd dimensional T with boundary.* If T is odd dimensional, we can restrict the L_i to the boundary ∂T, since we assumed that $\partial T \cap T_{deg} = \emptyset$. Then the *boundary charge* is defined to be $\int_{\partial T} Q_c(t)|_{\partial T_0}$.

In the differentiable case, we represent Q_c by a closed form $\omega = d\phi$ of even degree; strictly speaking this is a polynomial form. Also, since B is odd dimensional, we have by Stokes' Theorem

[3] I.e. N is a smooth submanifold of the same dimension as T, which can be deformation retracted onto $T_{\deg}$. If $T_{\deg}$ is smooth N can be chosen to be a standard small tubular neighborhood. Otherwise we assume that T_{deg} is sufficiently nice that we have a stratification that allows for Thom–Mather theory [36, 31].

that $0 = \int_B \omega = \int_{\partial B} \phi = \int_{\partial T} \phi + \int_{-\partial \bar{N}} \phi$.

$$\int_{\partial T} \phi = - \int_{-\partial \bar{N}} \phi = \int_{\partial \bar{N}} \phi \tag{9}$$

where $\partial \bar{N}$ has the outward orientation viewed from $\bar{N}$. Else we just use the usual pairing between the corresponding homologies and cohomologies.

If the boundary is empty, then we have that $\int_{\partial \bar{N}} \phi = 0$.

3.5.7. *Local charges and codimension 3.* For each component $\bar{N}_k$ of $\bar{N}$, we can consider the restriction $i^*_{\partial \bar{N}_i}(Q_c)$ of Q_c to ∂N_k, where k indexes the components, and define the local charge of that component to be $\int_{\partial \bar{N}_k} i^*_{\partial \bar{N}_i}(Q_c)$. This is of course only useful if T is odd–dimensional, so that $\partial \bar{N}$ is even–dimensional.

A special situation arises, if the smooth part $T^{\mathrm{sm}}_{\mathrm{deg}}$ of T_{deg} is of codimension 3. Recall that $\partial \bar{N}$ is an S^{r-1} bundle over $T^{\mathrm{sm}}_{\mathrm{deg}}$, where $r = \mathit{codim}(T^{\mathrm{sm}}_{\mathrm{deg}})$ which in this case is 3 and hence $\partial \bar{N}$ is a 2–sphere bundle. Thus in this case, we can restrict the L_i to the fiber $S^2 = S^2(p)$ over any point p of $T^{\mathrm{sm}}_{\mathrm{deg}}$. We call $\int_{S^2(p)} \beta_i|_{S^2}(p)$ the i–th local charge at p and $\int_{S^2(p)} Q_c|_{S^2}(p)$ the local charge.

3.5.8. *Isolated critical points in dimension* 3. For *isolated* critical points of T_{deg} the local charges are just given by integrating over small spheres around these points, which is what $\partial \bar{N}$ is. If T_{deg} consists only of isolated critical points, then formula (9) states that the boundary charge is the sum over the local charges. If moreover the boundary is empty, this means that the sum of all the i–th local charges is 0. This is the case for the gyroid.

3.5.9. *Slicing.* A slicing for T is a smooth codimension 1 foliation by compact oriented manifolds of T which has a global transverse section S and the leaves of the foliation generically do not intersect T_{deg}. For this we need the Euler characteristic to be 0, which is in particular the case for all odd dimensional compact manifolds.

For $s \in S$ let F_s be the leaf of s and i_s be the inclusion, if $F_s \cap T_{deg} = \emptyset$, we can consider the pullback of C and consider

$$Q_s := \int_{F_s} i^* Q_c \tag{10}$$

which is the total Chern class of the slice. An interesting commonly encountered situation arises if

(1) F_s generically does not intersect T_{deg}
(2) Each component of $\bar{N}$ contains only one component for T_{deg}.
(3) Any component of $\bar{N}$ is contained between some pair of slices. That is for a component $T'_{\mathrm{deg}} \subset T_{\mathrm{deg}}$ there are s_1, s_2 and an n–dimensional closed submanifold M of the n–dimensional manifold T, such that $M \cap T_{\mathrm{deg}} = T'_{\mathrm{deg}}$, and $\partial M \cap T_{\mathrm{deg}} = \emptyset$, $\partial M = F_{s_1} - F_{s_2}$ and the component of $\bar{N}$ corresponding to T'_{deg} is entirely contained in M.

In this case, by using Stokes' Theorem we get that the total contribution of T'

$$\int_{\partial \bar{N} \cap M} Q_s|_{\partial \bar{N} \cap M} = Q_{s_1} - Q_{s_2} \tag{11}$$

Now (12) is a great tool to numerically find T_{deg}. For this one just runs through the $s \in S$ and looks for jumps in Q_s.

3.5.10. *T_{deg} of codimension* 3. If T' is smooth then the total charge is

$$\int_{T_{deg}} (\int_{S^2(p)} Q_c|_{S^2}(p))dp = Q_{s_1} - Q_{s_2} \tag{12}$$

If we are in dimension 3 then codimension 3 means that the degenerate locus consists of only isolated critical points. Here the equation (12) simplifies to just a finite sum over the critical points.

If furthermore the critical points are A_1 singularities, see §3.3.1, then the jumps in the charge are from ± 1 to ∓ 1, as calculated in [34, 17] depending on if one calculates for the upper or lower band and the chosen orientation/parameterization.

3.5.11. *3–dimensional torus models.* If we have that $T = T^3$ the situation is especially nice. It is fibered by T^2s via any of the three projections $\pi_j : S^1 \times S^1 \times S^1 \to S^1, j = 1, 2, 3$. The inclusion of fibers of the three coordinate projections actually generates the whole cohomology of T^3. It has non–vanishing 2nd cohomology $H^2(T^3) \simeq \mathbb{Z}^3$. In contrast to the two–torus where puncturing kills the 2nd cohomology a punctured three torus actually still has second cohomology. It is given explicitly in the proof of the theorem below. This is a main difference between graphene and the gyroid, see below. One has to be sure however, that the condition of generically not intersecting the degenerate locus is not violated. A counterexample is the case for the D–surface, see below. Given finitely many points $p_i \in T^3$, we say that they are in generic position with respect to an identification $T^3 \simeq S^1 \times S^1 \times S^1$, if all their coordinates (viz. projections) are pairwise distinct. By changing the identification with automorphisms of T^3, we can always obtain this situation.

Notice that the slicing along any of the three co–ordinate foliations given by the projections $\pi_j, j = 1, 2, 3$ only gives a finite set of numbers $Q_{s,i}$ for each Eigenbundle L_i, since the integral over the Chern–class is constant as s varies in a component of $S^1 \setminus \{\pi_j(p_i)\}$.

Theorem 3.13. *For a smooth variation with base T^3 and only finitely many degenerate points, which we may assume to be in generic position, the slicing method applied to all three coordinate projections completely determines the K–theoretic charges and hence the line bundles L_i up to isomorphism.*

Proof. For no degenerate points this is clear as $H^2(T^3, \mathbb{Z}) \simeq \mathbb{Z}^3$ and generators are given by the three coordinate embeddings of T^2.

If there are $m \geq 1$ degenerate points p_i and pick a coordinate projection π_j and let

$$z_1, \dots, z_m \in S^1$$

be the images of the p_i. Let $t_1, \dots, t_m$ be points in between the z_i, that is one point per component of $S^1 \setminus \{z_i\}$. Consider the CW model of the torus, which has one 2–cell at height t_i and 3–cells in between and 0 and 1 cells accordingly. Then $T_0 = T \setminus \{p_i\}$ deformation retracts onto the 2–skeleton of this complex. And the homology of T_0 can be calculated either (a) via the standard Meyer–Vietoris sequence for T covered by N and a slightly enlarged B, or (b) using cellular chains for the above CW complex. Using the former, we see that there are $m - 1 + 3$ classes in $H^2(T_0, \mathbb{Z}) \simeq \mathbb{Z}^{m+2}$. Namely the three original classes, plus the classes of the m little spheres minus the diagonal class of all the spheres. In the CW basis of (b) this is given by a set of m horizontal slices separating the m points and the images of the two other coordinate embeddings of T^2.

Now the slicing method will give the paring with these two cells and as the Poincaré paring is non–degenerate, the cohomology class of $c_1(L_i)$ is determined by these numbers and hence the line bundle up to isomorphism. ∎

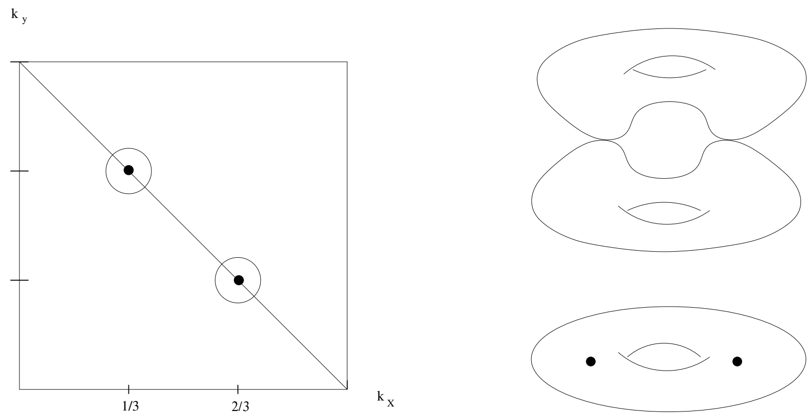


FIGURE 8. I. The base two torus with the singular points (as a square with opposite sides identified), the anti–diagonal $(\phi, \bar{\phi})$, the 2 points of T_{deg} and the discs making up N. II. A picture of the ramified double cover X.

Remark 3.14. In fact, one only needs one complete slicing and then one slice each in the other coordinate directions as determining data.

3.6. **Topological Stability.** Having non–vanishing topological charges produces topological stability. If we perturb the Hamiltonian slightly by adding a small perturbation term λH_1 and continuously vary λ starting at 0, then T_0 does not move much —for instance as a submanifold of $T \times R$, see §3.3.1. In particular, there will be no new singular points in T_0 for small perturbation. The Eigenbundles over T_0 also vary continuously and hence so do their Chern classes. Since these are defined over $\mathbb{Z}$ they are actually locally constant, so that all the non–vanishing charges, scalar, K-theoretic or cohomological, must be preserved.

4. Results for the commutative and non–commutative C^*–geometries of wire networks

In this section, we summarize our results for the different quantum wire networks, honeycomb, P (or more generally any Bravais lattice), D and G. The basis are the results from [21, 22, 23, 24] and a new analysis for the topological charges using slicing.

The first set of results are on the singular geometry of the momentum space in the commutative situation. These include all the three aspects developed above, the branched cover and its singularities, the symmetries and the topological invariants.

The second set of results are on the classification of the C^*–algebras that appear when one allows a constant background magnetic field.

4.1. **Singular geometry of the momentum space for periodic wire networks.**

4.1.1. *The Honeycomb Lattice.* In this case the space X is a double cover of the torus T^2 ramified at two points $(e^{2\pi i \frac{1}{3}}, e^{-2\pi i \frac{1}{3}})$ and $(e^{-2\pi i \frac{1}{3}}, e^{2\pi i \frac{1}{3}})$. These two points are A_1 singularities and Dirac points. This is depicted in Figure 8. T_0 is T^2 with two points removed, so $H^2(T_0) = 0$ and so all the charges vanish and all bundles are trivial, thus the two Dirac points are in general not topologically stable. Along the anti–diagonal $(\phi, \bar{\phi})$ we have the equation for the two sheets of the cover $E(\phi) = \pm(1 + 2\cos(\phi))$. This is depicted in Figure 9.

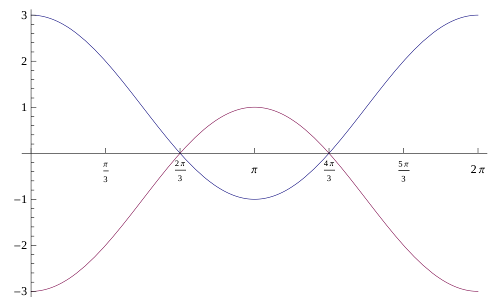


FIGURE 9. The cover along the anti–diagonal $(\phi, \bar{\phi})$

Remark 4.1. There has been an investigation of deformation directions which do not destroy these points [12]. In our setup this means the following: the characteristic map has its image in $[-9, 0]$ where the swallowtail for A_1 is the point 0. One only considers deformations which still have 0 in the image of the characteristic map.

At the Dirac points there is an enhanced symmetry which is Abelian, so it does not have any higher dimensional irreps, but the isotypical decomposition is fully decomposed and forces the double degeneracy at the Dirac points due to the form of the Hamiltonian.

4.1.2. *The primitive cubic (P) case, and other Bravais cases.* The cover $X \to T^k$ is trivial and so is the line bundle of Eigenvectors.

Remark 4.2. The analysis of [5] of the quantum Hall effect, however, suggests that there is a non–trivial noncommutative line bundle in the case of $k = 2$ for *non–zero B–field.* Furthermore, in this case there is a non–trivial bundle, not using the noncommutative geometry, but rather the Eigenfunctions constructed in [37] for the full Hilbert space $\mathscr{H}$. This is what is also considered in [34]. We will study this phenomenon in the gyroid and the other cases in the future.

4.1.3. *The Diamond (D) case.* In this case, we see that that $1 - \chi(\Gamma) = 3$ and T is the 3–Torus T^3. The space X defined by $\mathscr{B}$ in the commutative situation is a generically 2–fold cover of T^3 where the ramification locus $T_{\deg}$ is along three circles on T^3 given by the equations $\phi_i = \pi, \phi_j \equiv \phi_k + \pi \mod 2\pi$ with $\{i, j, k\} = \{1, 2, 3\}$. $T_{\deg} = \Xi^{-1}(0)$ is the inverse image —of the characteristic map— of the only singular point (the origin) of the miniversal unfolding of A_1. The characteristic region is the interval $[-16, 0]$.

Thus the singularities are of type A_1 but they are not discrete, but rather pulled back to the entire $T_{\deg}$, hence there are also no Dirac points. Figure 10 depicts the base 3–torus with the singular locus, which is of codimension 2.

The space $T_0 = T^3 \setminus T_{\deg}$ contracts onto a 1–dimensional CW–complex and hence has $H^2(T_0) = 0$. Thus there are no non–vanishing topological charges associated to this geometry and no stability.

Analogous to the honeycomb case there are Abelian enhanced symmetries with 1–dimensional isotypical components, which force the double degeneracy in view of the structure of the Hamiltonian.

4.1.4. *The Gyroid (G) case.* For the gyroid, the commutative geometry is given by a generically unramified 4-fold cover of the three torus, see [21]. There are only 4 ramification points. This means that the locus is of real codimension 3 contrary to the D case where it was of codimension 2. Furthermore the degenerations are 3 branches coming together at 2 points —$(0, 0, 0)$ and

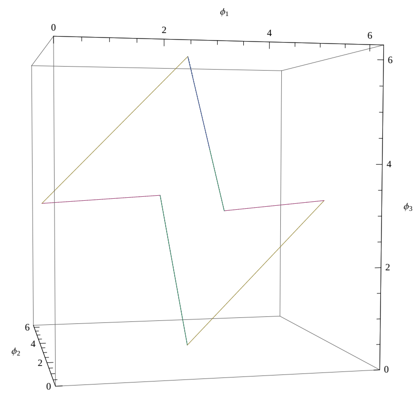


FIGURE 10. The base 3–torus as a cube with opposite sides identified and the singular locus consisting of three S^1s mutually intersecting in two points

(π,π,π)— and 2 pairs of branches coming together at the other two points —$(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{2})$ and $(\frac{3\pi}{2},\frac{3\pi}{2},\frac{3\pi}{2})$. The latter furnish double Dirac points.

Using the characteristic map the first type of singular point corresponds to an A_2 singularity and the second type corresponds to the type (A_1, A_1) stratum of the swallowtail. All the inverse images have discrete fibers. There are two image points on the A_2 stratum each with one inverse image under Ξ and there is one point on the (A_1, A_1) stratum, with two inverse images.

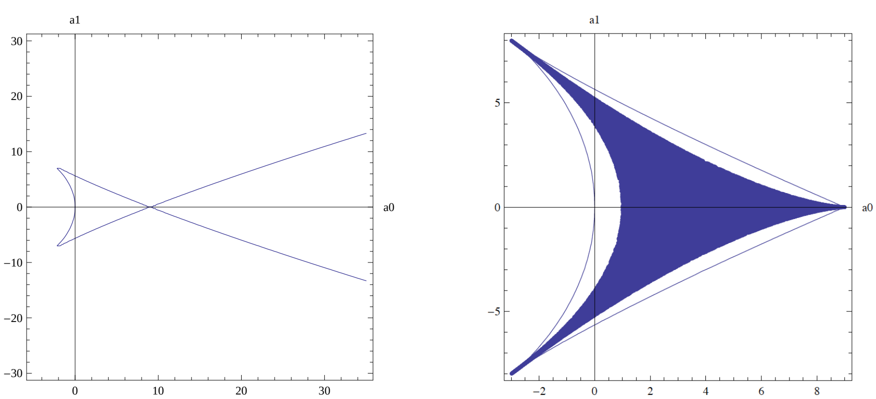


FIGURE 11. The -6 slice of the swallowtail of A_3 and the region occupied by the gyroid

All the A_1 singularities in the fibers are Dirac points. That is there are four of these points. Furthermore at all points there are enhanced symmetries by non–Abelian groups.

At $(0,0,0)$ the enhanced symmetry group is the symmetric group $\mathbb{S}_4$ —the full symmetry group of $\bar{\Gamma}$ which entirely lifts to $Aut(T^3)$— yielding one 1–dim irrep and one 3–dim irrep which forces the triple degeneracy. At (π,π,π) we have an *a priori* projective representation of $\mathbb{S}_4$, which we showed however to be equivalent to the standard representation of $\mathbb{S}_4$ and hence we again get one 1–dim irrep and one 3–dim irrep which forces the triple degeneracy. At the other two points things are really interesting. The stabilizer symmetry group is A_4 and it yields a projective representation which is carried by the double cover of A_4 aka. $2A_4$, 2T, the binary

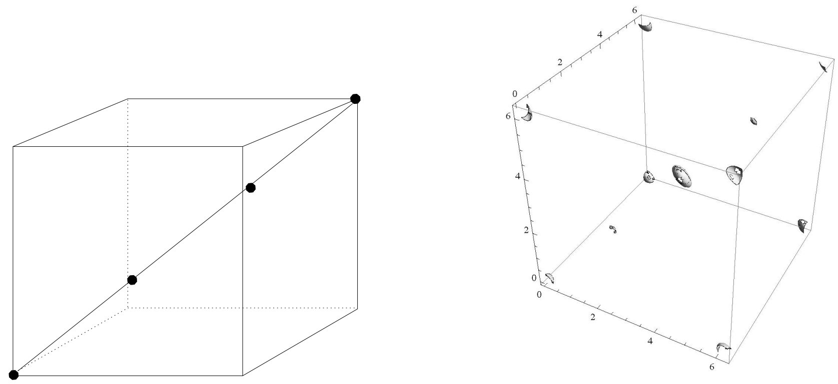


FIGURE 12. The 4 points of T_{deg} along the diagonal (again T^3 is depicted as a cube with opposite sides identified) and the 4 spheres making up $\partial\bar{N}$

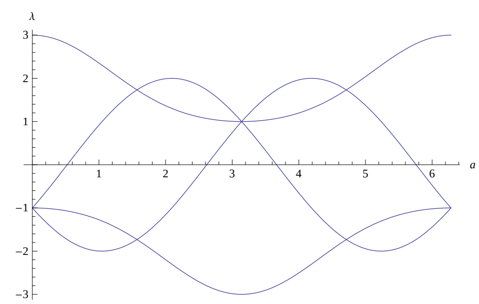


FIGURE 13. The spectrum/cover along the diagonal

tetrahedral group or $SL(2,3)$. The representation decomposes into two 2–dim irreps forcing the two double degeneracies.

Notice that we essentially need a projective representation, since A_4 itself has no 2–dim irreps.

Now T_{deg} is the set of the four points above and $T_0 = T^3 \setminus T_{\text{deg}}$ contracts onto a 2–dim CW complex with non–trivial second homology.

Thus there are K–theoretic and cohomological charges. This is the special case of dimension 3 with codimension 3 degenerate points and moreover we have a slicing of T^3 by the fiber bundle $T^3 \to S^1$ by any of the tree coordinate projections. In fact the homology is generated by any four slices which sit in between the 4 slices that contain the degenerate points. Pairing with these surfaces completely determines the Chern class of the line bundles and hence the line bundles up to isomorphism.

Figure 12 depicts T^3 with the 4 singular point as well as the 4 spheres making up $\partial\bar{N}$. Figure 13 shows the cover along the diagonal of T^3 which contains T_{deg}.

Figure 14 shows the different slices which on the one hand make up the CW complex and on the other give the submanifolds the curvature is integrated over to yield the $Q_{s,i}$. The relevant numerics were carried out in [25].

The result of the numerical slicing is contained in Figure 15 as well as the analytic values.

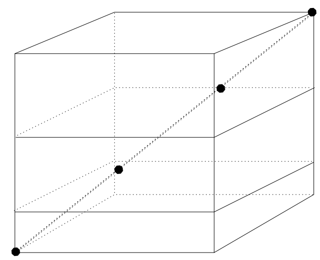


FIGURE 14. Slices corresponding to the first coordinate projection

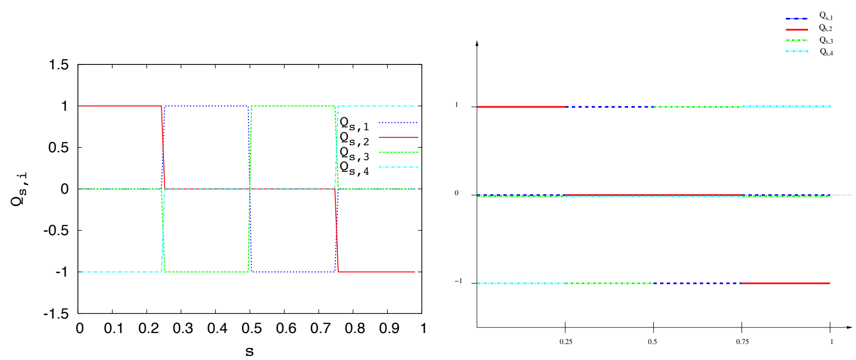


FIGURE 15. The charges $Q_{s,i}, i = 1, \ldots 4$ as functions of s, the position of the slice (compare to Fig. 14). Left: the result of a numerical computation. Right: the analytical values

In accordance with the analytic calculations of [34, 17] the Dirac points yield jumps in the charge by ± 1 for the two bands that cross. A new result is that the A_2 points yield jumps by $-2, 0, 2$ for the three bands that cross. This behavior is typical of a standard example of a 3–dimensional degenerating family of Hamiltonians with a single triple crossing considered in [7, 34] and we conjecture that indeed locally the family of the Gyroid is diffeomorphic to this family.

All these charges are topologically stable. In preliminary numerical simulations introducing symmetry breaking deformations we found that the A_2 points each split into four A_1 points in compliance with the jumps given above. We expect to explain this behavior using time reversal symmetry.

4.2. **Results on the non–commutative geometry.** In this section we summarise our results for the non-commutative geometry of the PDG, Bravais and Honeycomb wire networks, resulting form a constant magnetic field B (see 2.1.2 and [21, 22]).

4.2.1. *Honeycomb.* Generically $\mathscr{B}_\Theta = \mathbb{T}^2_\Theta$. In order to give the degenerate points, let

$$-e_1 := (1,0), e_2 = \frac{1}{2}(1, \sqrt{3}), e_3 := \frac{1}{2}(1, -\sqrt{3})$$

be the lattice vectors and $f_2 := e_2 - e_1 = \frac{1}{2}(-3, \sqrt{3})$, $f_3 := e_3 - e_1 = \frac{1}{2}(3, \sqrt{3})$ the period vectors of the honeycomb. The parameters we need are

$$(13) \qquad \theta := \hat{\Theta}(f_2, f_3), \quad q := e^{2\pi i\theta} \quad \textit{and} \quad \phi = \hat{\Theta}(-e_1, e_2), \quad \chi := e^{i\pi\phi}, \textit{thus} \quad q = \bar{\chi}^6$$

where $\hat{\Theta}$ is the quadratic from corresponding to the B–field $B = 2\pi\hat{\Theta}$.

Theorem 4.3. [21] *The algebra $\mathscr{B}_\Theta$ is the full matrix algebra of $M_2(\mathbb{T}^2_\theta)$ except in the following finite list of cases*

(1) $q = 1$.
(2) $q = -1$ *and* $\chi^4 = 1$.

The precise algebras are given in [21]. We wish to point out that $q = \chi = 1$ is the commutative case and $q = -\chi = 1$ is isomorphic to the commutative case, while the other cases give non–commutative proper subalgebras of $M_2(\mathbb{T}^2_\theta)$.

4.2.2. *P and Bravais cases.* For the simple cubic lattice and any other Bravais lattice of rank k (P is the rank 3 case): if $\Theta \neq 0$ then $\mathscr{B}_\Theta$ is simply the noncommutative torus $\mathbb{T}^k_\Theta$ and if $\Theta = 0$ then this $\mathscr{B}_0$ is the C^* algebra of T^k. There are no degenerate points.

4.2.3. *Diamond.* In the non–commutative case, we express our results in terms of parameters q_i and ξ_i defined as follows: Set $e_1 = \frac{1}{4}(1,1,1), e_2 = \frac{1}{4}(-1,-1,1), e_3 = \frac{1}{4}(-1,1,-1)$ for $B = 2\pi\Theta$ let

$$\Theta(-e_1, e_2) = \varphi_1 \quad \Theta(-e_1, e_3) = \varphi_2 \quad \Theta(e_2, e_3) = \varphi_3 \text{ and } \chi_i = e^{i\varphi_i} \text{ for } i = 1, 2, 3 \tag{14}$$

There are three operators U, V, W, given explicitly in [22], which span $\mathbb{T}^3_\Theta$ and have commutation relations

$$UV = q_1 VU \quad UW = q_2 WU \quad VW = q_3 WV \tag{15}$$

where the q_i expressed in terms of the χ_i are:

$$q_1 = \bar{\chi_1}^2 \chi_2^2 \chi_3^2 \quad q_2 = \bar{\chi_1}^6 \bar{\chi_2}^2 \bar{\chi_3}^2 \quad q_3 = \bar{\chi_1}^2 \bar{\chi_2}^6 \chi_3^2 \tag{16}$$

Vice versa, fixing the values of the q_i fixes the χ_i up to eighth roots of unity:

$$\chi_1^8 = \bar{q}_1 \bar{q}_2 \quad \chi_2^8 = q_1 \bar{q}_3 \quad \chi_3^8 = q_1^2 \bar{q}_2 q_3 \tag{17}$$

Other useful relations are $q_2\bar{q}_3 = \bar{\chi}_1^4 \chi_2^4 \bar{\chi}_3^4$ and $q_2 q_3 = \bar{\chi}_1^8 \bar{\chi}_2^8$. the algebra $\mathscr{B}_\Theta$ is the *full* matrix algebra *except* in the following cases in which it is a proper subalgebra.

(1) $q_1 = q_2 = q_3 = 1$ (the special bosonic cases) and one of the following is true:
   (a) All $\chi_i^2 = 1$ then $\mathscr{B}_\Theta$ is isomorphic to the commutative algebra in the case of no magnetic field above.
   (b) Two of the $\chi_i^4 = -1$, the third one necessarily being equal to 1.
(2) If $q_i = -1$ (special fermionic cases) and $\chi_i^4 = 1$. This means that either
   (a) all $\chi_i^2 = -1$ or
   (b) only one of the $\chi_i^2 = -1$ the other two being 1.
(3) $\bar{q}_1 = q_2 = q_3 = \bar{\chi}_2^4$ and $\chi_1^2 = 1$ it follows that $\chi_2^4 = \chi_3^4$. This is a one-parameter family.
(4) $q_1 = q_2 = q_3 = \bar{\chi}_1^4$ and $\chi_2^2 = 1$ it follows that $\chi_1^4 = \bar{\chi}_3^4$. This is a one-parameter family.
(5) $q_1 = q_2 = \bar{q}_3 = \bar{\chi}_1^4$ and $\chi_1^2 = \bar{\chi}_2^2$. It follows that $\chi_3^4 = 1$. This is a one-parameter family.

4.2.4. *Gyroid.* To state the results of [21] we use the bcc lattice vectors

$$g_1 = \frac{1}{2}(1,-1,1), \quad g_2 = \frac{1}{2}(-1,1,1), \quad g_3 = \frac{1}{2}(1,1,-1) \tag{18}$$

$$\theta_{12} = \frac{1}{2\pi} B \cdot (g_1 \times g_2), \quad \theta_{13} = \frac{1}{2\pi} B \cdot (g_1 \times g_3), \quad \theta_{23} = \frac{1}{2\pi} B \cdot (g_2 \times g_3)$$

$$\alpha_1 := e^{2\pi i \theta_{12}} \bar{\alpha}_2 := e^{2\pi i \theta_{13}} \alpha_3 := e^{2\pi i \theta_{23}}$$

$$\phi_1 = e^{\frac{\pi}{2} i \theta_{12}}, \quad \phi_2 = e^{\frac{\pi}{2} i \theta_{31}}, \quad \phi_3 = e^{\frac{\pi}{2} i \theta_{23}}, \quad \Phi = \phi_1 \phi_2 \phi_3$$

Classification Theorem.

(1) If $\Phi \neq 1$ or not all α_i are real then $\mathscr{B}_\Theta = M_4(\mathbb{T}^3_\Theta)$.
(2) If $\Phi = 1$, all $\alpha_i = \pm 1$, at least one $\alpha_i \neq 1$ and all ϕ_i are different then $\mathscr{B}_\Theta = M_4(\mathbb{T}^3_\Theta)$.
(3) If $\phi_i = 1$ for all i then the algebra is the same as in the commutative case.
(4) In all other cases (this is a finite list) $\mathscr{B}$ is non–commutative and $\mathscr{B}_\Theta \subsetneq M_4(\mathbb{T}^3_\Theta)$.

5. Outlook

5.1. **Observation and conjecture.** Looking at the cases above, we observe several regularities. First and foremost, there is agreement on the dimension of the degenerate locus in T^k between the commutative and the non–commutative case. In the commutative case, this locus is $T_{\deg} \subset T^k$; in the non–commutative case, it is the locus $T^{nc}_{\deg} \subset T^k$ of values of the B–field, where the matrix algebra is not the full matrix algebra. Here T^k parameterizes the entries of Θ mod $\mathbb{Z}$, which parameterize the non–commutative tori.

We conjecture that this is always the case.

There are several possible points of attack here. The first is through the symmetries: as we have seen, the re–gauging groupoid exists already in the non–commutative case. Another is to consider how, in the presence of a conserved topological charge, larger representations, such as A_2 in the gyroid case, break into smaller pieces. Using the slicing method described above, one can readily see how that happens under a deformation of the Hamiltonian in the commutative case. The question is whether the effect of non-commutativity is something similar.

5.2. **Stability, local structure and perturbations.** We furthermore plan to analyze the topological invariants further by studying local models for the crossings. In three dimensions, a double crossing has a unique local model up to orientation as already remarked in [7, 34]. In *loc. cit.* there are also examples of three-dimensional families with a unique singular point that corresponds to an n-fold crossing. Given the jumps in Chern classes we conjecture that for the Gyroid near the triple crossing the family of Hamiltonians restricted to the three bands that are involved is indeed diffeomorphic to that standard family. If this is established, we can show using an additional symmetry argument that the Chern class functions Q_s for the slicing and hence the whole geometry of line bundles is entirely determined by the singularities. A further study will then be how higher (more than double) topologically protected crossings dissolve under perturbations.

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ROTA–BAXTER ALGEBRAS, SINGULAR HYPERSURFACES, AND RENORMALIZATION ON KAUSZ COMPACTIFICATIONS

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Abstract. We consider Rota-Baxter algebras of meromorphic forms with poles along a (singular) hypersurface in a smooth projective variety and the associated Birkhoff factorization for algebra homomorphisms from a commutative Hopf algebra. In the case of a normal crossings divisor, the Rota-Baxter structure simplifies considerably and the factorization becomes a simple pole subtraction. We apply this formalism to the unrenormalized momentum space Feynman amplitudes, viewed as (divergent) integrals in the complement of the determinant hypersurface. We lift the integral to the Kausz compactification of the general linear group, whose boundary divisor is normal crossings. We show that the Kausz compactification is a Tate motive and that the boundary divisor and the divisor that contains the boundary of the chain of integration are mixed Tate configurations. The regularization of the integrals that we obtain differs from the usual renormalization of physical Feynman amplitudes, and in particular it may give mixed Tate periods in some cases that have non-mixed Tate contributions when computed with other renormalization methods.

1. Introduction

In this paper, we consider the problem of extracting periods of algebraic varieties from a class of divergent integrals arising in quantum field theory. The method we present here provides a regularization and extraction of finite values that differs from the usual (renormalized) physical Feynman amplitudes, but whose mathematical interest lies in the fact that it gives a period of a mixed Tate motive, for all graphs for which the amplitude can be computed using (global) forms with logarithmic poles. For more general graphs, one also obtains a period, where the nature of the motive involved depends on how a certain hyperplane arrangement intersects the big cell in a compactification of the general linear group. More precisely, the motive considered here is provided by the Kausz compactification of the general linear group and by a hyperplane arrangement that contains the boundary of the chain of integration.

The regularization procedure we propose is modeled on the algebraic renormalization method, based on Hopf algebras of graphs and Rota–Baxter algebras, as originally developed by Connes and Kreimer [22] and by Ebrahmi-Fard, Guo, and Kreimer [31]. The main difference in our approach is that we apply the formalism to a Rota–Baxter algebra of (even) meromorphic differential forms instead of applying it to a regularization of the integral. The procedure becomes especially simple in cases where the de Rham cohomology of the singular hypersurface complement is all realized by forms with logarithmic poles, in which case we replace the divergent integral with a family of convergent integrals obtained by a pole subtraction on the form and by (iterated) Poincaré residues. A similar approach was developed for integrals in configuration spaces by Ceyhan and the first author [21].

In Section 2 we introduce Rota–Baxter algebras of even meromorphic forms, along the lines of [21], and we formulate a general setting for extraction of finite values (regularization and renormalization) of divergent integrals modeled on algebraic renormalization applied to these Rota–Baxter algebras of differential forms.

In Section 3 we discuss the Rota–Baxter algebras of even meromorphic forms in the case of a smooth hypersurface $Y \subset X$. We show that, when restricted to forms with logarithmic poles, the Rota–Baxter operator becomes simply a derivation, and the Birkhoff factorization collapses to a simple pole subtraction, as in the case of log divergent graphs. We show that this simple pole subtraction can lead to too much loss of information about the unrenormalized integrand and we propose considering the additional information of the Poincaré residue and an additional integral associated to the residue.

In Section 4 we consider the case of singular hypersurfaces $Y \subset X$ given by a simple normal crossings divisor. We show that, in this case, the Rota–Baxter operator satisfies a simplified form of the Rota–Baxter identity, which however is not just a derivation. We show that this modified identity still suffices to have a simple pole subtraction $\phi_+(\Gamma) = (1-T)\phi(\Gamma)$ in the Birkhoff factorization, even though the negative piece $\phi_-(\Gamma)$ becomes more complicated. Again, to avoid too much loss of information in passing from $\phi(\Gamma)$ to $\phi_+(\Gamma)$, we consider, in addition to the renormalized integral $\int_\sigma \phi_+(\Gamma)$, the collection of integrals of the form $\int_{\sigma\cap Y_I} \mathrm{Res}_{Y_I}(\phi(\Gamma))$, where Res_{Y_I} is the iterated Poincaré residue, [25], along the intersection $Y_I = \cap_{j\in I} Y_j$ of components of Y. These integrals are all periods of mixed Tate motives if $\{Y_I\}$ is a mixed Tate configuration, in the sense of [33]. We discuss the question of further generalizations to more general types of singularities, beyond the normal crossings case, via Saito's theory of forms with logarithmic poles [58], by showing that one can also define a Rota–Baxter structure on the Saito complex of forms with logarithmic poles.

In Section 5 we present our main application, which is a regularization (different from the physical one) of the Feynman amplitudes in momentum space, computed on the complement of the determinant hypersurface as in [4]. Since the determinant hypersurface has worse singularities than what we need, we pull back the integral computation to the Kausz compactification [47] of the general linear group, where the boundary divisor that replaces the determinant hypersurface is a simple normal crossings divisor. We show that the motive of the Kausz compactification is Tate, and that the components of the boundary divisor form a mixed Tate configuration. We discuss how one can replace the form η_Γ of the Feynman amplitude with a form with logarithmic poles. In general, this form is defined on the big cell of the Kausz compactification. For certain graphs, it is possible to show, using the mixed Hodge structure, that the form with logarithmic poles extends globally to the Kausz compactification, with poles along the boundary divisor.

2. Rota–Baxter algebras of meromorphic forms

We generalize the algebraic renormalization formalism to a setting based on Rota–Baxter algebras of algebraic differential forms on a smooth projective variety with poles along a hypersurface.

2.1. **Rota–Baxter algebras.** A Rota–Baxter algebra of weight λ is a unital commutative algebra $\mathcal{R}$ over a field K such that $\lambda \in K$, together with a linear operator $T : \mathcal{R} \to \mathcal{R}$ satisfying the Rota–Baxter identity

$$T(x)T(y) = T(xT(y)) + T(T(x)y) + \lambda T(xy). \tag{2.1}$$

For example, Laurent polynomials $\mathcal{R} = \mathbb{C}[z, z^{-1}]$ with T the projection onto the polar part are a Rota–Baxter algebra of weight -1.

The Rota–Baxter operator T of a Rota–Baxter algebra of weight -1, satisfying

$$T(x)T(y) + T(xy) = T(xT(y)) + T(T(x)y), \tag{2.2}$$

determines a splitting of $\mathcal{R}$ into $\mathcal{R}_+ = (1-T)\mathcal{R}$ and $T\mathcal{R}$, where $(1-T)\mathcal{R}$ and $T\mathcal{R}$ are not just vector spaces but algebras, because of the Rota–Baxter relation (2.2). The algebra $T\mathcal{R}$ is non-unital. In order to work with unital algebras, one defines $\mathcal{R}_-$ to be the unitization of $T\mathcal{R}$, that is, $T\mathcal{R} \oplus K$ with multiplication $(x,t)(y,s) = (xy + ty + sx, ts)$. For an introduction to Rota–Baxter algebras we refer the reader to [38].

2.2. **Rota–Baxter algebras of even meromorphic forms.** Let Y be a hypersurface in a projective variety X, with defining equation $Y = \{f = 0\}$. We denote by $\mathcal{M}^\star_X$ the sheaf of meromorphic differential forms on X, and by $\mathcal{M}^\star_{X,Y}$ the subsheaf of meromorphic forms on with poles (of arbitrary order) along Y, that is, $\mathcal{M}^\star_{X,Y} = j_*\Omega^1_U$, where $U = X \smallsetminus Y$ and $j : U \hookrightarrow X$ is the inclusion. Passing to global sections of $\mathcal{M}^\star_{X,Y}$ gives a graded-commutative algebra over the field of definition of the varieties X and Y, which, for simplicity, we will still denote by $\mathcal{M}^\star_{X,Y}$. We can write forms $\omega \in \mathcal{M}^\star_{X,Y}$ as sums $\omega = \sum_{p\geq 0} \alpha_p/f^p$, where the α_p are holomorphic forms.

In particular, we consider forms of even degrees, so that $\mathcal{M}^{\text{even}}_{X,Y}$ is a commutative algebra under the wedge product.

Lemma 2.1. *The commutative algebra $\mathcal{M}^{\text{even}}_{X,Y}$, together with the linear operator $T : \mathcal{M}^{\text{even}}_{X,Y} \to \mathcal{M}^{\text{even}}_{X,Y}$, defined as the polar part*

$$T(\omega) = \sum_{p\geq 1} \alpha_p/f^p, \tag{2.3}$$

is a Rota–Baxter algebra of weight -1.

Proof. For $\omega_1 = \sum_{p\geq 0} \alpha_p/f^p$ and $\omega_2 = \sum_{q\geq 0} \beta_q/f^q$, we have

$$T(\omega_1 \wedge \omega_2) = \sum_{p\geq 0, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}} + \sum_{p\geq 1, q\geq 0} \frac{\alpha_p \wedge \beta_q}{f^{p+q}} - \sum_{p\geq 1, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

$$T(T(\omega_1) \wedge \omega_2) = \sum_{p\geq 1, q\geq 0} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

$$T(\omega_1 \wedge T(\omega_2)) = \sum_{p\geq 0, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

$$T(\omega_1) \wedge T(\omega_2) = \sum_{p\geq 1, q\geq 1} \frac{\alpha_p \wedge \beta_q}{f^{p+q}},$$

so that (2.2) is satisfied. ☐

Note that the restriction to even form is introduced only in order to ensure that the resulting Rota–Baxter algebra is commutative, while (2.3) satisfies (2.2) regardless of the restriction on degrees.

Remark 2.2. Equivalently, we have the following description of the Rota–Baxter operator, which we will use in the following. The linear operator

$$T(\omega) = \alpha \wedge \xi, \quad \text{for } \omega = \alpha \wedge \xi + \eta, \tag{2.4}$$

acting on forms $\omega = \alpha \wedge \xi + \eta$, with α a meromorphic form on X with poles on Y and ξ and η holomorphic forms on X, is a Rota–Baxter operator of weight -1.

The Rota–Baxter identity is equivalently seen then as follows. For $\omega_i = \alpha_i \wedge \xi_i + \eta_i$, with $i = 1, 2$, we have

$$T(\omega_1 \wedge \omega_2) = (-1)^{|\alpha_2|\,|\xi_1|}\alpha_1 \wedge \alpha_2 \wedge \xi_1 \wedge \xi_2 + \alpha_1 \wedge \xi_1 \wedge \eta_2 + (-1)^{|\eta_1|\,|\alpha_2|}\alpha_2 \wedge \eta_1 \wedge \xi_2$$

while

$$T(T(\omega_1)\wedge\omega_2) = (-1)^{|\alpha_2|\,|\xi_1|}\alpha_1\wedge\alpha_2\wedge\xi_1\wedge\xi_2 + \alpha_1\wedge\xi_1\wedge\eta_2$$
$$T(\omega_1\wedge T(\omega_2)) = (-1)^{|\alpha_2|\,|\xi_1|}\alpha_1\wedge\alpha_2\wedge\xi_1\wedge\xi_2 + (-1)^{|\eta_1|\,|\alpha_2|}\alpha_2\wedge\eta_1\wedge\xi_2$$

and

$$T(\omega_1)\wedge T(\omega_2) = (-1)^{|\alpha_2|\,|\xi_1|}\alpha_1\wedge\alpha_2\wedge\xi_1\wedge\xi_2,$$

where all signs are positive if the forms are of even degree. Thus, the operator T satisfies (2.2).

The proof automatically extends to the following slightly more general setting.

Lemma 2.3. *Let (X_ℓ, Y_ℓ) for $\ell \geq 1$ be a collection of smooth projective varieties X_ℓ with hypersurfaces Y_ℓ, all defined over the same field. Then the commutative algebra $\bigwedge_\ell \mathcal{M}^{\rm even}_{X_\ell,Y_\ell}$ is a Rota–Baxter algebra of weight -1 with the polar projection operator T determined by the T_ℓ on each $\mathcal{M}^{\rm even}_{X_\ell,Y_\ell}$.*

A similar setting was considered in Theorem 6.4 of [21].

2.3. **Renormalization via Rota–Baxter algebras.** In [22], the BPHZ renormalization procedure of perturbative quantum field theory was reinterpreted as a Birkhoff factorization of loops in the pro-unipotent group of characters of a commutative Hopf algebra of Feynman graphs. This procedure of *algebraic renormalization* was reformulated in more general and abstract terms in [31], using Hopf algebras and Rota–Baxter algebras.

We summarize here quickly the basic setup of algebraic renormalization. We refer the reader to [22], [23], [31], [52] for more details.

The Connes–Kreimer Hopf algebra of Feynman graphs $\mathcal{H}$ is a commutative, non-cocommutative, graded, connected Hopf algebra over $\mathbb{Q}$ associated to a given Quantum Field Theory (QFT). A theory is specified by assigning a Lagrangian and the corresponding action functional, which in turn determines which graphs occur as Feynman graphs of the theory. For instance, the only allowed valences of vertices in a Feynman graph are the powers of the monomials in the fields that appear in the Lagrangian. The generators of the Connes–Kreimer Hopf algebra of a given QFT are the 1PI Feynman graphs Γ of the theory, namely those Feynman graphs that are 2-egde connected. As a commutative algebra, $\mathcal{H}$ is then just a polynomial algebra in the 1PI graphs Γ. A grading on $\mathcal{H}$ is given by the loop number (first Betti number) of graphs. In the case where Feynman graphs also have vertices of valence 2, one uses the number of internal edges instead of loop number, to have finite dimensional graded pieces, but we ignore this subtlety for the present purposes. The grading satisfies

$$\deg(\Gamma_1\cdots\Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0.$$

The connectedness property means that the degree zero part is just $\mathbb{Q}$. The coproduct in $\mathcal{H}$ is given by

$$\Delta(\Gamma) = \Gamma\otimes 1 + 1\otimes\Gamma + \sum_{\gamma\in\mathcal{V}(\Gamma)} \gamma\otimes\Gamma/\gamma, \tag{2.5}$$

where the class $\mathcal{V}(\Gamma)$ consists of all (not necessarily connected) divergent subgraphs γ such that the quotient graph (identifying each component of γ to a vertex) is still a 1PI Feynman graph of the theory. As in any graded connected Hopf algebra, the antipode is constructed inductively as

$$S(\Gamma) = -\Gamma - \sum S(\Gamma')\Gamma''$$

for $\Delta(\Gamma) = \Gamma\otimes 1 + 1\otimes\Gamma + \sum \Gamma'\otimes\Gamma''$, with the terms Γ', Γ'' of lower degrees.

Remark 2.4. The general element in the Hopf algebra $\mathcal{H}$ is not a graph Γ but a polynomial function $P = \sum a_{i_1,\ldots,i_k} \Gamma_{i_1}^{n_{i_1}} \cdots \Gamma_{i_k}^{n_{i_k}}$ with $\mathbb{Q}$ coefficients in the generators given by the graphs. However, for simplicity of notation, in the following we will just write Γ to denote an arbitrary element of $\mathcal{H}$.

An *algebraic Feynman rule* $\phi : \mathcal{H} \to \mathcal{R}$ is a *homomorphism of commutative algebras* from the Hopf algebra $\mathcal{H}$ of Feynman graphs to a Rota–Baxter algebra $\mathcal{R}$ of weight -1,

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R}).$$

The set $\mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R})$ has a group structure, where the multiplication $\star$ is dual to the coproduct in the Hopf algebra, $\phi_1 \star \phi_2(\Gamma) = \langle \phi_1 \otimes \phi_2, \Delta(\Gamma)\rangle$.

Algebra homomorphisms $\phi : \mathcal{H} \to \mathcal{R}$ between a Hopf algebra $\mathcal{H}$ and a Rota–Baxter algebra $\mathcal{R}$ are also often referred to as "characters" in the renormalization literature.

The morphism ϕ by itself does not know about the coalgebra structure of $\mathcal{H}$ and the Rota–Baxter structure of $\mathcal{R}$. These enter in the factorization of ϕ into divergent and finite part.

A Birkhoff factorization of an algebraic Feynman rule consists of a pair of commutative algebra homomorphisms

$$\phi_\pm \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R}_\pm)$$

where $\mathcal{R}_\pm$ is the splitting of $\mathcal{R}$ induced by the Rota–Baxter operator T, with $\mathcal{R}_+ = (1-T)\mathcal{R}$ and $\mathcal{R}_-$ the unitization of $T\mathcal{R}$, satisfying

$$\phi = (\phi_- \circ S) \star \phi_+,$$

with the product $\star$ dual to the coproduct Δ as above. The Birkhoff factorization is unique if one also imposes the normalization condition $\epsilon_- \circ \phi_- = \epsilon$, where ϵ is the counit of $\mathcal{H}$ and ϵ_- is the augmentation in the algebra $\mathcal{R}_-$.

As shown in Theorem 4 of [22] (see equations (32) and (33) therein), there is an inductive formula for the Birkhoff factorization of an algebraic Feynman rule, of the form

$$\phi_-(\Gamma) = -T(\phi(\Gamma) + \sum \phi_-(\Gamma')\phi(\Gamma'')) \quad \text{and} \quad \phi_+(\Gamma) = (1-T)(\phi(\Gamma) + \sum \phi_-(\Gamma')\phi(\Gamma'')) \tag{2.6}$$

where $\Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum \Gamma' \otimes \Gamma''$.

The Birkhoff factorization (2.6) of algebra homomorphisms $\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R})$ is often referred to as "algebraic Birkhoff factorization", to distinguish it from the (analytic) Birkhoff factorization formulated in terms of loops (or infinitesimal loops) with values in Lie groups. We refer the reader to §6.4 of Chapter 1 of [23] for a discussion of the relation between these two kinds of Birkhoff factorization.

In the original Connes–Kreimer formulation, this approach is applied to the unrenormalized Feynman amplitudes regularized by dimensional regularization, with the Rota–Baxter algebra consisting of germs of meromorphic functions at the origin, with the operator of projection onto the polar part of the Laurent series.

In the following, we consider the following variant on the Hopf algebra of Feynman graphs.

Definition 2.5. *As an algebra, $\mathcal{H}_{\mathrm{even}}$ is the commutative algebra generated by Feynman graphs of a given scalar quantum field theory that have an even number of internal edges, $\#E(\Gamma) \in 2\mathbb{N}$. The coproduct (2.5) on $\mathcal{H}_{\mathrm{even}}$ is similarly defined with the sum over divergent subgraphs γ with even $\#E(\gamma)$, with 1PI quotient.*

Notice that in dimension $D \in 4\mathbb{N}$ all the log divergent subgraphs $\gamma \subset \Gamma$ have an even number of edges, since $Db_1(\gamma) = 2\#E(\gamma)$ in this case. This is a class of graphs that are especially interesting in physical applications.

Question 2.6. *Is there a graded-commutative version of Birkhoff factorization involving graded-commutative Rota–Baxter and Hopf algebras?*

Such an extension to the graded-commutative case would be necessary to include the more general case of differential forms of odd degree (associated to Feynman graphs with an odd number of internal edges).

One can approach the question above by using the general setting of [32]:

1. Let $\mathcal{H}$ be any connected filtered cograded Hopf algebra and let $\mathcal{R}$ be a (not necessarily commutative) associative algebra equipped with a Rota-Baxter operator of weight $\lambda \neq 0$. The algebraic Birkhoff factorization of any $\phi \in \mathrm{Hom}(\mathcal{H}, \mathcal{R})$ was obtained by Ebrahimi-Fard, Guo and Kreimer in [32].
2. However, if the target algebra $\mathcal{R}$ is not commutative, the set of characters $\mathrm{Hom}(\mathcal{H}, \mathcal{R})$ is not a group since it is not closed under convolution product, i.e. if $f, g \in \mathrm{Hom}(\mathcal{H}, \mathcal{R})$, then $f \star g$ does not necessarily belong to $\mathrm{Hom}(\mathcal{H}, \mathcal{R})$.

The usual proof (see Theorem 4 of [22] and Theorem 1.39 in Chapter 1 of [23]) of the fact that the two parts $\phi_\pm$ of the Birkhoff factorization are algebra homomorphisms uses explicitly both the commutativity of the target Rota–Baxter algebra $\mathcal{R}$ and the fact that $\mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R})$ is a group, and does not extend directly to the graded-commutative case. The argument given in Theorems 3.4 and 3.7 of [32] provides a more general form of Birkhoff factorization that applies to a graded-commutative (and more generally non-commutative) Rota–Baxter algebra. The resulting form of the factorization is more complicated than in the commutative case, in general. However, if the Rota–Baxter operator of weight -1 also satisfies $T^2 = T$ and $T(T(x)y) = T(x)y$ for all $x, y \in \mathcal{R}$, then the form of the Birkhoff factorization for not necessarily commutative Rota–Baxter algebras simplifies considerably, and the ϕ_+ part of the factorization consists of a simple pole subtraction, as we prove in Proposition 2.10 below.

2.4. **Rota–Baxter algebras and Atkinson factorization.** In the following we will discuss some interesting properties of algebraic Birkhoff decomposition when the Rota-Baxter operator satisfies the identity $T(T(x)y) = T(x)y$.

Let $e : \mathcal{H} \to \mathcal{R}$ be the unit of $\mathrm{Hom}(\mathcal{H}, \mathcal{R})$ (under the convolution product) defined by $e(1_{\mathcal{H}}) = 1_{\mathcal{R}}$ and $e(\Gamma) = 0$ on $\oplus_{n>0}\mathcal{H}_n$.

The main observation can be summarized as follows:

1. If the Rota-Baxter operator T on $\mathcal{R}$ also satisfies the identity $T(T(x)y) = T(x)y$, then on $\ker(e) = \oplus_{n>0}\mathcal{H}_n$, the negative part of the Birkhoff factorization ϕ_- takes the following form:
$$\phi_- = -T(\phi(\Gamma)) - \sum T(\phi(\Gamma'))\phi(\Gamma''), \quad \text{for } \Delta(\Gamma) = 1 \otimes \Gamma + \Gamma \otimes 1 + \sum \Gamma' \otimes \Gamma''.$$
2. If T also satisfies $T(xT(y)) = xT(y)$, $\forall x, y \in \mathcal{R}$, then the positive part is given by $\phi_+(\Gamma) = (1 - T)(\phi(\Gamma))$, $\forall \Gamma \in \ker(e) = \oplus_{n>0}\mathcal{H}_n$.

This follows from the properties of the Atkinson Factorization in Rota–Baxter algebras, which we recall below.

Proposition 2.7. (Atkinson Factorization, [7], see also [39]) *Let $(\mathcal{R}, T)$ be a Rota-Baxter algebra of weight $\lambda \neq 0$. Let $\tilde{T} = -\lambda\mathrm{id} - T$ and let $a \in \mathcal{R}$. Assume that b_l and b_r are solutions of the fixed point equations*

$$b_l = 1 + T(b_l a), \quad b_r = 1 + \tilde{T}(a b_r). \tag{2.7}$$

Then

$$b_l(1 + \lambda a)b_r = 1.$$

Thus

$$1 + \lambda a = b_l^{-1} b_r^{-1} \tag{2.8}$$

if b_l and b_r are invertible.

A Rota-Baxter algebra $(\mathcal{R}, T)$ is called complete if there are algebras $\mathcal{R}_n \subseteq \mathcal{R}, n \geq 0$, such that $(\mathcal{R}, \mathcal{R}_n)$ is a complete algebra and $T(\mathcal{R}_n) \subseteq \mathcal{R}_n$.

Proposition 2.8. (Existence and uniqueness of the Atkinson Factorization, [39]) *Let $(\mathcal{R}, T, \mathcal{R}_n)$ be a complete Rota-Baxter algebra of weight $\lambda \neq 0$. Let $\tilde{T} = -\lambda \mathrm{id} - T$ and let $a \in \mathcal{R}_1$.*

(1) *Equations (2.7) have unique solutions b_l and b_r. Further b_l and b_r are invertible. Hence the Atkinson Factorization (2.8) exists.*
(2) *If $\lambda \neq 0$ and $T^2 = -\lambda T$ (in particular if $T^2 = -\lambda T$ on $\mathcal{R}$), then there are unique $c_l \in 1 + T(\mathcal{R})$ and $c_r \in 1 + \tilde{T}(\mathcal{R})$ such that*

$$1 + \lambda a = c_l c_r.$$

Define

$$(Ta)^{[n+1]} := T((Ta)^{[n]} a) \quad \text{and} \quad (Ta)^{\{n+1\}} = T(a(Ta)^{\{n\}})$$

with the convention that $(Ta)^{[1]} = T(a) = (Ta)^{\{1\}}$ and $(Ta)^{[0]} = 1 = (Ta)^{\{0\}}$.

Proposition 2.9. *Let $(\mathcal{R}, \mathcal{R}_n, T)$ be a complete filtered Rota-Baxter algebra of weight -1 such that $T^2 = T$. Let $a \in \mathcal{R}_1$. If T also satisfies the following identity*

$$T(T(x)y) = T(x)y, \quad \forall x, y \in \mathcal{R}, \tag{2.9}$$

then the equation

$$b_l = 1 + T(b_l a). \tag{2.10}$$

has a unique solution

$$1 + T(a)(1 - a)^{-1}.$$

Proof. First, we have $(Ta)^{[n+1]} = T(a)a^n$ for $n \geq 0$. In fact, the case when $n = 0$ just follows from the definition. Suppose it is true up to n, then

$$(Ta)^{[n+2]} = T((Ta)^{[n+1]} a) = T((T(a)a^n)a) = T(T(a)a^{n+1}) = T(a)a^{n+1}.$$

Arguing as in [32], $b_l = \sum_{n=0}^{\infty} (Ta)^{[n]} = 1 + T(a) + T(T(a)a) + \cdots + (Ta)^{[n]} + \cdots$ is the unique solution of (2.10). So

$$\begin{aligned} b_l &= 1 + T(a) + T(a)a + T(a)a^2 + \cdots \\ &= 1 + T(a)(1 + a + a^2 + \cdots) \\ &= 1 + T(a)(1 - a)^{-1}. \end{aligned}$$

□

A bialgebra $\mathcal{H}$ over a field K is called a connected, filtered cograded bialgebra if there are subspaces $\mathcal{H}_n$ of $\mathcal{H}$ such that (a) $\mathcal{H}_p \mathcal{H}_q \subseteq \sum_{k \leq p+q} \mathcal{H}_k$; (b) $\Delta(\mathcal{H}_n) \subseteq \oplus_{p+q=n} \mathcal{H}_p \otimes \mathcal{H}_q$; (c) $\mathcal{H}_0 = \mathrm{im}(u) = K$, where $u : K \to \mathcal{H}$ is the unit of $\mathcal{H}$.

Proposition 2.10. *Let $\mathcal{H}$ be a connected filtered cograded bialgebra (hence a Hopf algebra) and let $(\mathcal{R}, T)$ be a (not necessarily commutative) Rota-Baxter algebra of weight $\lambda = -1$ with $T^2 = T$. Suppose that T also satisfies (2.9). Let $\phi : \mathcal{H} \to \mathcal{R}$ be a character, that is, an algebra* *homomorphism. Then there are unique maps* $\phi_- : \mathcal{H} \to T(\mathcal{R})$ *and* $\phi_+ : \mathcal{H} \to \tilde{T}(\mathcal{R})$, *where* $\tilde{T} = 1 - T$, *such that*

$$\phi = \phi_-^{*(-1)} * \phi_+,$$

where $\phi^{*(-1)} = \phi \circ S$, *with* S *the antipode.* ϕ_- *takes the following form on* $\ker(e) = \oplus_{n>0}\mathcal{H}_n$*:*

$$\begin{aligned}\phi_-(\Gamma) &= -T(\phi(\Gamma)) - \sum_{n=1}^{\infty}(-1)^n \sum T(\phi(\Gamma^{(1)}))\phi(\Gamma^{(2)})\phi(\Gamma^{(3)})\cdots\phi(\Gamma^{(n+1)}) \\ &= -T(\phi(\Gamma)) - \sum_{n=1}^{\infty}(-1)^n((T\phi)\tilde{*}\phi^{\tilde{*}^n})(\Gamma).\end{aligned}$$

Here we use the notation $\tilde{\Delta}^{n-1}(\Gamma) = \sum \Gamma^{(1)} \otimes \cdots \otimes \Gamma^{(n)}$, *and* $\tilde{\Delta}(\Gamma) := \Delta(\Gamma) - \Gamma \otimes 1 - 1 \otimes \Gamma$ *(which is coassociative), and* $\tilde{*}$ *is the convolution product defined by* $\tilde{\Delta}$. *Furthermore, if* T *satisfies*

$$T(xT(y)) = xT(y), \quad \forall x, y \in A, \tag{2.11}$$

then ϕ_+ *takes the form on* $\ker(e) = \oplus_{n>0}\mathcal{H}_n$*:*

$$\phi_+(\Gamma) = (1 - T)(\phi(\Gamma)).$$

Proof. Define $R := \mathrm{Hom}(\mathcal{H}, \mathcal{R})$ and

$$P : R \to R, \quad P(f)(\Gamma) = T(f(\Gamma)),\ f \in \mathrm{Hom}(\mathcal{H}, \mathcal{R}), \Gamma \in \mathcal{H}.$$

Then by [39], R is a complete algebra with filtration $R_n = \{f \in \mathrm{Hom}(\mathcal{H}, \mathcal{R}) | f(\mathcal{H}_{n-1}) = 0\}$, $n \geq 0$, and P is a Rota-Baxter operator of weight -1 and $P^2 = P$. Moreover, since T satisfies (2.9), it is easy to check that $P(P(f)g) = P(f)g$ for any $f, g \in \mathrm{Hom}(\mathcal{H}, \mathcal{R})$. Let $\phi : \mathcal{H} \to \mathcal{R}$ be a character. Then $(e - \phi)(1_{\mathcal{H}}) = e(1_{\mathcal{H}}) - \phi(1_{\mathcal{H}}) = 1_{\mathcal{R}} - 1_{\mathcal{R}} = 0$. So $e - \phi \in \mathcal{R}_1$. Set $a = e - \phi$, by Proposition 2.8, we know that there are unique $c_l \in T(\mathcal{R})$ and $c_r \in (1 - T)(\mathcal{R})$ such that $\phi = c_l c_r$. Moreover, by Proposition 2.9, we have

$$\phi_- = b_l = c_l^{-1} = e + T(a)(e - a)^{-1} = e + T(e - \phi)\sum_{n=0}^{\infty}(e - \phi)^n.$$

We also have $\sum_{n=0}^{\infty}(e - \phi)^n(1_{\mathcal{H}}) = 1_{\mathcal{R}}$ and for any $X \in \ker(e) = \oplus_{n>0}\mathcal{H}_n$, we have

$$\begin{aligned}&(e - \phi)^0(\Gamma) = e(\Gamma) = 0; \quad (e - \phi)^1(\Gamma) = -\phi(\Gamma); \\ &(e - \phi)^2(\Gamma) = \sum(e - \phi)(\Gamma')(e - \phi)(\Gamma'') = \sum \phi(\Gamma')\phi(\Gamma'').\end{aligned}$$

More generally, we have $(e-\phi)^n(\Gamma) = (-1)^n \sum \phi(\Gamma^{(1)})\phi(\Gamma^{(2)})\cdots\phi(\Gamma^{(n)}) = (-1)^n \phi^{\tilde{*}^n}(\Gamma)$. So for $X \in \ker(e) = \oplus_{n>0}\mathcal{H}_n$,

$$\begin{aligned}
\phi_-(\Gamma) &= (T(e-\phi)\sum_{n=0}^{\infty}(e-\phi)^n)(\Gamma) \\
&= T(e-\phi)(1_{\mathcal{H}})\sum_{n=0}^{\infty}(e-\phi)^n(\Gamma) + T(e-\phi)(\Gamma)\sum_{n=0}^{\infty}(e-\phi)^n(1_{\mathcal{H}}) \\
&\quad + \sum T((e-\phi)(\Gamma'))\sum_{n=1}^{\infty}(e-\phi)^n(\Gamma'') \\
&= -T(\phi(\Gamma)) - \sum T(\phi(\Gamma'))\sum_{n=1}^{\infty}(-1)^n \sum \phi((\Gamma'')^{(1)})\phi((\Gamma'')^{(2)})\cdots\phi((\Gamma'')^{(n)}) \\
&= -T(\phi(\Gamma)) - \sum_{n=1}^{\infty}(-1)^n \sum T(\phi(\Gamma^{(1)}))\phi(\Gamma^{(2)})\phi(X^{(3)})\cdots\phi(\Gamma^{(n+1)}) \\
&= -T(\phi(\Gamma)) - \sum_{n=1}^{\infty}(-1)^n((T\phi)\tilde{*}\phi^{\tilde{*}^n})(\Gamma).
\end{aligned}$$

Suppose that T also satisfies equation (2.11), then for any $a, b \in \mathcal{R}$, we have

$$\begin{aligned}
(1-T)(a)(1-T)(b) &= ab - T(a)b - aT(b) + T(a)T(b) \\
&= ab - T(T(a)b) - T(aT(b)) + T(a)T(b) \\
&= ab - T(ab) = (1-T)(ab),
\end{aligned}$$

as T is a Rota-Baxter operator of weight -1. As shown in [22] and [32],

$$\phi_+(\Gamma) = (1-T)(\phi(\Gamma) + \sum \phi_-(\Gamma')\phi(\Gamma'')).$$

So $\phi_+(\Gamma) = (1-T)(\phi(\Gamma)) + \sum(1-T)(\phi_-(\Gamma'))(1-T)(\phi(\Gamma''))$ by the previous computation. But ϕ_- is in the image of T and $T^2 = T$, so we must have $(1-T)(\phi_-(\Gamma')) = 0$, which shows that $\phi_+(\Gamma) = (1-T)(\phi(\Gamma))$. ☐

2.5. **A variant of algebraic renormalization.** We consider now a setting inspired by the formalism of the Connes–Kreimer renormalization recalled above. The setting generalizes the one considered in [21] for configuration space integrals and our main application will be to extend the approach of [21] to momentum space integrals.

The main difference with respect to the Connes–Kreimer renormalization is that, instead of renormalizing the Feynman amplitude (regularized so that it gives a meromorphic function), we propose to renormalize the differential form, before integration, and then integrate the renormalized form to obtain a period.

The result obtained by this method differs from the physical renormalization, as we will discuss further in Section 5.11 below. There are at present no explicit examples of periods that are known not to be expressible in terms of rational combinations of mixed Tate periods, just because no such general statement of algebraic independence of numbers is known. However, it is generally expected that motives that are not mixed Tate will have periods that are not expressible in terms of mixed Tate periods, for instance periods associated to H^1 of an elliptic curve. There are known examples ([18], [19]) of Feynman integrals that give periods of non-mixed Tate motives (a K3 surface, for instance). In our setting, the period obtained by applying the Birkhoff factorization to the Feynman integrand η_Γ is always a mixed Tate period. However, it is difficult to ensure that the result is non-trivial. As we will discuss in more detail in Section 5, one can ensure a non-trivial result by replacing the form η_Γ with a cohomologous form with logarithmic poles and taking into account both the result of the pole subtraction and all the Poincaré residues. However, passing to a form with logarithmic poles requires, in general, restricting to the big cell of the Kausz compactification, and this introduces a constraint on the nature of the period. If the intersection of the big cell of the Kausz compactification with the divisor $\Sigma_{\ell,g}$ that contains the boundary of the chain of integeration is a mixed Tate motive, then the convergent integral we obtain by replacing the integration form with a form with logarithmic poles is a mixed Tate period. For particular graphs, for which the form with logarithmic poles extends globally to the Kausz compactification, with poles along the boundary divisor, we obtain a mixed Tate period without any further assumption.

The main steps required for our setup are the following. For a variety X, we denote by $\mathfrak{m}(X)$ the motive in the Voevodsky category.

- For each $\ell \geq 1$, we construct a pair (X_ℓ, Y_ℓ) of a smooth projective variety X_ℓ (defined over $\mathbb{Q}$) whose motive $\mathfrak{m}(X_\ell)$ is mixed Tate (over $\mathbb{Z}$), together with a (singular) hypersurface $Y_\ell \subset X_\ell$.
- For each Feynman graph Γ with loop number ℓ we construct a map
$$\Upsilon : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \to X_\ell \smallsetminus Y_\ell,$$
where $\hat{X}_\Gamma \subset \mathbb{A}^n$ is the affine graph hypersurface, with n the number of edges of Γ.
- Using the map Υ, we describe the Feynman integrand as a morphism of commutative algebras
$$\phi : \mathcal{H}_{\text{even}} \to \bigwedge_\ell \mathcal{M}^{\text{even}}_{X_\ell, Y_\ell}, \quad \phi(\Gamma) = \eta_\Gamma,$$
with $\mathcal{H}$ the Connes–Kreimer Hopf algebra and with the Rota–Baxter structure of Lemma 2.3 on the target algebra, and with η_Γ an algebraic differential form on X_ℓ with polar locus Y_ℓ, for $\ell = b_1(\Gamma)$.
- We express the (unrenormalized) Feynman integrals as a (generally divergent) integral $\int_{\Upsilon(\sigma)} \eta_\Gamma$, over a chain $\Upsilon(\sigma)$ in X_ℓ that is the image of a chain σ in $\mathbb{A}^n$.
- We construct a divisor $\Sigma_\ell \subset X_\ell$, that contains the boundary $\partial\Upsilon(\sigma)$, whose motive $\mathfrak{m}(\Sigma_\ell)$ is mixed Tate (over $\mathbb{Z}$) for all $\ell \geq 1$.
- We perform the Birkhoff decomposition $\phi_\pm$ obtained inductively using the coproduct on $\mathcal{H}$ and the Rota–Baxter operator T (polar part) on $\mathcal{M}^*_{X_\ell, Y_\ell}$.
- This gives a holomorphic form $\phi_+(\Gamma)$ on X_ℓ. The divergent Feynman integral is then replaced by the integral
$$\int_{\Upsilon(\sigma)} \phi_+(\Gamma)$$
which is a period of the mixed Tate motive $\mathfrak{m}(X_\ell, \Sigma_\ell)$.
- In addition to the integral of $\phi_+(\Gamma)$ on X_ℓ we consider integrals on the strata of the complement $X_\ell \smallsetminus Y_\ell$ of the polar part $\phi_-(\Gamma)$, which under suitable conditions will be interpreted as Poincaré residues.

If convergent, the Feynman integral $\int_{\Upsilon(\sigma)} \eta_\Gamma$ would be a period of $\mathfrak{m}(X_\ell \smallsetminus Y_\ell, \Sigma_\ell \smallsetminus (\Sigma_\ell \cap Y_\ell))$. The renormalization procedure described above replaces it with a (convergent) integral that is a period of the simpler motive $\mathfrak{m}(X_\ell, \Sigma_\ell)$. By our assumptions on X_ℓ and Σ_ℓ, the motive $\mathfrak{m}(X_\ell, \Sigma_\ell)$ is mixed Tate for all ℓ.

Thus, this strategy eliminates the difficulty of analyzing the motive $\mathfrak{m}(X_\ell \smallsetminus Y_\ell, \Sigma_\ell \smallsetminus (\Sigma_\ell \cap Y_\ell))$ encountered for instance in [4]. The form of renormalization proposed here always produces a mixed Tate period, but at the cost of incurring in a considerable loss of information with respect to the original Feynman integral.

Indeed, a difficulty in the procedure described above is ensuring that the resulting regularized form

$$\phi_+(\Gamma) = (1-T)(\phi(\Gamma) + \sum_{\gamma\subset\Gamma} \phi_-(\gamma)\wedge\phi(\Gamma/\gamma))$$

is nontrivial. This condition may be difficult to control in explicit cases, although we will discuss below (see Section 5) conditions under which one can reduce the problem to forms with logarithmic poles, where using the pole subtraction together with Poincaré residues one can obtain nontrivial periods (although the result one obtains is not equivalent to the physical renormalization of the Feynman amplitude).

An additional difficulty that can cause loss of information with respect to the Feynman integral is coming from the combinatorial conditions on the graph given in [4] that we will use to ensure that the map Υ to the complement of the determinant hypersurface is an embedding, see Section 5.11.

3. Rota–Baxter algebras and forms with logarithmic poles

We now focus on the case of meromorphic forms with logarithmic poles, where the Rota–Baxter structure and the renormalization procedure described above drastically simplify.

Lemma 3.1. *Let X be a smooth projective variety and $Y\subset X$ a smooth hypersurface with defining equation $Y=\{f=0\}$. Let $\Omega^\star_X(\log(Y))$ be the sheaf of algebraic differential forms on X with logarithmic poles along Y. After passing to global sections, we obtain a graded-commutative algebra, which we still denote by $\Omega^\star_X(\log(Y))$, for simplicity. The Rota–Baxter operator T of Lemma 2.1 preserves the commutative subalgebra $\Omega^{\mathrm{even}}_X(\log(Y))$ and the pair $(\Omega^{\mathrm{even}}_X(\log(Y)),T)$ is a graded Rota–Baxter algebra of degree -1 with the property that, for all $\omega_1,\omega_2\in\Omega^{\mathrm{even}}_X(\log(Y))$, the wedge product $T(\omega_1)\wedge T(\omega_2)=0$.*

Proof. Forms $\omega\in\Omega^\star_X(\log(Y))$ can be written in canonical form

$$\omega = \frac{df}{f}\wedge\xi+\eta,$$

with ξ and η holomorphic, so that $T(\omega)=\frac{df}{f}\wedge\xi$. We then have (2.2) as in Remark 2.2 above, with $T(\omega_1)\wedge T(\omega_2)=(-1)^{|\xi_1|+1}\alpha\wedge\alpha\wedge\xi_1\wedge\xi_2$ where α is the 1-form $\alpha=df/f$ so that $\alpha\wedge\alpha=0$. ☐

Lemma 3.1 shows that, when restricted to $\Omega^\star_X(\log(Y))$, the operator T satisfies the simpler identity

$$T(xy)=T(T(x)y)+T(xT(y)). \tag{3.1}$$

This property greatly simplifies the decomposition of the algebra induced by the Rota–Baxter operator.

Let $\mathcal{R}_+=(1-T)\mathcal{R}$. For an operator T satisfying (3.1) and $T(x)T(y)=0$, for all $x,y\in\mathcal{R}$, the property that $\mathcal{R}_+\subset\mathcal{R}$ is a subalgebra follows immediately from the simple identity

$$(1-T)(x)\cdot(1-T)(y)=xy-T(x)y-xT(y)$$

$$=xy-T(x)y-xT(y)-(T(xy)-T(T(x)y)-T(xT(y)))=(1-T)(xy-T(x)y-xT(y)).$$

Moreover, we obtain a simplified form of the general result of Proposition 2.10, when taking into account the vanishing $T(x)T(y)=0$, as shown in Lemma 3.1.

Lemma 3.2. *Let $\mathcal{R}$ be a commutative algebra and $T : \mathcal{R} \to \mathcal{R}$ a linear operator that satisfies the identity (3.1) and such that, for all $x, y \in \mathcal{R}$, the product $T(x)T(y) = 0$. Then both T and $1 - T$ are idempotent, $T^2 = T$ and $(1 - T)^2 = 1 - T$.*

Proof. The identity (3.1) gives $T(1) = 0$, since taking $x = y = 1$ one obtains $T(1) = 2T^2(1)$ while taking $x = T(1)$ and $y = 1$ gives $T^2(1) = T^3(1)$. Then (3.1) with $y = 1$ gives

$$T(x) = T(xT(1)) + T(T(x)1) = T^2(x)$$

for all $x \in \mathcal{R}$. For $1 - T$ we then have $(1 - T)^2(x) = x - 2T(x) + T^2(x) = (1 - T)(x)$, for all $x \in \mathcal{R}$. ☐

Lemma 3.3. *Let $\mathcal{R}$ be a commutative algebra and $T : \mathcal{R} \to \mathcal{R}$ a linear operator that satisfies the identity (3.1) and such that, for all $x, y \in \mathcal{R}$ the product $T(x)T(y) = 0$. If, for all $x, y \in \mathcal{R}$, the identity $T(x)y + xT(y) = T(T(x)y) + T(xT(y))$ holds, then the operator $(1 - T) : \mathcal{R} \to \mathcal{R}_+$ is an algebra homomorphism and the operator T is a derivation on $\mathcal{R}$.*

Proof. We have

$$(1 - T)(xy) = xy - T(T(x)y) - T(xT(y)), \text{ while } (1 - T)(x) \cdot (1 - T)(y) = xy - T(x)y - xT(y).$$

Assuming that, for all $x, y \in \mathcal{R}$, we have $T(T(x)y) + T(xT(y)) = T(x)y + xT(y)$ gives

$$(1 - T)(xy) = (1 - T)(x) \cdot (1 - T)(y).$$

Moreover, the identity (3.1) can be rewritten as $T(xy) = T(x)y + xT(y)$; hence T is just a derivation on $\mathcal{R}$. ☐

Consider then the case of a smooth hypersurface Y in a smooth projective variety X. We have the following properties.

Proposition 3.4. *Let $Y \subset X$ be a smooth hypersurface in a smooth projective variety. The Rota–Baxter operator $T : \mathcal{M}^{\text{even}}_{X,Y} \to \mathcal{M}^{\text{even}}_{X,Y}$ of weight -1 on meromorphic forms on X with poles along Y restricts to a derivation on the graded algebra $\Omega^{\text{even}}_X(\log(Y))$ of forms with logarithmic poles. Moreover, the operator $1 - T$ is a morphism of commutative algebras from $\Omega^{\text{even}}_X(\log(Y))$ to the algebra of holomorphic forms Ω^{even}_X.*

Proof. It suffices to check that the polar part operator $T : \Omega^{\text{even}}_X(\log(Y)) \to \Omega^{\text{even}}_X(\log(Y))$ satisfies the hypotheses of Lemma 3.3. We have seen that, for all $\omega_1, \omega_2 \in \Omega^{\text{even}}_X(\log(Y))$, the product $T(\omega_1) \wedge T(\omega_2) = 0$. Moreover, for $\omega_i = d\log(f) \wedge \xi_i + \eta_i$, we have $T(\omega_1) \wedge \omega_2 = d\log(f) \wedge \xi_1 \wedge \eta_2$ and $\omega_1 \wedge T(\omega_2) = (-1)^{|\eta_1|} d\log(f) \wedge \eta_1 \wedge \xi_2$, where the ξ_i and η_i are holomorphic, so that we have $T(T(\omega_1) \wedge \omega_2) = T(\omega_1) \wedge \omega_2$ and $T(\omega_1 \wedge T(\omega_2)) = \omega_1 \wedge T(\omega_2)$. Thus, the hypotheses of Lemma 3.3 are satisfied. ☐

3.1. **Birkhoff factorization and forms with logarithmic poles.** In cases where the pair (X, Y) has the property that all de Rham cohomology classes in $H^*_{dR}(X \smallsetminus Y)$ are represented by global algebraic differential forms with logarithmic poles, the construction above simplifies significantly. Indeed, the Birkhoff factorization becomes essentially trivial, because of Proposition 3.4. In other words, all graphs behave "as if they were log divergent". This can be stated more precisely as follows.

Proposition 3.5. *Let $Y \subset X$ be a smooth hypersurface inside a smooth projective variety and let $\Omega^{\text{even}}_X(\log(Y))$ denote the commutative algebra of algebraic differential forms on X of even degree with logarithmic poles on Y. Let $\phi : \mathcal{H} \to \Omega^{\text{even}}_X(\log(Y))$ be a morphism of commutative algebras* *from a commutative Hopf algebra $\mathcal{H}$ to $\Omega_X^{\text{even}}(\log(Y))$ with the operator T of pole subtraction. Then for every $\Gamma \in \mathcal{H}$ one has*

$$\phi_+(\Gamma) = (1-T)\phi(\Gamma),$$

while the negative part of the Birkhoff factorization takes the form

$$\phi_-(\Gamma) = -T(\phi(\Gamma)) - \sum \phi_-(\Gamma')\phi(\Gamma''),$$

where $\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum \Gamma' \otimes \Gamma''$. Moreover, ϕ_- takes the following nonrecursive form on $\ker(e) = \oplus_{n>0}\mathcal{H}_n$:

$$\begin{aligned}\phi_-(\Gamma) &= -T(\phi(\Gamma)) - \sum_{n=1}^{\infty}(-1)^n \sum T(\phi(\Gamma^{(1)}))\phi(\Gamma^{(2)})\phi(\Gamma^{(3)})\cdots\phi(\Gamma^{(n+1)}) \\ &= -T(\phi(\Gamma)) - \sum_{n=1}^{\infty}(-1)^n((T\phi)\tilde{\star}\phi^{\tilde{\star}^n})(\Gamma).\end{aligned}$$

Proof. The operator T of pole subtraction is a derivation on $\Omega_X^{\text{even}}(\log(Y))$. By (2.6) we have $\phi_+(\Gamma) = (1-T)(\phi(\Gamma) + \sum \phi_-(\Gamma')\phi(\Gamma''))$. By Proposition 3.4 we know that, in the case of forms with logarithmic poles along a smooth hypersurface, $1-T$ is an algebra homomorphism, hence $\phi_+(\Gamma) = (1-T)(\phi(\Gamma)) + \sum(1-T)(\phi_-(\Gamma'))(1-T)(\phi(\Gamma'')))$, but $\phi_-(\Gamma')$ is in the range of T and, again by Proposition 3.4, we have $T^2 = T$, so that the terms in the sum all vanish, since $(1-T)(\phi_-(\Gamma')) = 0$. By (2.6) we have

$$\phi_-(\Gamma) = -T(\phi(\Gamma) + \sum \phi_-(\Gamma')\phi(\Gamma'')) = -T\phi(\Gamma) - \sum T(\phi_-(\Gamma'))\phi(\Gamma'') - \sum \phi_-(\Gamma')T(\phi(X\Gamma')),$$

because by Proposition 3.4 T is a derivation. The last sum vanishes because $\phi_-(\Gamma')$ is in the range of T and we have $T(\eta) \wedge T(\xi) = 0$ for all $\eta, \xi \in \Omega_X^*(\log(Y))$. Thus, we are left with $\phi_-(\Gamma) = -T\phi(\Gamma) - \sum T(\phi_-(\Gamma'))\phi(\Gamma'') = -T\phi(\Gamma) - \sum \phi_-(\Gamma')\phi(\Gamma'')$. The last part follows from Proposition 2.10, since $T(T(\eta) \wedge \xi) = T(\eta) \wedge \xi$. ∎

Notice that this is compatible with the property that $\phi(\Gamma) = (\phi_- \circ S \star \phi_+)(\Gamma)$ (with the $\star$-product dual to the Hopf algebra coproduct). In fact, this identity is equivalent to $\phi_+ = \phi_- \star \phi$, which means that $\phi_+(\Gamma) = \langle \phi_- \otimes \phi, \Delta(\Gamma)\rangle = \phi_-(\Gamma) + \phi(\Gamma) + \sum \phi_-(\Gamma')\phi(\Gamma'') = (1-T)\tilde{\phi}(\Gamma)$ as above. Equivalently, all the nontrivial terms $\phi_-(\Gamma')\phi(\Gamma'')$ in $\tilde{\phi}(\Gamma)$ satisfy

$$T(\phi_-(\Gamma')\phi(\Gamma'')) = \phi_-(\Gamma')\phi(\Gamma''),$$

because of the simplified form (3.1) of the Rota–Baxter identity.

Corollary 3.6. *Suppose given a character $\phi : \mathcal{H} \to \Omega_X^{\text{even}}(\log(Y))$ of the Hopf algebra of Feynman graphs, where $X = X_\ell$ and $Y = Y_\ell$ independently of the number of loops $\ell \geq 1$. Then the negative part of the Birkhoff factorization of Proposition 3.5 has the simple form*

$$\phi_-(\Gamma) = -\frac{dh}{h} \wedge \left(\xi_\Gamma + \sum_{N\geq 1}(-1)^N \sum_{\gamma_N \subset \cdots \subset \gamma_1 \subset \gamma_0 = \Gamma} \xi_{\gamma_N} \wedge \bigwedge_{j=1}^{N} \eta_{\gamma_{j-1}/\gamma_j} \right), \tag{3.2}$$

where $\phi(\Gamma) = \frac{dh}{h} \wedge \xi_\Gamma + \eta_\Gamma$, and $Y = \{h = 0\}$.

Proof. The result follows from the expression

$$\phi_-(\Gamma) = -T(\phi(\Gamma)) - \sum_{\gamma \subset \Gamma} \phi_-(\gamma)\phi(\Gamma/\gamma),$$

obtained in Proposition 3.5, where $\phi(\Gamma) = \omega_\Gamma = \frac{dh}{h} \wedge \xi_\Gamma + \eta_\Gamma$, so that $T(\phi(\Gamma)) = \frac{dh}{h} \wedge \xi_\Gamma$ and $\phi(\Gamma/\gamma) = \frac{dh}{h} \wedge \xi_{\Gamma/\gamma} + \eta_{\Gamma/\gamma}$. The wedge product of $\phi_-(\gamma) = -T(\phi(\gamma)) - \sum_{\gamma_2 \subset \gamma} \phi_-(\gamma_2)\phi(\gamma/\gamma_2)$

with $\phi(\Gamma/\gamma)$ will give a term $\frac{dh}{h}\wedge\xi_\gamma\wedge\eta_{\Gamma/\gamma}$ and additional terms $\phi_-(\gamma_2)\phi(\gamma/\gamma_2)\wedge\eta_{\Gamma/\gamma}$. Proceeding inductively on these terms, one obtains (3.2). □

Remark 3.7. In the geometric construction we consider here, one does not have a single pair (X, Y) for all loop numbers. Instead, we consider a more general situation, where X_ℓ and Y_ℓ depend on the loop number $\ell \geq 1$. In this case, the form of the negative piece $\phi_-(\Gamma)$ is more complicated than in Corollary 3.6, as it contains forms on the products $X_{\ell(\gamma)} \times X_{\ell(\Gamma/\gamma)}$ with logarithmic poles along $Y_{\ell(\gamma)} \times X_{\ell(\Gamma/\gamma)} \cup X_{\ell(\gamma)} \times Y_{\ell(\Gamma/\gamma)}$. However, the general form of the expression is similar, only more cumbersome to write explicitly.

3.2. **Polar subtraction and the residue.** We have seen that, in the case of a smooth hypersurface $Y \subset X$, the Birkhoff factorization in the algebra of forms with logarithmic poles reduces to a simple pole subtraction, $\phi_+(\Gamma) = (1-T)\phi(\Gamma)$. If the unrenormalized $\phi(\Gamma)$ is a form written as $\alpha + \frac{df}{f} \wedge \beta$, with α and β holomorphic, then $\phi_+(\Gamma)$ vanishes identically whenever $\alpha = 0$. In that case, all information about $\phi(\Gamma)$ is lost in the process of pole substraction. Suppose that $\int_\sigma \phi(\Gamma)$ is the original unrenormalized integral. To maintain some additional information, it is preferable to consider, in addition to the integral $\int_\sigma \phi_+(\Gamma)$, also an integral of the form

$$\int_{\sigma\cap Y} \mathrm{Res}_Y(\eta),$$

where $\mathrm{Res}_Y(\eta) = \beta$ is the Poincaré residue of $\eta = \alpha + \frac{df}{f} \wedge \beta$ along Y. It is dual to the Leray coboundary, in the sense that

$$\int_{\sigma\cap Y} \mathrm{Res}_Y(\eta) = \frac{1}{2\pi i}\int_{\mathcal{L}(\sigma\cap Y)} \eta,$$

where the Leray coboundary $\mathcal{L}(\sigma \cap Y)$ is a circle bundle over $\sigma \cap Y$. In this way, even when $\alpha = 0$, one can still retain the nontrivial information coming from the Poincaré residue, which is also expressed as a period.

4. Singular hypersurfaces and meromorphic forms

In our main application, we will need to work with pairs (X, Y) where X is smooth projective, but the hypersurface Y is singular. Thus, we now discuss extensions of the results above to more general situations where $Y \subset X$ is a singular hypersurface in a smooth projective variety X.

Again we denote by $\mathcal{M}^*_{X,Y}$ the sheaf of meromorphic differential forms on X with poles along Y, of arbitrary order, and by $\Omega^*_X(\log(Y))$ the sub-sheaf of forms with logarithmic poles along Y. Let h be a local determination of Y, so that $Y = \{h = 0\}$. We can then locally represent forms $\omega \in \mathcal{M}^*_{X,Y}$ as finite sums $\omega = \sum_{p\geq 0} \omega_p/h^p$, with the ω_p holomorphic. The polar part operator $T : \mathcal{M}^{\text{even}}_{X,Y} \to \mathcal{M}^{\text{even}}_{X,Y}$ can then be defined as in (2.3).

In the case we considered in the previous section, with $Y \subset X$ a smooth hypersurface, forms with logarithmic poles can be represented as

$$\omega = \frac{dh}{h} \wedge \xi + \eta, \tag{4.1}$$

with ξ and η holomorphic. The Leray residue is given by $\mathrm{Res}(\omega) = \xi$. It is well defined, as the restriction of ξ to Y is independent of the choice of a local equation for Y.

In the next subsection we discuss how the case of a smooth hypersurface generalizes to the case of a normal crossings divisor $Y \subset X$ inside a smooth projective variety X. The normal crossings divisor is a particularly nice case of a larger class of singular hypersurfaces. The complex of forms with logarithmic poles extends from the smooth hypersurface case to the normal crossings divisor case as in [25]. For more general singular hypersurfaces, an appropriate notion of forms with logarithmic poles was introduced by Saito in [58]. The construction of the residue was also generalized from the smooth hypersurface case to the case where Y is a normal crossings divisor in [25] and to more general singular hypersurfaces in [58].

4.1. **Normal crossings divisors.** The main case of singular hypersurfaces that we focus on for our applications will be simple normal crossings divisors. In fact, while our formulation of the Feynman amplitude in momentum space is based on the formulation of [4], where the unrenormalized Feynman integral lives on the complement of the determinant hypersurface, which has worse singularities, we will reformulate the integral on the Kausz compactification of GL_n where the boundary divisor of the compactification is normal crossings.

If $Y \subset X$ is a simple normal crossings divisor in a smooth projective variety, with Y_j the components of Y, with local equations $Y_j = \{f_j = 0\}$, the complex of forms with logarithmic poles $\Omega^*_X(\log(Y))$ is spanned by the forms $\frac{df_j}{f_j}$ and by the holomorphic forms on X.

As in Theorem 6.3 of [21], we obtain that the Rota–Baxter operator of polar projection $T : \mathcal{M}^{\mathrm{even}}_{X,Y} \to \mathcal{M}^{\mathrm{even}}_{X,Y}$ restricts to a Rota–Baxter operator $T : \Omega^{\mathrm{even}}_X(\log(Y)) \to \Omega^{\mathrm{even}}_X(\log(Y))$ given by

$$T : \eta \mapsto T(\eta) = \sum_j \frac{df_j}{f_j} \wedge \mathrm{Res}_{Y_j}(\eta), \tag{4.2}$$

where the holomorphic form $\mathrm{Res}_{Y_j}(\eta)$ is the Poincaré residue of η restricted to Y_j.

Unlike the case of a single smooth hypersurface, for a simple normal crossings divisor the Rota–Baxter operator operator T does not satisfy $T(x)T(y) \equiv 0$, since we now have terms like $\frac{df_j}{f_j} \wedge \frac{df_k}{f_k} \neq 0$, for $j \neq k$, so the Rota–Baxter identity for T does not reduce to a derivation, but some of the properties that simplify the Birkhoff factorization in the case of a smooth hypersurface still hold in this case.

Proposition 4.1. *The Rota–Baxter operator T of (4.2) satisfies $T^2 = T$ and the Rota–Baxter identity simplifies to the form*

$$T(\eta \wedge \xi) = T(\eta) \wedge \xi + \eta \wedge T(\xi) - T(\eta) \wedge T(\xi). \tag{4.3}$$

The operator $(1 - T) : \mathcal{R} \to \mathcal{R}_+$ is an algebra homomorphism, with $\mathcal{R} = \Omega^{\mathrm{even}}_X(\log(Y))$ and $\mathcal{R}_+ = (1-T)\mathcal{R}$. The Birkhoff factorization of a commutative algebra homomorphism $\phi : \mathcal{H} \to \mathcal{R}$, with $\mathcal{H}$ a commutative Hopf algebra, is given by

$$\begin{array}{l} \phi_+(\Gamma) = (1-T)\phi(\Gamma) \\ \phi_-(\Gamma) = -T(\phi(\Gamma) + \sum \phi_-(\Gamma')\phi(\Gamma'')). \end{array} \tag{4.4}$$

Moreover, ϕ_- takes the following form on $\ker(e) = \oplus_{n>0}\mathcal{H}_n$:

$$\begin{aligned} \phi_-(\Gamma) &= -T(\phi(\Gamma)) - \sum_{n=1}^{\infty} (-1)^n \sum T(\phi(\Gamma^{(1)}))\phi(\Gamma^{(2)})\phi(\Gamma^{(3)}) \cdots \phi(\Gamma^{(n+1)}) \\ &= -T(\phi(\Gamma)) - \sum_{n=1}^{\infty} (-1)^n ((T\phi) \tilde{*} \phi^{\tilde{*}^n})(\Gamma). \end{aligned}$$

Proof. The argument is the same as in the proof of Theorem 6.3 in [21]. It is clear by construction that T is idempotent and the simplified form (4.3) of the Rota–Baxter identity follows by observing that $T(T(\eta)\wedge\xi) = T(\eta)\wedge\xi$ and $T(\eta\wedge T(\xi)) = \eta\wedge T(\xi)$ as in Theorem 6.3 in [21]. Then one sees that

$$(1-T)(\eta)\wedge(1-T)(\xi) = \eta\wedge\xi - T(\eta)\wedge\xi - \eta\wedge T(\xi) + T(\eta)\wedge T(\xi) = \eta\wedge\xi - T(\eta\wedge\xi)$$

by (4.3). Consider then the Birkhoff factorization. We write $\tilde{\phi}(\Gamma) := \phi(\Gamma) + \sum \phi_-(\Gamma')\phi(\Gamma'')$. The fact that $(1-T)$ is an algebra homomorphism then gives

$$\phi_+(\Gamma) = (1-T)(\tilde{\phi}(\Gamma)) = (1-T)(\phi(\Gamma) + \sum \phi_-(\Gamma')\phi(\Gamma''))$$
$$= (1-T)(\phi(\Gamma)) + \sum (1-T)(\phi_-(\Gamma'))(1-T)(\phi(\Gamma''))),$$

with $(1-T)(\phi_-(\Gamma')) = -(1-T)T(\tilde{\phi}_-(\Gamma')) = 0$, because T is idempotent. The last statement again follows from Proposition 2.10, since we have $T(T(\eta)\wedge\xi) = T(\eta)\wedge\xi$. □

4.2. **Multidimensional residues.** In the case of a simple normal crossings divisor $Y \subset X$, we can proceed as discussed in Section 3.2 for the case of a smooth hypersurface. Indeed, as we have seen in Proposition 4.1, we also have in this case a simple pole subtraction $\phi_+(\Gamma) = (1-T)\phi(\Gamma)$, even though the negative term $\phi_-(\Gamma)$ of the Birkhoff factorization can now be more complicated than in the case of a smooth hypersurface.

The unrenormalized $\phi(\Gamma)$ is a form $\eta = \alpha + \sum_j \frac{df_j}{f_j}\wedge\beta_j$, with α and β_j holomorphic and $Y_j = \{f_j = 0\}$ the components of Y. Again, if $\alpha = 0$ we loose all information about $\phi(\Gamma)$ in our renormalization of the logarithmic form. To avoid this problem, we can again consider, instead of the single renormalized integral $\int_\sigma \phi_+(\Gamma)$, an additional family of integrals

$$\int_{\sigma\cap Y_I} \mathrm{Res}_{Y_I}(\eta),$$

where $Y_I = \cap_{j\in I} Y_j$ is an intersection of components of the divisor Y and $\mathrm{Res}_{Y_I}(\eta)$ is the iterated (or multidimensional, or higher) Poincaré residue of η, in the sense of [25]. These are dual to the iterated Leray coboundaries,

$$\int_{\sigma\cap Y_I} \mathrm{Res}_{Y_I}(\eta) = \frac{1}{(2\pi i)^n}\int_{\mathcal{L}_I(\sigma\cap Y_I)} \eta,$$

where $\mathcal{L}_I = \mathcal{L}_{j_i}\circ\cdots\circ\mathcal{L}_{j_n}$ for $Y_I = Y_{j_1}\cap\cdots\cap Y_{j_n}$.

If arbitrary intersections Y_I of components of Y are all mixed Tate motives, then all these integrals are also periods of mixed Tate motives.

4.3. **Saito's logarithmic forms.** Given a singular reduced hypersurface $Y \subset X$, a differential form ω with logarithmic poles along Y, in the sense of Saito [58], can always be written in the form ([58], (1.1))

$$f\,\omega = \frac{dh}{h}\wedge\xi + \eta, \tag{4.5}$$

where $f \in \mathcal{O}_X$ defines a hypersurface $V = \{f = 0\}$ with $\dim(Y\cap V) \leq \dim(X) - 2$, and with ξ and η holomorphic forms.

In the following, we use the notation ${}^S\Omega^\star_X(\log(Y))$ to denote the forms with logarithmic poles along Y in the sense of Saito, to distinguish it from the more restrictive notion of forms with logarithmic poles $\Omega^\star_X(\log(Y))$ considered above for the normal crossings case.

Following [2], we say that a (reduced) hypersurface $Y \subset X$ has *Saito singularities* if the modules of logarithmic differential forms and vector fields along Y are free. The condition that $Y \subset X$ has Saito singularities is equivalent to the condition that ${}^S\Omega^n_X(\log(Y)) = \bigwedge^n\ {}^S\Omega^1_X(\log(Y))$, [58].

Let Y be a hypersurface with Saito singularities and let $\mathcal{M}_Y$ denote the sheaf of germs of meromorphic functions on Y. Then setting

$$\mathrm{Res}(\omega) = \frac{1}{f}\,\xi\,|_Y \tag{4.6}$$

defines the residue as a morphism of $\mathcal{O}_X$-modules, for all $q \geq 1$,

$$\mathrm{Res} : {}^S\Omega^q_X(\log(Y)) \to \mathcal{M}_Y \otimes_{\mathcal{O}_Y} \Omega^{q-1}_Y. \tag{4.7}$$

Unlike the case of smooth hypersurfaces and normal crossings divisors, in the case of more general hypersurfaces with Saito singularities, the Saito residue of forms with logarithmic poles is not a holomorphic form, but only a *meromorphic* form on Y.

For $Y \subset X$ a reduced hypersurface that is quasihomogeneous with Saito singularities, a refinement of (4.7), which we view as the exact sequence

$$0 \to \Omega^q_X \to {}^S\Omega^q_X(\log(Y)) \stackrel{\mathrm{Res}}{\to} \mathcal{M}_Y \otimes_{\mathcal{O}_Y} \Omega^{q-1}_Y,$$

is given in [2], where the image of the Saito Poincaré residue is more precisely identified as $\mathrm{Res}^S\Omega^q_X(\log(Y)) \simeq \omega^{q-1}_Y$, where $\omega^\bullet_Y$ denotes the module of regular meromorphic differential forms in the sense of [8], with $\omega^\bullet_Y \subset j_*j^*\Omega^\bullet_Y$, where $j : S \hookrightarrow Y$ is the inclusion of the singular locus. Namely, it is shown in [2] that one has, for all $q \geq 2$, an exact sequence of $\mathcal{O}_X$-modules

$$0 \to \Omega^q_X \to {}^S\Omega^q_X(\log(Y)) \xrightarrow{\mathrm{Res}} \omega^{q-1}_Y \to 0. \tag{4.8}$$

It is natural to ask whether the extraction of polar part from forms with logarithmic poles that we considered here for the case of smooth hypersurfaces and normal crossings divisors extends to more general singular hypersurfaces using Saito's formulation.

Question 4.2. *Is there a Rota–Baxter operator T expressed in terms of the Saito residue, in the case of a singular hypersurfaces $Y \subset X$ with Saito singularities?*

We describe here a possible approach to this question. We introduce an analog of the Rota–Baxter operator considered above, given by the extraction of the polar part. The "polar part" operator, in this more general case, does not map $\Omega^{\mathrm{even}}_X(\log(Y))$ to itself, but we show below that it gives a well defined Rota-Baxter operator of weight -1 on the space of Saito forms ${}^S\Omega^{\mathrm{even}}_X(\log(Y))$, and that this operator is a derivation.

Lemma 4.3. *The set $S_Y := \{f \,|\, \dim(\{f = 0\} \cap Y) \leq \dim(X) - 2\}$ is a multiplicative set. Localization of the Saito forms with logarithmic poles gives $S_Y^{-1}\,{}^S\Omega_X(\log(Y)) = {}^S\Omega_X(\log(Y))$.*

Proof. We have $V_{12} = \{f_1f_2 = 0\} = \{f_1 = 0\} \cup \{f_2 = 0\}$ and

$$\dim(Y \cap V_{12}) = \dim((Y \cap \{f_1 = 0\}) \cup (Y \cap \{f_2 = 0\})) \leq \dim(X) - 2,$$

since $\dim(Y \cap \{f_i = 0\}) \leq \dim(X) - 2$ for $i = 1, 2$. Thus, for any $f_1, f_2 \in S_Y$, we have $f_1f_2 \in S_Y$. Moreover, we have $1 \in S_Y$, hence S_Y is a multiplicative set. The localization of ${}^S\Omega^\star_X(\log(Y))$ at S_Y is just ${}^S\Omega^\star_X(\log(Y))$ itself: in fact, for $\tilde{f}^{-1}\omega \in S_Y^{-1}\,{}^S\Omega^\star_X(\log(Y))$, with $\tilde{f} \in S_Y$ and $\omega \in {}^S\Omega^\star_X(\log(Y))$, expressed as in (4.5), we have

$$f\tilde{f}(\tilde{f}^{-1}\omega) = f\omega = \frac{dh}{h} \wedge \xi + \eta,$$

where $f\tilde{f} \in S_Y$, hence $\tilde{f}^{-1}\omega \in {}^S\Omega_X(\log(Y))$. $\square$

Given a form $\omega \in {}^S\Omega^\star_X(\log(Y))$, which we can write as in (4.5), the residue (4.6) is the image under the restriction map $S_Y^{-1}\Omega^\star_X \to S_Y^{-1}\Omega^\star_Y$ of the form $f^{-1}\xi \in S_Y^{-1}\Omega^\star_X$. Moreover, we have an inclusion $\Omega^\star_X \hookrightarrow {}^S\Omega^\star_X(\log(Y))$, which induces a map of the localizations

$$S_Y^{-1}\Omega^\star_X \hookrightarrow S_Y^{-1}\,{}^S\Omega^\star_X(\log(Y)) = {}^S\Omega^\star_X(\log(Y)).$$

We can then define a linear operator

$$T : {}^S\Omega^\star_X(\log(Y)) \to {}^S\Omega^\star_X(\log(Y)) \wedge S_Y^{-1}\,\Omega^\star_X \hookrightarrow {}^S\Omega^\star_X(\log(Y)) \wedge S_Y^{-1}\,{}^S\Omega^\star_X(\log(Y)) = {}^S\Omega^\star_X(\log(Y))$$

given by

$$T(\omega) = \frac{dh}{h} \wedge \frac{\xi}{f}, \quad \text{for} \quad f\,\omega = \frac{dh}{h} \wedge \xi + \eta. \tag{4.9}$$

Lemma 4.4. *The operator T of (4.9) is a Rota–Baxter operator of weight -1 on ${}^S\Omega^{\text{even}}_X(\log(Y))$, which is just given by a derivation, satisfying the Leibniz rule*

$$T(\omega_1 \wedge \omega_2) = T(\omega_1) \wedge \omega_2 + \omega_1 \wedge T(\omega_2).$$

Proof. Let

$$f_1\,\omega_1 = \frac{dh}{h} \wedge \xi_1 + \eta_1 \quad f_2\,\omega_2 = \frac{dh}{h} \wedge \xi_2 + \eta_2.$$

Then

$$f_1\,f_2\,\omega_1 \wedge \omega_2 = (\frac{dh}{h} \wedge \xi_1 + \eta_1) \wedge (\frac{dh}{h} \wedge \xi_2 + \eta_2) = \frac{dh}{h} \wedge (\xi_1 \wedge \eta_2 + (-1)^p \eta_1 \wedge \xi_2) + \eta_1 \wedge \eta_2,$$

where $\eta_1 \in \Omega^p(X)$. By Lemma 4.3, we know that $f_1 f_2 \in S_Y$. We have

$$T(\omega_1 \wedge \omega_2) = \frac{dh}{h} \wedge (\frac{\xi_1}{f_1} \wedge \frac{\eta_2}{f_2} + (-1)^p \frac{\eta_1}{f_1} \wedge \frac{\xi_2}{f_2}).$$

Since

$$T(\omega_1) = \frac{dh}{h} \wedge \frac{\xi_1}{f_1}, \quad \text{and} \quad T(\omega_2) = \frac{dh}{h} \wedge \frac{\xi_2}{f_2},$$

we obtain

$$T(\omega_1) \wedge T(\omega_2) = \frac{dh}{h} \wedge \frac{\xi_1}{f_1} \wedge \frac{dh}{h} \wedge \frac{\xi_2}{f_2} = 0.$$

Moreover, we have

$$T(\omega_1) \wedge \omega_2 = (\frac{dh}{h} \wedge \frac{\xi_1}{f_1}) \wedge \frac{dh}{h} \wedge \frac{\xi_2}{f_2} + \frac{dh}{h} \wedge \frac{\xi_1}{f_1} \wedge \frac{\eta_2}{f_2} = \frac{dh}{h} \wedge \frac{\xi_1}{f_1} \wedge \frac{\eta_2}{f_2},$$

with

$$f_1 f_2 (T(\omega_1) \wedge \omega_2) = \frac{dh}{h} \wedge \xi_1 \wedge \eta_2,$$

and similarly,

$$\omega_1 \wedge T(\omega_2) = (-1)^p \frac{dh}{h} \wedge \frac{\eta_1}{f_1} \wedge \frac{\xi_2}{f_2},$$

hence T satisfies the Leibniz rule. The operator T also satisfies $T(T(\omega_1) \wedge \omega_2) = T(\omega_1) \wedge \omega_2$, and $T(\omega_1 \wedge T(\omega_2)) = \omega_1 \wedge T(\omega_2)$, hence the condition that T is a derivation is equivalent to the condition that it is a Rota-Baxter operator of weight -1. $\square$

Correspondingly, we have

$$(1-T)\omega = \omega - \frac{dh}{h} \wedge \frac{\xi}{f} = \frac{\eta}{f} \in S_Y^{-1}\,\Omega_X^{\text{even}}.$$

Under the restriction map $S_Y^{-1}\,\Omega_X^{\text{even}} \to S_Y^{-1}\,\Omega_Y^{\text{even}}$ we obtain a form $(1-T)(\omega)|_Y$. It follows that we can define a "subtraction of divergences" operation on $\phi : \mathcal{H} \to {}^{S}\Omega_X^{\text{even}}(\log(Y))$ by taking $\phi_+ : \mathcal{H} \to {}^{S}\Omega_X^{\text{even}}(\log(Y))$ given by $\phi_+(a) = (1-T)\phi(a)|_Y$, for $a \in \mathcal{H}$, which maps $\phi(a) = \omega$ to $(1-T)\omega|_Y = f^{-1}\eta|_Y$, where $f\,\omega = \frac{dh}{h} \wedge \xi + \eta$. While this has subtracted the logarithmic pole along Y, it has also created a new pole along $V = \{f = 0\}$. Thus, it results again in a meromorphic form. If we consider the restriction to Y of $\phi_+(a) = f^{-1}\,\eta|_Y$, we obtain a meromorphic form with first order poles along a subvariety $V \cap Y$, which is by hypothesis of codimension at least one in Y. Thus, we can conceive of a more complicated renormalization method that progressively subtracts poles on subvarieties of increasing codimension, inside the polar locus of the previous pole subtraction, by iterating this procedure. A more detailed account of this iterative procedure and of possible applications to the setting of renormalization will be discussed elsewhere.

5. Compactifications of GL_n and momentum space Feynman integrals

In this section, we restrict our attention to the case of compactifications of PGL_ℓ and of GL_ℓ and we use a formulation of the parametric Feynman integrals of perturbative quantum field theory in terms of (possibly divergent) integrals on a cycle in the complement of the determinant hypersurface [4], to obtain a new method of regularization and renormalization. This gives rise to a renormalized integral that is a period of a mixed Tate motive, under certain conditions on the graph and on the intersection of the big cell of the compactification with a divisor $\Sigma_{\ell,g}$. We show that a certain loss of information can occur with respect to the usual physical Feynman integral.

5.1. **The determinant hypersurface.** In the following we use the notation $\hat{\mathcal{D}}_\ell$ and $\mathcal{D}_\ell$, respectively, for the affine and the projective determinant hypersurfaces. Namely, we consider in the affine space $\mathbb{A}^{\ell^2}$, identified with the space of all $\ell \times \ell$-matrices, with coordinates $(x_{ij})_{i,j=1,\dots,\ell}$, the hypersurface

$$\hat{\mathcal{D}}_\ell = \{\det(X) = 0 \,|\, X = (x_{ij})\} \subset \mathbb{A}^{\ell^2}.$$

Since $\det(X) = 0$ is a homogeneous polynomial in the variables (x_{ij}), we can also consider the projective hypersurface $\mathcal{D}_\ell \subset \mathbb{P}^{\ell^2-1}$.

The complement $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$ is identified with the space of invertible $\ell \times \ell$-matrices, namely with GL_ℓ.

5.2. **The Kausz compactification of GL_n.** We recall here some basic facts about the Kausz compactification $K\mathrm{GL}_n$ of GL_n, following [47] and the exposition in §12 of [53].

We first recall the Vainsencher compactification [60] of PGL_ℓ. Let $X_0 = \mathbb{P}^{\ell^2-1}$ be the projectivization of the vector space of square $\ell \times \ell$-matrices. Let Y_i be the locus of matrices of rank i and consider the iterated blowups $X_i = \mathrm{Bl}_{\bar{Y}_i}(X_{i-1})$, with $\bar{Y}_i$ the closure of Y_i in X_{i-1}. The Y_i are PGL_i-bundles over a product of Grassmannians. It is shown in Theorem 1 and (2.4) of [60] that the X_i are smooth, and that $X_{\ell-1}$ is a wonderful compactification of PGL_ℓ, in the sense of [24]. One denotes by $\overline{\mathrm{PGL}_\ell}$ the wonderful compactification of PGL_ℓ obtained in this way. We also refer the reader to §12 of [53] for a quick overview of the main properties of the Vainsencher compactification.

The Kausz compactification [47] of GL_ℓ is similar. One regards $\mathbb{A}^{\ell^2}$ as the big cell in $\mathcal{X}_0 = \mathbb{P}^{\ell^2}$. The iterated sequence of blowups is given in this case by setting $\mathcal{X}_i = \mathrm{Bl}_{\bar{\mathcal{Y}}_{i-1}\cup\bar{\mathcal{H}}_i}(\mathcal{X}_{i-1})$, where $\mathcal{Y}_i \subset \mathbb{A}^{\ell^2}$ are the matrices of rank i and $\mathcal{H}_i$ are the matrices at infinity (that is, in $\mathbb{P}^{\ell^2-1} = \mathbb{P}^{\ell^2} \smallsetminus \mathbb{A}^{\ell^2}$) of rank i. The Kausz compactification is $K\mathrm{GL}_\ell = \mathcal{X}_{\ell-1}$. It is shown in Corollary 4.2 of [47] that the $\mathcal{X}_i$ are smooth and in Corollary 4.2 and Theorem 9.1 of [47] that the blowup loci are disjoint unions of loci with the following structure: the closure $\bar{\mathcal{Y}}_{i-1}$ in $\mathcal{X}_{i-1}$ is a $K\mathrm{GL}_{i-1}$-bundle over a product of Grassmannians and the closure $\bar{\mathcal{H}}_i$ in in $\mathcal{X}_{i-1}$ is a $\overline{\mathrm{PGL}_i}$-bundle over a product of Grassmannians. Theorem 9.1 of [47] also shows that these compactifications have a moduli space interpretation. An overview of these properties and of the relation between the Vainsencher and the Kausz compactifications is given in §12 of [53].

As observed in [53], the Kausz compactification is then the closure of GL_ℓ inside the wonderful compactification of $\mathrm{PGL}_{\ell+1}$, see also [43], Chapter 3, §1.4. The compactification $K\mathrm{GL}_\ell$ is smooth and projective over $\mathrm{Spec}(\mathbb{Z})$ (Corollary 4.2 [47]).

The other property of the Kausz compactification that we will be using in the following is the fact that the complement of the dense open set GL_ℓ inside the compactification $K\mathrm{GL}_\ell$ is a normal crossing divisor (Corollary 4.2 [47]).

5.3. **The virtual motive of the Kausz compactification.** We organize the computation of the motive of the Kausz compactification in three subsections, respectively dealing with the virtual motive (Grothendieck class), the numerical motive, and the Chow motive. The main reason for providing separate arguments, instead of giving only the strongest result about the Chow motive, is a pedagogical illustration of the difference between these three levels of motivic structure, where one can see in a very explicit case what is needed to improve from one level to the next, and what are the implications (conditional and unconditional). We begin with the Grothendieck class, which is usually more familiar, especially in the mathematical physics setting.

We use the description recalled above of the Kausz compactification, together with the blowup formula, to check that the virtual motive (class in the Grothendieck ring) of the Kausz compactification is Tate.

Proposition 5.1. *Let $K_0(\mathcal{V})$ be the Grothendieck ring of varieties (defined over $\mathbb{Q}$ or over $\mathbb{Z}$) and let $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$ be the Tate subring generated by the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$. For all $\ell \geq 1$ the class $[K\mathrm{GL}_\ell]$ is in $\mathbb{Z}[\mathbb{L}]$. Moreover, let $\mathcal{Z}_\ell$ be the normal crossings divisor $\mathcal{Z}_\ell = K\mathrm{GL}_\ell \smallsetminus \mathrm{GL}_\ell$. Then all the unions and intersections of components of $\mathcal{Z}_\ell$ have Grothendieck classes in $\mathbb{Z}[\mathbb{L}]$.*

Proof. We use the blowup formula for classes in the Grothendieck ring: if $\tilde{\mathcal{X}} = \mathrm{Bl}_{\mathcal{Y}}(\mathcal{X})$, where $\mathcal{Y}$ is of codimension $m+1$ in $\mathcal{X}$, then the classes satisfy

$$[\tilde{\mathcal{X}}] = [\mathcal{X}] + \sum_{k=1}^{m} [\mathcal{Y}]\mathbb{L}^k. \tag{5.1}$$

The Kausz compactification is obtained as an iterated blowup, starting with a projective space, whose class is in $\mathbb{Z}[\mathbb{L}]$ and blowing up at each step a smooth locus that is a bundle over a product of Grassmannians with fiber either a $K\mathrm{GL}_i$ or a $\overline{\mathrm{PGL}}_i$ for some $i < \ell$. The Grothendieck class of a bundle is the product of the class of the base and the class of the fiber. Classes of Grassmannians (and products of Grassmannians) are in $\mathbb{Z}[\mathbb{L}]$. The classes of the wonderful compactifications $\overline{\mathrm{PGL}}_i$ of PGL_i are also in $\mathbb{Z}[\mathbb{L}]$, since it is known that the motive of these wonderful compactifications are mixed Tate (this follows, for instance, from the cell decomposition given in Proposition 4.4. of [41]). Thus, it suffices to assume, inductively, that the classes $[K\mathrm{GL}_i] \in \mathbb{Z}[\mathbb{L}]$ for all $i < \ell$, and conclude via the blowup formula that $[K\mathrm{GL}_\ell] \in \mathbb{Z}[\mathbb{L}]$.

Consider then the boundary divisor $\mathcal{Z}_\ell = K\mathrm{GL}_\ell \smallsetminus \mathrm{GL}_\ell$. The geometry of the normal crossings divisor $\mathcal{Z}_\ell$ is described explicitly in §4 of [47]. It has components Y_i and Z_i, for $0 \leq i \leq \ell$, that correspond to the blowup loci described above. The multiple intersections $\cap_{i\in I} Y_i \cap \cap_{j\in J} Z_j$ of these components of $\mathcal{Z}_\ell$ are described in turn in terms of bundles over products of flag varieties with fibers that are lower dimensional compactifications $K\mathrm{GL}_i$ and $\overline{\mathrm{PGL}}_i$ and products. Again, flag varieties have cell decompositions, hence their Grothendieck classes are in $\mathbb{Z}[\mathbb{L}]$ and the rest of the argument proceeds as in the previous case. If arbitrary intersections of the components of $\mathcal{Z}_\ell$ have classes in $\mathbb{Z}[\mathbb{L}]$ then arbitrary unions and unions of intersections also do by inclusion-exclusion in $K_0(\mathcal{V})$. □

5.4. **The numerical motive of the Kausz compactification.** Knowing that the Grothendieck class $[K\mathrm{GL}_\ell]$ is in the Tate subring $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$ implies that the motive is of Tate type in the category of pure motives with respect to the numerical equivalence. More precisely, we have the following.

Proposition 5.2. *Let $h_{\mathrm{num}}(K\mathrm{GL}_\ell)$ denote the motive of the Kausz compactification $K\mathrm{GL}_\ell$ in the category of pure motives over $\mathbb{Q}$, with the numerical equivalence relation. Then $h_{\mathrm{num}}(K\mathrm{GL}_\ell)$ is in the (tensor) subcategory generated by the Tate object. The same is true for arbitrary unions and intersections of the components of the boundary divisor $\mathcal{Z}_\ell$ of the compactification.*

Proof. The same argument used in Proposition 5.1 can be upgraded at the level of numerical motives. We replace the blowup formula (5.1) for Grothendieck classes with the corresponding formula for motives, which follows (already at the level of Chow motives) from Manin's identity principle, [51]:

$$h(\tilde{X}) = h(X) \oplus \bigoplus_{r=1}^{m} h(Y) \otimes \mathbb{L}^{\otimes r}, \tag{5.2}$$

with $\tilde{X} = \mathrm{Bl}_Y(X)$ the blowup of a smooth subvariety $Y \subset X$ of codimension $m+1$ in a smooth projective variety X, and where $\mathbb{L} = h^2(\mathbb{P}^1)$ is the Lefschetz motive. Moreover, we use the fact that, for numerical motives, the motive of a locally trivial fibration $X \to S$ with fiber Y is given by the product

$$h_{\mathrm{num}}(X) = h_{\mathrm{num}}(Y) \otimes h_{\mathrm{num}}(S), \tag{5.3}$$

see Exercise 13.2.2.2 of [5]. The decomposition (5.3) allows us to describe the numerical motives of the blowup loci of the iterated blowup construction of $K\mathrm{GL}_\ell$ as products of numerical motives of Grassmannians and of lower dimensional compactifications $K\mathrm{GL}_i$ and $\overline{\mathrm{PGL}}_i$. The motive of a Grassmannian can be computed explicitly as in [49], already at the level of Chow motives. If $G(d,n)$ denotes the Grassmannian of d-planes in k^n, the Chow motive $h(G(d,n))$ is given by

$$h(G(d,n)) = \bigoplus_{\lambda\in W^d} \mathbb{L}^{\otimes|\lambda|}, \tag{5.4}$$

where

$$W^d = \{\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{N}^d \,|\, n-d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0\}$$

and $|\lambda| = \sum_i \lambda_i$, see Theorem 2.1 and Lemma 3.1 of [49]. The same decomposition into powers of the Lefschetz motive holds at the numerical level. Moreover, we know (also already for Chow motives) that the motives $h(\overline{\mathrm{PGL}}_i)$ of the wonderful compactifications are Tate (see [41]), and we conclude the argument as in Proposition 5.1 by assuming inductively that the motives $h_{\mathrm{num}}(K\mathrm{GL}_i)$ are Tate, for $i < \ell$. The argument for the loci $\cap_{i\in I} Y_i \cap \cap_{j\in J} Z_j$ in $\mathcal{Z}_\ell$ is analogous. □

Remark 5.3. Proposition 5.2 also follows from Proposition 5.1 using the general fact that two numerical motives that have the same class in $K_0(\mathrm{Num}(k)_\mathbb{Q})$ are isomorphic as objects in $\mathrm{Num}(k)_\mathbb{Q}$, because of the semi-simplicity of the category of numerical motives [44], together with the existence, for $char(k) = 0$, of a unique ring homomorphism (the motivic Euler characteristic) $\chi_{\mathrm{mot}} : K_0(\mathcal{V}_k) \to K_0(\mathrm{Num}(k)_\mathbb{Q})$, such that $\chi_{\mathrm{mot}}([X]) = [h_{\mathrm{num}}(X)]$, for X a smooth projective variety, see Corollary 13.2.2.1 of [5].

5.5. **The Chow motive of the Kausz compactification.** Manin's blowup formula (5.2) and the computation of the motive of Grassmannians and of the wonderful compactifications $\overline{\mathrm{PGL}}_i$ already hold at the level of Chow motives. However, if we want to extend the argument of Proposition 5.2 to Chow motives, we run into the additional difficulty that one no longer necessarily has the decomposition (5.3) for the motive of a locally trivial fibration. Under some hypotheses on the existence of a cellular structure on the fibers, one can still obtain a decomposition for motives of bundles, and more generally locally trivial fibrations, the fibers of which have cell decompositions with suitable properties, see [46], and also [40], [41], [45], [56]. We obtain an unconditional result on the Chow motive of the Kausz compactification, by analyzing its cellular structure.

Recall that, for G a connected reductive algebraic group and B a Borel subgroup, a *spherical variety* is a normal algebraic variety on which G acts with a dense orbit of B, [14]. Spherical varieties can be regarded as a generalization of toric varieties: when G is a torus, one recovers the usual notion of toric variety.

Proposition 5.4. *The Chow motive $h(K\mathrm{GL}_\ell)$ of the Kausz compactification is a Tate motive.*

Proof. The result follows by showing that $K\mathrm{GL}_\ell$ has a cellular structure for all $\ell \geq 1$, which allows us to extend the decomposition of the motive used in Proposition 5.2 from the numerical to the Chow case.

As shown in §3.1 of [14], it follows from the work of Bialynicki–Birula [11] that any complete, smooth and spherical variety X has a cellular decomposition. This is determined by the decomposition of the spherical variety into B-orbits and is obtained by considering a one-parameter subgroup $\lambda : \mathbb{G}_m \hookrightarrow X$, such that the set of fixed points X^λ is finite. The cells are given by

$$X(\lambda, x) = \{z \in X \mid \lim_{t\to 0} \lambda(t)z = x\}, \quad \text{for} \quad x \in X^\lambda. \tag{5.5}$$

The Kausz compactification $K\mathrm{GL}_\ell$ is a smooth toroidal equivariant compactification of GL_ℓ, see Proposition 1.15 of §3 of [43] and also Proposition 10.1 and Proposition 12.1 of [53]. In particular, it is a spherical variety (see Proposition 10.1 of [53]), hence it has a cellular structure as above.

A relative cellular variety, in the sense of [46], is a smooth and proper variety with a decomposition into affine fibrations over proper varieties. The blowup loci of the Kausz compactification $K\mathrm{GL}_\ell$ are relative cellular varieties in this sense, since they are bundles over products of Grassmannians, with fiber a lower dimensional compactification $K\mathrm{GL}_i$, with $i < \ell$. Using the cell decomposition of the fibers $K\mathrm{GL}_i$, we obtain a decomposition of these blowup loci as relative cellular varieties, with pieces of the decomposition being fibrations over a product of Grassmannians, with fibers the cells of the cellular structure of $K\mathrm{GL}_i$.

There is an embedding of the category of pure Chow motives in the category of mixed motives, see [5]. By viewing the Chow motives of these blowup loci as elements in the Voevodsky category of mixed motives, Corollary 6.11 of [46] shows that they are direct sums of motives of products of Grassmannians (which are Tate motives), with twists and shifts which depend on the dimensions of the cells of $K\mathrm{GL}_i$. We conclude from this that all the blowup loci are Tate motives. We can then repeatedly apply the blowup formula for Chow motives to conclude (unconditionally) that the Chow motive of $K\mathrm{GL}_\ell$ is itself a Tate motive. Note that the blowup formula also holds in the Voevodsky category, Proposition 3.5.3 of [61], in the form

$$\mathfrak{m}(\mathrm{Bl}_Y(X)) = \mathfrak{m}(X) \oplus \bigoplus_{r=1}^{\mathrm{codim}_X(Y)-1} \mathfrak{m}(Y)(r)[2r],$$

which corresponds to the usual formula of [51] in the case of pure motives, after viewing them as objects in the category of mixed motives. The result can also be obtained, in a similar way, using Theorem 2.10 of [41] instead of Corollary 6.11 of [46]. □

Remark 5.5. Given the existence of a cellular decomposition of $K\mathrm{GL}_\ell$, as above, it is possible to give a quicker proof that the Chow motive is Tate, by using distinguished triangles in the Voevodsky category associated to the inclusions of unions of cells, showing that $\mathfrak{m}(K\mathrm{GL}_\ell)$ is mixed Tate, then using the inclusion of pure motives in the mixed motives to conclude that $h(K\mathrm{GL}_\ell)$ is Tate. In Proposition 5.4 above we chose to maintain the structure of the argument more similar to the cases of the virtual and the numerical motive, for better direct comparison.

Remark 5.6. Notice that a *conditional* result about the Chow motive would follow directly from Proposition 5.2 or Remark 5.3, if one assumes the Kimura–O'Sullivan finiteness conjecture (or Voevodsky's nilpotence conjecture, which implies it). For the precise statement and implications of the Kimura–O'Sullivan finiteness conjecture, and its relation to Voevodsky's nilpotence, we refer the reader to the survey [6]. By arguing as in Lemma 13.2.1.1 of [5], that would extend the result of Proposition 5.2 to the Chow motive. At the level of Grothendieck classes, the conjecture in fact implies that the K_0 of Chow motives and the K_0 of numerical motives coincide, hence one can argue as in Remark 5.3 and conclude that, in order to know that the Chow motive is mixed Tate, it suffices to know that the Grothendieck class is mixed Tate.

5.6. **Feynman integrals in momentum space and non-mixed-Tate examples.** It was shown in [12] that the parametric form of Feynman integrals in perturbative quantum field theory can be formulated as a (possibly divergent) period integral on the complement of a hypersurface defined by the vanishing of a combinatorial polynomial associated to Feynman graphs. Namely, one writes the (unrenormalized) Feynman amplitudes for a *massless* scalar quantum field theory as integrals

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}} \tag{5.6}$$

where $n = \#E_\Gamma$ is the number of internal edges, $\ell = b_1(\Gamma)$ is the number of loops, and D is the spacetime dimension. Here we consider the "unregularized" Feynman integral, where D is just the integer valued dimension, without performing the procedure of dimensional regularization that analytically continues D to a complex number. The domain of integration is a simplex $\sigma_n = \{t \in \mathbb{R}^n_+ | \sum_i t_i = 1\}$. In the integrand, ω_n is the volume form, and P_Γ and Ψ_Γ are polynomials defined as follows. The graph polynomial is defined as

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin T} t_e$$

where the summation is over spanning trees (assuming the graph Γ is connected). The polynomial P_Γ is given by

$$P_\Gamma(t,p) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e \tag{5.7}$$

with the sum over cut-sets C (complements of a spanning tree plus one edge) and with variables s_C depending on the external momenta of the graph, $s_C = (\sum_{v\in V(\Gamma_1)} P_v)^2$, where Γ_1 is one of the connected components after the cut (by momentum conservation, it does not matter which). The variables P_v are given by combinations of the external momenta $p = (p_e) \in \mathbb{Q}^{\# E_{ext}(\Gamma)\cdot D}$, in the form $P_v = \sum_{e\in E_{ext}(\Gamma), t(e)=v} p_e$, where $\sum_{e\in E_{ext}(\Gamma)} p_e = 0$.

In the range $-n + D\ell/2 \geq 0$, which includes the log divergent case $n = D\ell/2$, the Feynman amplitude is therefore the integral of an algebraic differential form defined on the complement of the graph hypersurface $\hat{X}_\Gamma = \{t \in \mathbb{A}^n \,|\, \Psi_\Gamma(t) = 0\}$. Divergences occur due to the intersections of the domain of integration σ_n with the hypersurface. Some regularization and renormalization procedure is required to separate the chain of integration from the divergence locus. We refer the reader to [12] (or to [52] for an introductory exposition).

It was originally conjectured by Kontsevich that the graph hypersurfaces $\hat{X}_\Gamma$ would always be mixed Tate motives, which would have explained the pervasive occurrence of multiple zeta values in Feynman integral computations observed in [16]. A general result of [10] disproved the conjecture, while more recent results of [18], [19], [29] showed explicit examples of Feynman graphs that give rise to non-mixed-Tate periods.

5.7. **Determinant hypersurface and parametric Feynman integrals.** In [4] the computation of parametric Feynman integrals was reformulated by replacing the graph hypersurface complement by the complement of the determinant hypersurface.

More precisely, the (affine) graph hypersurface $\hat{X}_\Gamma$ is defined by the vanishing of the graph polynomial Ψ_Γ. It follows from the matrix-tree theorem that this polynomial can be written as a determinant

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e\notin T} t_e \,,$$

with $M_\Gamma(t)$ the $\ell \times \ell$ matrix

$$(M_\Gamma)_{kr}(t) = \sum_{i=1}^n t_i \eta_{ik} \eta_{ir}, \tag{5.8}$$

where the matrix η is given by

$$\eta_{ik} = \begin{cases} \pm 1 & \text{edge } \pm e_i \in \text{ loop } \ell_k \\ 0 & \text{otherwise.} \end{cases}$$

This definition of the matrix η involves the choice of a basis $\{\ell_k\}$ of the first homology $H_1(\Gamma;\mathbb{Z})$ and the choice of an orientation of the edges of the graph, with $\pm e$ denoting the matching/reverse orientation on the edge e. The resulting determinant $\Psi_\Gamma(t)$ is independent of both choices.

One considers then the map

$$\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik} \eta_{ir}$$

that realizes the graph hypersurface as the preimage

$$\hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

of the determinant hypersurface $\hat{\mathcal{D}}_\ell = \{\det(x_{ij}) = 0\}$.

It is shown in [4] that the map

$$\Upsilon : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \hookrightarrow \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell \tag{5.9}$$

is an embedding whenever the graph Γ is 3-edge-connected with a closed 2-cell embedding of face width ≥ 3.

Remark 5.7. As discussed in §3 of [4], the 3-edge-connected condition on graphs can be viewed as a strengthening of the usual 1PI (one-particle-irreducible) condition assumed in physics, since the 1PI condition corresponds to 2-edge-connectivity. In perturbative quantum field theory, one considers 1PI graphs when computing the asymptotic expansion of the effective action. Similarly, one can consider the 2PI effective action (which is related to non-equilibrium phenomena in quantum field theory, see §10.5.1 of [57]) and restrict to 3-edge-connected graphs. The condition of having a closed 2-cell embedding of face width ≥ 3, on the other hand, is a strengthening of the analogous property with face width ≥ 2, which conjecturally is satisfied for all 2-vertex-connected graphs (strong orientable embedding conjecture, see Conjecture 5.5.16 of [54]). 2-vertex-connectivity is again a natural strengthening of the 1PI condition. A detailed discussion of equivalent formulations and implications of these combinatorial conditions, as well as specific examples of graphs that fail to satisfy them, are given in §3 of [4].

Let $\mathcal{P}_\Gamma(x,p)$ denote a homogeneous polynomial in $x \in \mathbb{A}^{\ell^2}$, with $p \in \mathbb{Q}^{\# E_{ext}(\Gamma)\cdot D}$, with the property that the restriction to the image of the map $\Upsilon = \Upsilon_\Gamma$ agrees with the second Symanzik polynomial P_Γ defined in (5.7),

$$\mathcal{P}_\Gamma(x,p)|_{x=\Upsilon(t)\in\Upsilon(\mathbb{A}^n)} = P_\Gamma(t,p).$$

When the map Υ_Γ is an embedding, one can, without loss of information, rewrite the parametric Feynman integral as (see Lemma 2.3 of [4])

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}. \tag{5.10}$$

Here $\omega_\Gamma(x)$ is an n-form on $\mathbb{A}^{\ell^2}$ such that the restriction of $\omega_\Gamma(x)$ to the subspace $\Upsilon(\mathbb{A}^n)$ satisfies $\omega_\Gamma(\Upsilon(t)) = \omega_n(t)$, with ω_n the volume form on $\mathbb{A}^n$. Under the condition that Υ is an embedding, the restriction of the integrand to the image $\Upsilon(\sigma_n)$ then agrees with the original Feynman integral.

The question on the nature of periods is then reformulated in [4] by considering a normal crossings divisor $\hat{\Sigma}_\Gamma$ in $\mathbb{A}^{\ell^2}$ with $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$ and considering the motive

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)). \tag{5.11}$$

The motive $\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell)$ of the determinant hypersurface complement belongs to the category of mixed Tate motives (see Theorem 4.1 of [4]), with Grothendieck class

$$[\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (\mathbb{L}^i - 1).$$

However, as shown in [4], the nature of the motive (5.11) is much more difficult to discern, because of the nature of the intersection between the divisor $\hat{\Sigma}_\Gamma$ and the determinant hypersurface. Assuming the previous conditions on the graph (3-edge-connectedness with a closed 2-cell embedding of face width at least 3), it is shown in Proposition 5.1 of [4] that one can consider a divisor $\hat{\Sigma}_{\ell,g}$ that only depends on $\ell = b_1(\Gamma)$ and on the minimal genus g of the surface S_g realizing the closed 2-cell embedding of Γ,

$$\hat{\Sigma}_{\ell,g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}, \tag{5.12}$$

where $f = \ell - 2g + 1$ and the irreducible components $L_1, \dots, L_{\binom{f}{2}}$ are linear subspaces defined by the equations

$$\begin{cases} x_{ij} &= 0 \quad 1 \leq i < j \leq f-1 \\ x_{i1} + \cdots + x_{i,f-1} &= 0 \quad 1 \leq i \leq f-1. \end{cases}$$

For a given choice of subspaces $V_1, \dots, V_\ell$ of a fixed ℓ-dimensional space, one defines the *variety of frames* as

$$\mathbb{F}(V_1, \dots, V_\ell) := \{(v_1, \dots, v_\ell) \in \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell \,|\, v_k \in V_k\}.$$

In other words, the variety of frames $\mathbb{F}(V_1, \dots, V_\ell)$ consists of the set of ℓ-tuples $(v_1, \dots, v_\ell)$, with $v_i \in V_i$, such that $\dim \operatorname{span}\{v_1, \dots, v_\ell\} = \ell$. It is shown in [4] that the motives (5.11) are mixed Tate if the varieties of frames $\mathbb{F}(V_1, \dots, V_\ell)$ are mixed Tate. This question is closely related to the geometry of intersections of unions of Schubert cells in flag varieties and Kazhdan–Lusztig theory.

In this paper we will follow a different approach, which uses the same reformulation of parametric Feynman integrals in momentum space in terms of determinant hypersurfaces, as in [4], but instead of computing the integral in the determinant hypersurface complement, pulls it back to the Kausz compactification of GL_ℓ, following the model of computations of Feynman integrals in configuration spaces described in [21].

5.8. **Cohomology and forms with logarithmic poles.** Let $\mathcal{X}$ be a smooth projective variety and $\mathcal{Z} \subset \mathcal{X}$ a divisor. Let $\mathcal{M}^\star_{\mathcal{X},\mathcal{Z}}$ denote, as before, the complex of meromorphic differential forms on $\mathcal{X}$ with poles (of arbitrary order) along $\mathcal{Z}$, and let $\Omega^\star_{\mathcal{X}}(\log(\mathcal{Z}))$ be the complex of forms with logarithmic poles along $\mathcal{Z}$. Let $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{Z}$ and $j : \mathcal{U} \hookrightarrow \mathcal{X}$ be the inclusion.

Grothendieck's Comparison Theorem, [37], shows that the natural morphism (de Rham morphism)

$$\mathcal{M}^\star_{\mathcal{X},\mathcal{Z}} \to Rj_* \mathbb{C}_{\mathcal{U}}$$

is a quasi-isomorphism, hence de Rham cohomology $H^\star_{dR}(\mathcal{U})$ is computed by the hypercohomology of the meromorphic de Rham complex. In particular, for $\mathcal{U}$ affine, the hypercohomology is not necessary and all classes are represented by closed global differential forms, with hypercohomology replaced by the cohomology of the complex of global sections.

The *Logarithmic Comparison Theorem* consists of the statement that, for certain classes of divisors $\mathcal{Z}$, the natural morphism

$$\Omega^\star_{\mathcal{X}}(\log(\mathcal{Z})) \to \mathcal{M}^\star_{\mathcal{X},\mathcal{Z}}$$

is also a quasi-isomorphism. This is known to hold for simple normal crossings divisors by [25], and for strongly quasihomogeneous free divisors by [20], and for a larger class of locally quasihomogeneous divisors in [42]. For our purposes, we will focus only on the case of simple normal crossings divisors.

In combination with Grothendieck's Comparison Theorem, the Logarithmic Comparison Theorem of [25] for a simple normal crossings divisor implies that the de Rham cohomology of the divisor complement is computed by the hypercohomology of the logarithmic de Rham complex,

$$H^\star_{dR}(\mathcal{U}) \simeq \mathbb{H}^\star(\mathcal{X}, \Omega^\star_{\mathcal{X}}(\log \mathcal{Z})). \tag{5.13}$$

Remark 5.8. Even under the assumption that the complement $\mathcal{U}$ is affine, the hypercohomology on the right hand side of (5.13) cannot always be replaced by global sections and cohomology. For example, if $\mathcal{X}$ is a smooth projective curve of genus g, and $\mathcal{U}$ is the complement of n points in $\mathcal{X}$, then $H^1_{dR}(\mathcal{U})$ has dimension $2g+n-1$, but the dimension of the space of global sections of the sheaf of logarithmic differentials is only $g+n-1$ by Riemann-Roch.

Some direct comparisons between de Rham cohomology $H^\star_{dR}(\mathcal{U})$ and the cohomology of the logarithmic de Rham complex are known. We discuss in the coming subsections how these apply to our specific case. Our purpose is to replace the meromorphic form that arises in the Feynman integral computation with a cohomologous form with logarithmic poles along the divisor of the Kausz compactification. In doing so, we need to maintain explicit control of the motive of the variety over which cohomology is taken, and also maintain the algebraic nature of all the differential forms involved.

5.9. **Pullback to the Kausz compactification, forms with logarithmic poles, and renormalization.** For fixed $D, \ell \in \mathbb{N}$ (respectively the integer spacetime dimension and the loop number) and for assigned external momenta $p \in \mathbb{Q}^D$, we now consider the algebraic differential form

$$\eta_{\Gamma,D,\ell,p}(x) := \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}. \tag{5.14}$$

For simplicity, we write the above as $\eta_\Gamma(x)$. This is defined on the complement of the determinant hypersurface, $\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell = \mathrm{GL}_\ell$. Thus, by pulling back to the Kausz compactification, we can regard it as an algebraic differential form on

$$K\mathrm{GL}_\ell \smallsetminus \mathcal{Z}_\ell = \mathrm{GL}_\ell,$$

where $\mathcal{Z}_\ell$ is the normal crossings divisor at the boundary of the Kausz compactification.

5.9.1. *Cellular decomposition approach.* We consider a special case of a simple normal crossings divisor $\mathcal{Z}$ in a smooth projective variety $\mathcal{X}$, under the additional assumption that $\mathcal{X}$ has a cell decomposition. We denote by $\{X_{\alpha,i}\}$ the finite collection of cells of dimension i, and in particular we simply write $X_\alpha = X_{\alpha,\dim\mathcal{X}}$ for the top dimensional cells.

Proposition 5.9. *Let $\mathcal{Z} \subset \mathcal{X}$ be a pair as above, with $\{X_\alpha\}$ the top dimensional cells of the cellular decomposition. Given a meromorphic form $\eta \in \mathcal{M}^m_{\mathcal{X},\mathcal{Z}}$, there exist forms $\beta^{(\alpha)}$ on X_α with logarithmic poles along the normal crossings divisor $\mathcal{Z}$, such that*

$$[\beta^{(\alpha)}] = [\eta|_{X_\alpha}] \in H^*_{dR}(X_\alpha \smallsetminus \mathcal{Z}). \tag{5.15}$$

Proof. Lemma 2.5 of [20] shows that the Logarithmic Comparison Theorem is equivalent to the statement that, for all Stein open sets $\mathcal{V} \subset \mathcal{X}$, there are isomorphisms

$$H^\star(\Gamma(\mathcal{V}, \Omega^\star_{\mathcal{X}}(\log \mathcal{Z}))) \simeq H^\star_{dR}(\mathcal{V} \smallsetminus \mathcal{Z}).$$

Namely, the hypercohomology in the Logarithmic Comparison Theorem can be replaced by cohomology of the complex of sections, when restricted to Stein open sets. □

Remark 5.10. The forms $\beta^{(\alpha)}$ do not match consistently on the closures of the cells X_α, because of nontrivial Čech cocycles, hence they are not restrictions of a unique form with logarithmic poles β defined on all of $\mathcal{X}$. In particular, the forms $\beta^{(\alpha)}$ obtained in this way depend on the cellular decomposition used.

Lemma 5.11. *Let $\mathcal{Z} \subset \mathcal{X}$ and $\{X_\alpha\}$ be as above, and suppose given a meromorphic form $\eta \in \mathcal{M}^N_{\mathcal{X},\mathcal{Z}}$, with $N = \dim\mathcal{X}$, and an N-chain $\sigma \subset \mathcal{X}$ with $\partial\sigma \subset \Sigma$, for a divisor Σ in $\mathcal{X}$.*

After performing a pole subtraction on the logaritmic forms on each cell X_α one can replace the integral $\int_\Sigma \eta$ with a renormalized version

$$\int_\sigma \beta^+ := \sum_\alpha \int_{X_\alpha \cap \sigma} \beta^{(\alpha),+}, \tag{5.16}$$

where $\beta^{(\alpha),+}$ is a simple pole subtraction on $\beta^{(\alpha)}$. The integral (5.16) is a sum of periods of motives $\mathfrak{m}(X_\alpha, X_\alpha \cap \Sigma)$. The information contained in the subtracted polar part is recovered by the Poincaré residues

$$\int_{\sigma \cap \mathcal{Z}_I} \mathrm{Res}_{\mathcal{Z}_I}(\beta) := \sum_\alpha \int_{\sigma \cap \mathcal{Z}_I \cap X_\alpha} \mathrm{Res}_{\mathcal{Z}_I}(\beta^{(\alpha)}) \tag{5.17}$$

along the intersections of components $\mathcal{Z}_I = Z_{i_1} \cap \cdots \cap Z_{i_k}$, $I = \{i_1, \ldots, i_k\}$ of the divisor $\mathcal{Z}$. These are sums of periods of the motives $\mathfrak{m}(\mathcal{Z}_I \cap X_\alpha)$.

Proof. Given the cell decomposition as above, we can write the integral as

$$\int_\sigma \eta = \sum_\alpha \int_{X_\alpha \cap \sigma} \eta|_{X_\alpha} = \sum_\alpha \int_{X_\alpha \cap \sigma} \beta^{(\alpha)}, \tag{5.18}$$

where each $\eta|_{X_\alpha}$ is replaced by the cohomologous $\beta^{(\alpha)}$ with logarithmic poles. After performing a pole subtraction on each $\beta^{(\alpha)}$ we obtain holomorphic forms $\beta^{(\alpha),+}$, hence the resulting integral is a period of $\mathfrak{m}(X_\alpha, X_\alpha \cap \Sigma)$. For the relation between polar subtraction and the Poincaré residues, see the discussion in §3.2 and §4.2 above. □

In both (5.16) and (5.17), we use the notation on the left-hand-side, with a global integral and a global form β, purely as a formal shorthand notation for the sum of the integrals on the cells of the $\beta^{(\alpha)}$, since the latter are not restrictions of a global form β.

Remark 5.12. Notice that the resulting integral (5.16) obtained in this way can be identified with a period of $\mathfrak{m}(\mathcal{X}, \Sigma)$ only in the case where the forms $\beta^{(\alpha),+}$ are restrictions $\beta^{(\alpha),+} = \beta^+|_{X_\alpha}$ of a single holomorphic form β^+ on $\mathcal{X}$. More generally, the resulting (5.16) is only a sum of periods of the motives $\mathfrak{m}(X_\alpha, X_\alpha \cap \Sigma)$.

Remark 5.13. If the cellular decomposition of $\mathcal{X}$ has a single top dimensional cell X, then a unique form with logarithmic poles $\beta \in \Omega^\star_X(\log \mathcal{Z})$, satisfying $[\eta|_X] = [\beta] \in H^\star_{dR}(X \smallsetminus \mathcal{Z})$, suffices to regularize the integral $\int_\sigma \eta$, with regularized value $\int_{\sigma \cap X} \beta^+$.

As we discussed in Proposition 5.4, the Kausz compactification is a spherical variety (Proposition 1.15 of §3 of [43] and also Propositions 9.1, 10.1 and 12.1 of [53]), hence it has a cellular decomposition (§3.1 of [14]) into cells $X(\lambda, x)$ as in (5.5). Thus, we can apply the procedure described above, to regularize the integral $\int_{\Upsilon(\sigma)} \eta_\Gamma$. While this regularization procedure depends on the choice of the cell decomposition, the construction of [14] for spherical varieties provides a cellular structure that is intrinsically defined by the orbit structure of $K\mathrm{GL}_\ell$ and is quite naturally reflecting its geometry. We can then perform a renormalization procedure based on the pole subtraction procedure for forms with logarithmic poles described above.

Corollary 5.14. *The cell decomposition $\{X(\lambda, x)\}$ of $K\mathrm{GL}_\ell$ has a single big cell X. Given $\eta_\Gamma = \eta_{\Gamma,D,\ell,p}$ as in (5.14), there is a form $\beta_\Gamma = \beta_{\Gamma,D,\ell,p}$ on the big cell X, with logarithmic poles along $\mathcal{Z}_\ell$, such that $[\eta_\Gamma|_X] = [\beta_\Gamma] \in H^\star_{dR}(X \smallsetminus \mathcal{Z})$. Applying the Birkhoff factorization for forms with logarithmic poles to β_Γ, we obtain a renormalized integral of the form*

$$R(\Gamma) = \int_{\tilde{\Upsilon}(\sigma_n) \cap X} \beta^+_{\Gamma,D,\ell,p}, \tag{5.19}$$

where β_Γ^+ *is a simple pole subtraction on* β_Γ*.*

Proof. As mentioned in Proposition 5.4, the spherical variety $K\mathrm{GL}_\ell$ is a smooth toroidal equivariant compactification of GL_ℓ (Proposition 1.15 of §3 of [43] and Propositions 9.1 and 12.1 of [53]). By §2.3 of [15] and Proposition 9.1 of [53], it then follows that there is just one big cell X. We can then write the integral as

$$\int_{\tilde{\Upsilon}(\sigma_n)} \eta_\Gamma = \int_{X\cap\tilde{\Upsilon}(\sigma_n)} \eta_\Gamma|_X, \tag{5.20}$$

where $\tilde{\Upsilon}(\sigma_n)$ is the pullback to $K\mathrm{GL}_\ell$ of the domain of integration $\Upsilon(\sigma_n)$.

Let $\mathcal{H}$ be the Hopf algebra of Feynman graphs. The morphism $\phi : \mathcal{H} \to \mathcal{M}^*_{X,\mathcal{Z}_\ell\cap X}$ assigns to a Feynman graph Γ a meromorphic differential form $\beta_\Gamma = \beta_{\Gamma,D,\ell,p}$ with logarithmic poles along $\mathcal{Z}_\ell$ satisfying $[\eta_\Gamma|_X] = [\beta_\Gamma] \in H^\star_{dR}(X \smallsetminus \mathcal{Z})$.

We then perform the Birkhoff factorization, and we denote by β_Γ^+ the regular differential form on $X \subset K\mathrm{GL}_\ell$ given by $\phi^+(\Gamma) = \beta_\Gamma^+$. Since we only have logarithmic poles, by Proposition 4.1 the operation becomes a simple pole subtraction and we have $\beta_\Gamma^+ = (1-T)\beta_\Gamma$. □

If we assume, as above, that the external momenta p in the polynomial $\mathcal{P}_\Gamma(x,p)$ are rational, then the form $\eta_\Gamma = \eta_{\Gamma,D,\ell,p}(x)$ is an algebraic differential form defined over $\mathbb{Q}$, hence we can also assume that the form with logarithmic poles β_Γ is also defined over $\mathbb{Q}$.

In addition to the integral (5.19), one also has the collection of the iterated Poincaré residues along the intersections of components of the divisor $\mathcal{Z}_\ell$. Namely, for any $\mathcal{Z}_{I,\ell} = \cap_{j\in I} Z_{j,\ell}$, with $Z_{j,\ell}$ the components of $\mathcal{Z}_\ell$, we have the additional integrals

$$\mathcal{R}(\Gamma)_I = \int_{\tilde{\Upsilon}(\sigma_n)\cap\mathcal{Z}_{I,\ell}\cap X} \mathrm{Res}_{\mathcal{Z}_I}(\beta_\Gamma). \tag{5.21}$$

5.9.2. *Griffiths-Schmid approach.* A global replacement of η_Γ by a single form $\beta_{\Gamma,D,\ell,p}$ on $K\mathrm{GL}_\ell$ with logarithmic poles along $\mathcal{Z}_\ell$ can be obtained if we use the $\mathcal{C}^\infty$-logarithmic de Rham complex instead of the algebraic or analytic one.

Proposition 5.15. *Consider the class* $[\eta_\Gamma]$ *in the analytic de Rham cohomology* $H^*_{dR}(\mathrm{GL}_\ell;\mathbb{C})$*. There is a* $\mathcal{C}^\infty$*-form* β_Γ^∞ *on* $K\mathrm{GL}_\ell$ *with logarithmic poles along* $\mathcal{Z}_\ell$ *such that*

$$[\beta_\Gamma^\infty] = [\eta_\Gamma] \in H^*_{dR}(K\mathrm{GL}_\ell \smallsetminus \mathcal{Z}_\ell;\mathbb{C}) = H^*_{dR}(\mathrm{GL}_\ell;\mathbb{C}). \tag{5.22}$$

Applying the Birkhoff factorization yields a renormalized integral

$$R^\infty(\Gamma) = \int_{\tilde{\Upsilon}(\sigma_n)} \beta^{\infty,+}_{\Gamma,D,\ell,p}, \tag{5.23}$$

where $\beta_\Gamma^{\infty,+}$ *is a simple pole subtraction on* β_Γ^∞*, and iterated residues*

$$\mathcal{R}^\infty(\Gamma)_I = \int_{\tilde{\Upsilon}(\sigma_n)\cap\mathcal{Z}_{I,\ell}} \mathrm{Res}_{\mathcal{Z}_I}(\beta_\Gamma^\infty). \tag{5.24}$$

Proof. For $\mathcal{X}$ a complex smooth projective variety and $\mathcal{Z}$ a simple normal crossings divisor, let $\Omega_{\mathcal{C}^\infty(\mathcal{X})}(\log\mathcal{Z})$ be the $\mathcal{C}^\infty$-logarithmic de Rham complex. The Griffiths-Schmid theorem (Proposition 5.14 of [36]) shows that there is an isomorphism $H^*_{dR}(\mathcal{U}) = H^*(\Omega_{\mathcal{C}^\infty(\mathcal{X})}(\log\mathcal{Z}))$. □

Remark 5.16. With the Griffiths-Schmid theorem one looses the algebraicity of differential forms. Namely, the forms β^∞_Γ and $\beta^{\infty,+}_\Gamma$ are only smooth and not algebraic or analytic differential forms. Even if the resulting form $\beta^{\infty,+}_\Gamma$, after pole subtraction, can then be replaced by an algebraic de Rham form in the same cohomology class in $H^*_{dR}(K\mathrm{GL}_\ell;\mathbb{C})$, it will remain, in general, only a form with $\mathbb{C}$-coefficients and not one defined over $\mathbb{Q}$. Thus, following this approach one obtains a consistent renormalization procedure, but one can lose control on the description of the resulting integrals as periods of motives defined over a number field.

5.9.3. *The Hodge filtration approach.* There is another case in which a form can be replaced globally by a cohomologous one with logarithmic poles on the complement of a normal crossings divisor, while only using algebraic or analytic forms. Indeed, there is a particular piece of the de Rham cohomology that is always realized by global sections of the (algebraic) logarithmic de Rham complex. This is the piece $F^n H^n_{dR}(\mathcal{U})$ of the Hodge filtration of Deligne's mixed Hodge structure, with $n = \dim \mathcal{X}$. This Hodge filtration on $\mathcal{U}$ is given by

$$F^p H^k_{dR}(\mathcal{U}) = \mathrm{Im}(\mathbb{H}^k(\mathcal{X}, \Omega^{\geq p}_{\mathcal{X}}(\log \mathcal{Z})) \to \mathbb{H}^k(\mathcal{X}, \Omega^\star_{\mathcal{X}}(\log \mathcal{Z}))).$$

Proposition 5.17. *Let $\mathcal{X}$ be a smooth projective variety with $N = \dim \mathcal{X}$, and let $\mathcal{Z}$ be a simple normal crossings divisor with affine complement $\mathcal{U} = \mathcal{X} \smallsetminus \mathcal{Z}$. Then, for $n \leq N$, the Hodge filtration satisfies*

$$H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}(\log \mathcal{Z})) = F^n H^n_{dR}(\mathcal{U})/F^{n+1} H^n_{dR}(\mathcal{U}). \tag{5.25}$$

In the case where $n = N$ the right-hand-side is reduced to $F^N H^N_{dR}(\mathcal{U})$.

Proof. The Hodge filtration $F^p H^k_{dR}(\mathcal{U})$ is induced by the naive filtration on $\Omega^\star_{\mathcal{X}}(\log \mathcal{Z})$. Recall that (see Theorem 8.21 and Proposition 8.25 of [62]) the spectral sequence of a filtration F on a complex $K^\star$ that comes from a double complex $K^{p,q}$, with

$$F^p K^n = \oplus_{r\geq p, r+s=n} K^{r,s}$$

has terms

$$E^{p,q}_0 = \mathrm{Gr}^F_p K^{p+q} = F^p K^{p+q}/F^{p+1}K^{p+q} = K^{p,q}$$

$$E^{p,q}_1 = H^{p+q}(\mathrm{Gr}^F_p K^\star) = H^q(K^{p,\star})$$

$$E^{p,q}_\infty = \mathrm{Gr}^F_p H^{p+q}(K^\star).$$

The spectral sequence above, applied to the Hodge filtration $F^p H^k_{dR}(\mathcal{U})$, is referred to as the Frölicher spectral sequence. It has

$$E^{p,q}_1 = H^q(\mathcal{X}, \Omega^p_{\mathcal{X}}(\log \mathcal{Z}))$$

$$E^{p,q}_\infty = F^p H^{p+q}_{dR}(\mathcal{U})/F^{p+1} H^{p+q}_{dR}(\mathcal{U}).$$

In particular,

$$E^{n,0}_1 = H^0(\mathcal{X}, \Omega^n_{\mathcal{X}}(\log \mathcal{Z})) \text{ and } E^{n,0}_\infty = F^n H^n_{dR}(\mathcal{U})/F^{n+1} H^n_{dR}(\mathcal{U}).$$

When $n = N = \dim \mathcal{X}$, the term $F^{N+1} H^N_{dR}(\mathcal{U})$ vanishes for dimensional reasons.

Deligne proved in [25] (see also the formulation of the result given in Theorem 8.35 of [62]) that, in the case where $\mathcal{Z}$ is a normal crossings divisor, the Frölicher spectral sequence of the Hodge filtration degenerates at the E_1 term. Thus, in particular, we obtain (5.25). ☐

Corollary 5.18. *Given a meromorphic form η with $[\eta] \in F^n H^n_{dR}(\mathrm{GL}_\ell)/F^{n+1} H^n_{dR}(\mathrm{GL}_\ell)$, with $n \leq \ell^2 = \dim K\mathrm{GL}_\ell$, there is a form β on $K\mathrm{GL}_\ell$ with logarithmic poles along the normal crossings divisor $\mathcal{Z}_\ell$, such that*

$$[\beta] = [\eta] \in H^n_{dR}(K\mathrm{GL}_\ell \smallsetminus \mathcal{Z}_\ell) = H^n_{dR}(\mathrm{GL}_\ell). \tag{5.26}$$

Then after pole subtraction one obtains

$$\int_{\tilde{\Upsilon}(\sigma_n)} \beta^+, \tag{5.27}$$

which is a period of $\mathfrak{m}(K\mathrm{GL}_\ell, \Sigma_{\ell,g})$.

In this case also, in addition to the integral (5.27), we also have the iterated residues (which in this case exist globally),

$$\int_{\tilde{\Upsilon}(\sigma_n)\cap \mathcal{Z}_{I,\ell}} \mathrm{Res}_{\mathcal{Z}_I}(\beta). \tag{5.28}$$

In general, it is difficult to estimate where the form η_Γ lies in the Hodge filtration. One can give an estimate, based on the relation between the filtration by order of pole and the Hodge filtration, but it need not be accurate because exact forms can cancel higher order poles. The same issue was discussed, in the original formulation in the graph hypersurface complement, in §9.2 and Proposition 9.8 of [13].

Let $\mathcal{X}$ be a smooth projective variety and $\mathcal{Z} \subset \mathcal{X}$ a simple normal crossings divisor. As before, let $\mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}$ denote the complex of meromorphic differential forms on $\mathcal{X}$ with poles (of arbitrary order) along $\mathcal{Z}$. This complex has a filtration $P^\star \mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}$ by order of poles (polar filtration), where $P^k \mathcal{M}^m_{\mathcal{Z},\mathcal{X}}$ consists of the m-forms with pole of order at most $m-k+1$, if $m-k \geq 0$ and zero otherwise. Deligne showed in §II.3, Proposition 3.13 of [26] and Proposition 3.1.11 of [25], that the filtration induced on the subcomplex $\Omega^\star_{\mathcal{X}}(\log \mathcal{Z})$ by the polar filtration on $\mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}$ is the naive filtration (that is, the Hodge filtration), and that the natural morphism

$$(\Omega^\star_{\mathcal{X}}(\log \mathcal{Z}), F^\star) \to (\mathcal{M}^\star_{\mathcal{Z},\mathcal{X}}, P^\star)$$

is a filtered quasi-isomorphism. In particular (Theorem 2 of [27]) the image of $\mathbb{H}^\star(\mathcal{X}, P^k \mathcal{M}^\star_{\mathcal{X},\mathcal{Z}})$ inside $H^\star_{dR}(\mathcal{U})$ contains $F^k H^\star_{dR}(\mathcal{U})$. This means that we can use the order of pole to obtain at least an estimate of the position of $[\eta_\Gamma]$ in the Hodge filtration. We need to compute the order of pole of the pullback of the form η_Γ along the blowups in the construction of the compactification $K\mathrm{GL}_\ell$.

Proposition 5.19. *For a graph Γ with $n = \#E_\Gamma$ and $\ell = b_1(\Gamma)$, such that $n \geq \ell - 2$, and with spacetime dimension $D \in \mathbb{N}$, the position of $[\eta_\Gamma]$ in the Hodge filtration $F^k H^n_{dR}(\mathrm{GL}_\ell)$ is estimated by $k \geq n - (\ell-1)(-n + (\ell+1)D/2) + (\ell-1)^2$.*

Proof. At the first step in the construction of the compactification $K\mathrm{GL}_\ell$ we blow up the locus of matrices of rank one. We need to compare the order of vanishing of $\det(x)^{-n+(\ell+1)D/2}$ along this locus, with the order of zero acquired by the form ω_Γ along the exceptional divisor of this blowup. The determinant vanishes at order $\ell - 1$ on that stratum. The form ω_Γ, on the other hand, acquires a zero of order $c-1$ where c is the codimension of the blowup locus. This can be seen in a local model: when blowing up a locus $L = \{z_1 = \cdots = z_c = 0\}$ in $\mathbb{C}^N$, the local coordinates w_i in the blowup can be taken as $w_i w_c = z_i$ for $i < c$ and $w_i = z_i$ for $i \geq c$, with $E = \{w_c = 0\}$ the exceptional divisor. Then for $n \geq c$, and a form $dz_1 \wedge \cdots \wedge dz_n$, the pullback satisfies

$$\pi^*(dz_1 \wedge \cdots \wedge dz_n) = d(w_c w_1) \wedge \cdots \wedge d(w_c w_{c-1}) \wedge d(w_c) \wedge \cdots \wedge d(w_n) = w_c^{c-1} dw_1 \wedge \cdots \wedge dw_n.$$

The codimension of the locus of rank one matrices is $c = (\ell-1)^2$. Thus, when performing the first blowup in the construction of $K\mathrm{GL}_\ell$, the pullback of the form η_Γ acquires a pole of order $(\ell-1)(-n+(\ell+1)D/2)-(\ell-1)^2+1$ along the exceptional divisor. Further blowups do not alter this pole order, hence we can estimate that the pullback of the n-form η_Γ to the Kausz compactification is in the term P^k of the polar filtration, with

$$n-k+1 = (\ell-1)(-n+(\ell+1)D/2)-(\ell-1)^2+1.$$

Taking into account the possibility of reductions of the order of pole, due to cancellations coming from exact forms, we obtain an estimate for the position in the polar and in the Hodge filtration, with $k \geq n-(\ell-1)(-n+(\ell+1)D/2)+(\ell-1)^2$. □

5.10. **Nature of the period.** We then discuss the nature of the period obtained by the evaluation of (5.27). We need a preliminary result. We define a mixed Tate configuration Y in an ambient variety X as follows.

Definition 5.20. *Let X be a smooth projective variety and $Y \subset X$ a divisor with irreducible components $\{Y_i\}_{i=1}^N$. Let $\mathcal{C}_Y = \{Y_I = Y_{i_1}\cap\cdots\cap Y_{i_k} \,|\, I=(i_1,\ldots,i_k),\, k\leq N\}$. Then Y is a mixed Tate configuration if all unions $Y_{I_1}\cup\cdots\cup Y_{I_r}$ of elements of the set $\mathcal{C}_Y$ have motives $\mathfrak{m}(Y_{I_1}\cup\cdots\cup Y_{I_r})$ contained in the Voevodsky derived category of mixed Tate motives.*

Remark 5.21. Note that in Definition 5.20 we do not require that Y is necessarily a normal crossings divisor. However, in the case of the boundary divisor $\mathcal{Z}_\ell$ of $K\mathrm{GL}_\ell$, we will use in Proposition 5.29 the fact that it is also normal crossings, in addition to satisfying the condition of Definition 5.20 (see Lemma 5.23).

Let $\Sigma_{\ell,g}$ be the proper transform of the divisor given by the projective version of $\hat{\Sigma}_{\ell,g}$ described in (5.12), defined by the same equations.

Lemma 5.22. *The divisor $\Sigma_{\ell,g}$ is a mixed Tate configuration.*

Proof. By (5.12), $\Sigma_{\ell,g}$ and any arbitrary union of components are hyperplane arrangements. It is known from [9] that motives of hyperplane arrangements are mixed Tate, see also §1.7.1–1.7.2 and §3.1.1 of [30], where the computation of the motive in the Voevodsky category can be obtained in terms of Orlik–Solomon models. Using a characterization of the mixed Tate condition in terms of eigenvalues of Frobenius, the mixed Tate nature of hyperplane arrangements was also proved in Proposition 3.1.1 of [48]. The mixed Tate property can be seen very explicitly at the level of the virtual motive. In fact, the Grothendieck class of an arrangement A in $\mathbb{P}^n$ is explicitly given (Theorem 1.1. of [3]) by

$$[A] = [\mathbb{P}^n] - \frac{\chi_{\hat{A}}(\mathbb{L})}{\mathbb{L}-1},$$

where $\chi_{\hat{A}}(t)$ is the characteristic polynomial of the associated central arrangement $\hat{A}$ in $\mathbb{A}^{n+1}$. It then follows by inclusion-exclusion in the Grothendieck ring that all unions and intersections of components of A are mixed Tate. □

Lemma 5.23. *The boundary divisor $\mathcal{Z}_\ell$ of the Kausz compactification $K\mathrm{GL}_\ell$ is a mixed Tate configuration.*

Proof. The motives of unions of intersections of components of $\mathcal{Z}_\ell$ can be described in terms of motives of bundles over products of flag varieties with fibers that are lower dimensional compactifications $K\mathrm{GL}_i$ and $\overline{\mathrm{PGL}}_i$, and we proved in Proposition 5.4 (see also Propositions 5.1 and 5.2) that all these motives are Tate, hence the condition of Definition 5.20 is satisfied. □

Proposition 5.24. *When the form $\beta_{\Gamma,D,\ell,p}$ on the big cell extends to a logarithmic form in $\Omega^\star_{K\mathrm{GL}_\ell}(\log \mathcal{Z}_\ell)$, the integral $R(\Gamma) = \int_{\tilde{\Upsilon}(\sigma_n)} \beta^+_{\Gamma,D,\ell,p}$ is a period of a mixed Tate motive.*

Proof. In the globally defined case, this is an integral of an algebraic differential form defined on the compactification $K\mathrm{GL}_\ell$, hence a genuine period, in the sense of algebraic geometry, of $K\mathrm{GL}_\ell$. By Proposition 5.4, we know that the Chow motive $h(K\mathrm{GL}_\ell)$ is Tate. We also know from Lemma 5.22 that the motive $\mathfrak{m}(\Sigma_{\ell,g})$ is mixed Tate. Under the embedding of pure motives into mixed motives we obtain objects $\mathfrak{m}(K\mathrm{GL}_\ell)$ and $\mathfrak{m}(\Sigma_{\ell,g})$ in the derived category of mixed Tate motives, $\mathcal{DMTM}(\mathbb{Q})$, that is, the smallest triangulated subcategory of the Voevodsky triangulated category of mixed motives $\mathcal{DM}(\mathbb{Q})$ containing all the Tate objects $\mathbb{Q}(n)$. It then follows that the relative motive $\mathfrak{m}(K\mathrm{GL}_\ell, \Sigma_{\ell,g})$ is also mixed Tate, as it sits in a distinguished triangle in the Voevodsky triangulated category, where the other two terms are mixed Tate. □

Remark 5.25. In the proof of Proposition 5.24 here above, we viewed the motive $\mathfrak{m}(K\mathrm{GL}_\ell, \Sigma_{\ell,g})$ as an element in the derived category $\mathcal{DMTM}(\mathbb{Q})$ of mixed Tate motives. All the varieties we are considering are defined over a number field, in fact over $\mathbb{Q}$. In the number field case, an abelian category of mixed Tate motives can be constructed as the heart of a t-structure in $\mathcal{DMTM}(\mathbb{Q})$: this is possible because the Beilinson-Soulé vanishing conjecture holds over a number field, see [50]. We denote by $\mathcal{MTM}(\mathbb{Q})$ this abelian category of mixed Tate motives. The obtain objects in $\mathcal{MTM}(\mathbb{Q})$ one applies the cohomology functor with respect to the t-structure. For example, for a projective space $\mathbb{P}^n$, one has the motive $\mathfrak{m}(\mathbb{P}^n) = \mathbb{Q}(0) \oplus \mathbb{Q}(-1)[2] \oplus \cdots \oplus \mathbb{Q}(-n)[2n]$ in $\mathcal{DMTM}(\mathbb{Q})$; its cohomology with respect to the t-structure is $h_{2i}(\mathfrak{m}(\mathbb{P}^n)) = \mathbb{Q}(-i)$, which is an object in $\mathcal{MTM}(\mathbb{Q})$, while the shifts are not. In the following, with an abuse of notation, we will write the motive simply as $\mathfrak{m}(K\mathrm{GL}_\ell, \Sigma_{\ell,g})$, and more generally $\mathfrak{m}(K\mathrm{GL}_\ell \smallsetminus A, B)$ for the cases considered in Proposition 5.29, although when we refer to the motive in $\mathcal{MTM}(\mathbb{Q})$ what we are really considering are the pieces of the cohomology with respect to the t-structure, and in particular, for the conclusion about the period, the piece that corresponds to the degree of the differential form.

Proposition 5.26. *Let $\beta_{\Gamma,D,\ell,p}$ be the form with logarithmic poles on the top cell X of the cellular decomposition of $K\mathrm{GL}_\ell$, as in Corollary 5.14. If the motive $\mathfrak{m}(\Sigma_{\ell,g} \cap X)$ is mixed Tate, then the integral $R(\Gamma) = \int_{\tilde{\Upsilon}(\sigma_n)} \beta^+_{\Gamma,D,\ell,p}$ is a period of a mixed Tate motive.*

Proof. Using distinguished triangles in the Voevodsky category, we see that, if the motive $\mathfrak{m}(\Sigma_{\ell,g} \cap X)$ is mixed Tate, then the motive $\mathfrak{m}(X, \Sigma_{\ell,g} \cap X)$ also is, since the big cell has $\mathfrak{m}(X) = \mathbb{L}^{\ell^2}$. The result then follows, since the integral is by construction a period of the motive $\mathfrak{m}(X, \Sigma_{\ell,g} \cap X)$. □

Remark 5.27. The central difficulty in the approach of [4], which was to analyze the nature of the motive of $\mathfrak{m}(\Sigma_{\ell,g} \cap \mathcal{D}_\ell)$, is here replaced by the problem of identifying the nature of the motive $\mathfrak{m}(\Sigma_{\ell,g} \cap X)$, where X is the big cell of $K\mathrm{GL}_\ell$.

It may seem at first that we have simply substituted the problem of understanding for which range of (ℓ, g) the intersection of the divisor $\Sigma_{\ell,g}$ with GL_ℓ remains mixed Tate, with the very similar problem of when the intersection of $\Sigma_{\ell,g}$ with the big cell X of $K\mathrm{GL}_\ell$ remains mixed Tate. However, this reformulation makes it possible to use the explicit description of the cells $X(\lambda, x)$ of spherical varieties in terms of limits as in (5.5), to analyze this question. We do not discuss this further in the present paper, and we simply state it as a question.

Question 5.28. *Let X be the big cell of the cellular decomposition (5.5) of $K\mathrm{GL}_\ell$ and let $\Sigma_{\ell,g}$ be the divisor described above. For which pairs (ℓ, g) is the motive $\mathfrak{m}(\Sigma_{\ell,g} \cap X)$ mixed Tate?*

One defines the category $\mathcal{MTM}(\mathbb{Z})$ of mixed Tate motives over $\mathbb{Z}$ as mixed Tate motives in $\mathcal{MTM}(\mathbb{Q})$ that are unramified over $\mathbb{Z}$. An object of $\mathcal{MTM}(\mathbb{Q})$ is unramified over $\mathbb{Z}$ if and only if, for any prime p, there exists a prime $\ell \neq p$ such that the ℓ-adic realization is unramified at p, see Proposition 1.8 of [28]. In the following statement our notation for the motives is meant as in Remark 5.25 above.

Proposition 5.29. *The motives $\mathfrak{m}(K\mathrm{GL}_\ell)$ are unramified over $\mathbb{Z}$. More generally, if A and B are unions of two disjoint sets of components of the boundary divisor $\mathcal{Z}_\ell$ of the compactification $K\mathrm{GL}_\ell$, the motives $\mathfrak{m}(K\mathrm{GL}_\ell \smallsetminus A, B)$ are unramified over $\mathbb{Z}$.*

Proof. This question can be approached in a way analogous to our previous discussion of the Chow motive, namely using the description of $K\mathrm{GL}_\ell$ as an iterated blowup and the properties of the divisor of the compactification. The argument is similar to the one used in Theorem 4.1 and Proposition 4.3 of [35] to prove the analogous statement for the moduli spaces $\overline{\mathcal{M}}_{0,n}$ of rational curved with marked points. There, it is shown that $\overline{\mathcal{M}}_{0,n}$ is unramified over $\mathbb{Z}$ by showing that the combinatorics of the normal crossings divisor of the compactification is not altered by reductions mod p, see Definition 4.2 of [35]. For a pair $(\mathcal{X}, \mathcal{Z})$, with $\mathcal{Z} \subset \mathcal{X}$ a normal crossing divisor, the condition that the reduction mod p does not alter the combinatorics means that $\mathcal{X}$ and all the strata of $\mathcal{Z}$ are smooth over $\mathbb{Z}_p$ and the reduction mod p gives a bijection from the strata of $\mathcal{Z}$ to those of the special fiber (see Definition 4.2 of [35] and Definition 3.9 of [34]). In our case we have $\mathcal{X} = K\mathrm{GL}_\ell$, with $\mathcal{Z} = \mathcal{Z}_\ell$ the boundary divisor of the Kausz compactification. As we showed in §§5.3, 5.4, and 5.5, the motive $\mathfrak{m}(K\mathrm{GL}_\ell)$ is a Tate motive. More generally, the motives $\mathfrak{m}(K\mathrm{GL}_\ell \smallsetminus A, B)$ are mixed Tate over $\mathbb{Q}$: this can be seen as in Proposition 3.6 of [34], using Lemma 5.23, which shows that $\mathcal{Z}_\ell$ is both a normal crossings divisor and a mixed Tate configuration in the sense of Definition 5.20. This implies, by the argument of Proposition 3.6 of [34], that all the motives $\mathfrak{m}(K\mathrm{GL}_\ell \smallsetminus A, B)$ are mixed Tate over $\mathbb{Q}$.

To check the condition that the reduction map preserves the combinatorics of $(K\mathrm{GL}_\ell, \mathcal{Z}_\ell)$, first note that both $K\mathrm{GL}_\ell$ and the strata of the normal crossing divisor $\mathcal{Z}_\ell$ are smooth over $\mathbb{Z}$, by Corollary 4.2 [47]. Moreover, the description of the Kausz compactification and of the strata of its boundary divisor given in Theorems 9.1 and 9.3 of [47] also holds over fields of characteristic p and is compatible with reduction, so that the set of strata is matched under the reduction map. The argument of Proposition 4.3 of [35] showing that the ℓ-adic realization is then unramified, for all ℓ with $\ell \neq p$, is based on the argument of Proposition 3.10 of [34]. Following this reasoning, the cohomologies $H^*(\mathcal{X} \smallsetminus A, B)$ can be computed using a simplicial resolution $\mathcal{S}_\bullet(\mathcal{X} \smallsetminus A, B)$, whose simplexes correspond to unions of intersections of components of the divisor. The argument of Proposition 3.10 of [34] then shows that the reduction map applied to the simplicial schemes $\mathcal{S}_\bullet(\mathcal{X} \smallsetminus A, B)$ induces an isomorphism in étale cohomology, $H^*_{et}(\bar{\mathcal{X}} \smallsetminus \bar{A}, \bar{B}, \mathbb{Q}_\ell) \simeq H^*_{et}(\bar{\mathcal{X}}^0 \smallsetminus \bar{A}^0, \bar{B}^0, \mathbb{Q}_\ell)$, where $\bar{\mathcal{X}} = \mathcal{X} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and $\mathcal{X}^0$ is the special fiber of the reduction. This shows that the étale realization is unramified for $\ell \neq p$. By Proposition 1.8 of [28] this means that the motives $\mathfrak{m}(K\mathrm{GL}_\ell \smallsetminus A, B)$ are mixed Tate over $\mathbb{Z}$. ☐

Remark 5.30. Given that the unramified condition holds, one can conclude from Brown's theorem [17] and the previous Proposition 5.24 (and Proposition 5.26, when $\mathfrak{m}(\Sigma_{\ell,g} \cap X)$ is mixed Tate) that the integral (5.27) is a $\mathbb{Q}[\frac{1}{2\pi i}]$-linear combination of multiple zeta values.

5.11. **Comparison with Feynman integrals.** The result obtained in this way clearly differs from the usual computation of Feynman integrals, where the methods used are based on regularization and pole subtraction of the integral (dimensional regularization, cutoff, zeta regularization, etc.) There are several reasons behind this difference, which we now discuss briefly.

In the usual physical renormalization non-mixed-Tate periods are known to occur, [18], [19]. In the setting we discussed here, the only possible source of non-mixed-Tate cases is the motive of the intersection $\Sigma_{\ell,g} \cap X$, where X is the big cell of the Kausz compactification $K\mathrm{GL}_\ell$. In particular, this locus is the same for all graphs with fixed loop number ℓ and fixed genus g. However, in the usual physical renormalization, not all graphs with the same ℓ and g have periods of the same nature, as one can see from the examples analyzed in [29], [59].

There is loss of information in mapping the computation of the Feynman integral from the complement of the graph hypersurface (as in [12], [18], [19]) to the complement of the determinant hypersurface (as in [4]), when the combinatorial conditions on the graph recalled in §5.7 are not satisfied. Explicit examples of graphs that violate those conditions are given in §3 of [4]. In such cases the map (5.9) need not be an embedding, hence part of the information contained in the Feynman integral calculation (5.6) will be lost in passing to (5.10).

However, this type of loss of information does not affect some of the cases where non-mixed Tate motives are known to appear in the momentum space Feynman amplitude.

Example 5.31. Let Γ be the graph with 14 edges that gives a counterexample to the Kontsevich polynomial countability conjecture, in Section 1 of [29]. The map $\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}$ of (5.9) has $n = \#E(\Gamma) = 14$ and $\ell = b_1(\Gamma) = 7$. Let Υ_i denote the composition of the map Υ with the projection onto the i-th row of the matrix M_Γ of (5.8). In order to check if the embedding condition for Υ is satisfied, we know from Lemma 3.1 of [4] that it suffices to check that Υ_i is injective for i ranging over a set of loops such that every edge of Γ is part of a loop in that set. This can then be checked by computer verification for the matrix M_Γ of this particular graph.

The example above is a log divergent graph in dimension four. It is known to give a non-mixed Tate contribution with the usual method of computation of the Feynman integral, [29], [18]. The same verification method we used for this case can be applied to the other currently known explicit counterexamples in [29], [18], [59], [19].

Even for integrals (including the example above) where the map (5.9) is an embedding, the regularization and renormalization procedure described here, using the Kausz compactification and subtraction of residues for forms with logarithmic poles, is not equivalent to the usual renormalization procedures of the regularized integrals. For instance, our regularized form (hence our regularized integral) can be trivial in cases where the usual regularization and renormalization would give a non-trivial result. This may occur if the form β with logarithmic poles happens to have a nontrivial residue, but a trivial holomorphic part β^+.

In such cases, part of the information loss coming from pole subtraction on the differential form is compensated by keeping track of the residues. However, in our setting these also deliver only mixed Tate periods, so that even when this information is included, one still loses the richer structure of the periods arising from other methods of regularization and renormalization, adopted in the physics literature.

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INTERSECTION SPACES, PERVERSE SHEAVES AND STRING THEORY

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Abstract. We survey recent results describing a perverse sheaf realization of Banagl's intersection space homology in the context of projective hypersurfaces with only isolated singularities. Intersection space homology has been recently proved to be relevant in type IIB string theory, as it provides the correct count of massless 3-branes arising during a Calabi-Yau conifold transition.

1. Introduction

In addition to the four dimensions that model our space-time, string theory requires six dimensions for a string to vibrate. By supersymmetry, these six real dimensions must be realized by a Calabi-Yau space. However, given the multitude of known topologically distinct Calabi-Yau 3-folds, the string model remains undetermined. Therefore, it is important to have mechanisms that allow one to move from one Calabi-Yau space to another. In Physics, a solution to this problem was first proposed by Green-Hübsch [GH1, GH2] who, motivated by Reid's "fantasy" [Re87], conjectured that topologically distinct Calabi-Yau 3-folds are connected to each other by means of conifold transitions, which induce a phase transition between the corresponding string models.

A conifold transition starts out with a smooth Calabi-Yau 3-fold, passes through a singular variety — *the conifold* — by a deformation of complex structure, and arrives at a topologically distinct smooth Calabi-Yau 3-fold by a small resolution of singularities. The deformation collapses embedded three-spheres (called *vanishing cycles*) to isolated ordinary double points, while the resolution resolves the singular points by replacing each of them with a $\mathbb{CP}^1$. In Physics, the topological change resulted from passing from one of the Calabi-Yau's to the conifold was interpreted by Strominger [Str95] by the condensation of massive black holes to massless ones. In type IIA string theory, there are charged two-branes that wrap around the $\mathbb{CP}^1$ 2-cycles, and which become massless when these 2-cycles are collapsed to points by the resolution map. Goresky-MacPherson's intersection homology [GM80, GM83] of the conifold accounts for all of these massless two-branes, and since it also satisfies Poincaré duality, it may be viewed as a physically correct homology theory for type IIA string theory. Similarly, in type IIB string theory there are charged three-branes wrapped around the vanishing cycles, and which become massless as these vanishing cycles are collapsed by the deformation of complex structure. Neither ordinary homology nor intersection homology of the conifold account for these massless three-branes; see [Ba10][Section 3.7] for more details. So a natural problem is to find a physically correct homology theory for type IIB string theory. A solution to this question was suggested by Banagl in [Ba10] via his intersection space homology theory.

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In [Ba10], Banagl developed a homotopy-theoretic method which associates to certain types of singular spaces X (e.g., a conifold) a CW complex IX, called the *intersection space* of X, whose reduced rational homology groups satisfy Poincaré Duality. Roughly speaking, the intersection space IX associated to a singular space X is constructed by replacing links of singularities of X by their corresponding Moore approximations, a process called *spatial homology truncation*. The *intersection space homology*

$$HI_*(X;\mathbb{Q}) := H_*(IX;\mathbb{Q}) \tag{1}$$

is not isomorphic to the intersection homology of the space X, and in fact it can be seen that in the middle degree and for isolated singularities, this new theory takes more cycles into account than intersection homology. For a conifold X, Banagl showed that the dimension of $HI_3(X)$ equals the number of physically present massless 3-branes in IIB theory, so intersection space homology can be viewed as a physically correct homology theory for type IIB string theory.

Our approach for studying intersection space homology is motivated by *mirror symmetry*. In mirror symmetry, given a Calabi-Yau 3-fold X, the mirror map associates to it another Calabi-Yau 3-fold Y so that type IIB string theory on $\mathbb{R}^4 \times X$ corresponds to type IIA string theory on $\mathbb{R}^4 \times Y$. If X and Y are smooth, their Betti numbers are related by precise algebraic identities (e.g., see [CK99]), e.g.,

$$\beta_3(Y) = \beta_2(X) + \beta_4(X) + 2, \tag{2}$$

etc. Morrison [Mor99] conjectured that the mirror of a conifold transition is again a conifold transition, but performed in the reverse order (i.e., by exchanging resolutions and deformations). Thus, if X and Y are mirrored conifolds (in mirrored conifold transitions), the intersection space homology of one space and the intersection homology of the mirror space form a *mirror-pair*, in the sense that

$$\beta_3(IY) = I\beta_2(X) + I\beta_4(X) + 2, \tag{3}$$

etc., where $I\beta_i$ denotes the i-th intersection homology Betti number (see [Ba10] for details). This suggests that it should be possible to compute the intersection space homology $HI_*(X;\mathbb{Q})$ of a variety X in terms of the topology of a smoothing deformation, by "mirroring" known results (e.g., [BBD, dCM, GM82]) relating the intersection homology groups $IH_*(X;\mathbb{Q})$ of X to the topology of a resolution of singularities.

This point of view was successfully exploited in [BM11, BBM], where we considered the case of a hypersurface $X \subset \mathbb{CP}^{n+1}$ with only isolated singularities, this being the main source of examples for conifold transitions. In this note, we review some of the main constructions and results from these works.

Convention: By "manifold" we mean a "complex projective manifold", and by "singular space" we mean a "complex projective variety of pure complex dimension n". We are only interested in "middle-perversity" calculations, so any mentioning of other perversity functions will be ignored. Unless otherwise specified, all (intersection) (co)homology groups will be computed with rational coefficients. Spaces considered in this paper will have at most isolated singularities.

Some of the properties of *intersection (co)homology* of a singular space X which are relevant for the above-mentioned "mirror" approach are:

(a) Intersection homology $IH_*(X)$ satisfies Poincaré duality.
(b) If $\widetilde{X}$ is a resolution of singularities of X, then $IH_*(X)$ is a sub-vector space of $H_*(\widetilde{X})$. Moreover, if $\widetilde{X}$ is a *small* resolution, then $IH_*(X) \cong H_*(\widetilde{X})$.

(c) $IH^*(X)$ is realized by a *perverse*, *self-dual*, *constructible* sheaf complex $\mathcal{IC}_X$, i.e.

$$IH^i(X) \cong \mathbb{H}^i(X, \mathcal{IC}_X[-n]). \tag{4}$$

(d) $IH^*(X)$ carries the *Kähler package*, including Hodge structures, as well as weak and hard Lefschetz theorems.

Based on the above considerations, it is therefore natural to try to "mirror" such properties in the context of intersection space (co)homology. As already mentioned, the Poincaré duality property is satisfied by the intersection spaces. We will thus focus on the properties (b), (c) and (d).

2. Hypersurface singularities and smoothing invariance of intersection space homology

Let X be a complex projective hypersurface of dimension $n > 2$, which, for simplicity, is assumed to have only one isolated singular point x. Let L_x, F_x and $T_x : H_n(F_x) \to H_n(F_x)$ denote the link, Milnor fiber and local monodromy operator of the isolated hypersurface singularity germ (X, x), respectively. By [Mi68], the link L_x is an $(n-2)$-connected closed oriented $(2n-1)$-dimensional manifold. Moreover, the Milnor fiber F_x is a parallelizable $(n-1)$-connected $2n$-dimensional manifold, which has the homotopy type of $\bigvee S^n$, a wedge of n-spheres. The number $\mu_x = \mathrm{rk} H_n(F_x)$ of these n-spheres (also called vanishing cycles) is the local Milnor number at x. It is known that all eigenvalues of T_x are roots of unity. We say that the local monodromy operator T_x is trivial if all eigenvalues of T_x are equal to 1.

The assumption on the dimension of X is needed to assure that the link L_x of x is simply-connected, so the intersection space IX can be defined as in [Ba10]. The actual definition of an intersection space is not needed here, only the calculation of Betti numbers, as described in Theorem 2.1 below, will be used in the sequel. Nevertheless, let us indicate briefly how IX is obtained from X. Let M be the complement of an open cone neighborhood of x so that M is a compact manifold with boundary $\partial M = L_x$. The spatial homology n-truncation of L_x is a topological space $L_x^{<n}$ such that $H_i(L_x^{<n}) = 0$ for $i \geq n$, together with a continuous map $f : L_x^{<n} \to L$ which induces a homology isomorphism in degrees $i < n$. The intersection space IX is then defined as the homotopy cofiber of the composition

$$L_x^{<n} \xrightarrow{f} L_x = \partial M \overset{\mathrm{incl}}{\hookrightarrow} M$$

(see [Ba10] for complete details, and [BM12] for a mild introduction).

The following result can be viewed as a generalization of the Betti calculation from [Ba10] in the context of conifold transitions:

Theorem 2.1. ([BM11][Thm.4.1, Thm.5.2]) *Let X_s be a nearby smoothing of X. Then, under the above assumptions and notations, the following holds:*

$$\dim HI^i(X; \mathbb{Q}) = \begin{cases} \dim H^i(X_s; \mathbb{Q}) & \text{if } i \neq n, 2n; \\ \dim H^i(X_s; \mathbb{Q}) - rk(T_x - 1) & \text{if } i = n; \\ 0 & \text{if } i = 2n. \end{cases} \tag{5}$$

Moreover, under some mild technical assumption on the homology of the link (that is, if $H_{n-1}(L_x; \mathbb{Z})$ is torsion-free), the above identities are derived via a continuous map $IX \to X_s$, and we obtain a smoothing invariance of the intersection space (co)homology $\widetilde{H}^*(IX)$ if, and only if, the local monodromy operator T_x is trivial. So this result can be viewed as mirroring property (b) of intersection homology, expressing the intersection cohomology of X as a sub-vector space of the cohomology of any resolution, with an isomorphism in the case of a small resolution.

Moreover, the local trivial monodromy condition (or the existence of "small deformations") should be regarded as mirroring that of the existence of small resolutions.

3. Perverse sheaf approach to intersection space homology

3.1. **Summary of results.** Guided by a similar philosophy derived from mirror symmetry, in [BBM] we constructed a perverse sheaf[1] $\mathcal{IS}_X$, the *intersection-space complex*, whose global hypercohomology calculates (abstractly) the intersection space cohomology groups of a projective hypersurface $X \subset \mathbb{CP}^{n+1}$ with one isolated singular point.

Theorem 3.1. *([BBM]) Let X_s be a nearby smoothing of X. Then there exists a perverse sheaf complex $\mathcal{IS}_X$ on X so that there are (abstract) isomorphisms*

$$\mathbb{H}^i(X;\mathcal{IS}_X[-n]) \simeq \begin{cases} H^i(IX) & \text{if } i \neq 2n \\ H^{2n}(X_s) = \mathbb{Q} & \text{if } i = 2n. \end{cases} \tag{6}$$

Our construction (see Section 3.2.2 for a sketch) can be viewed as mirroring the fact that the intersection cohomology groups can be computed from a perverse sheaf, namely the intersection cohomology complex $\mathcal{IC}_X$. We would like to point out that for general X there cannot exist a perverse sheaf $\mathcal{P}$ on X such that $HI^*(X;\mathbb{Q})$ can be computed from the hypercohomology group $\mathbb{H}^*(X;\mathcal{P}[-n])$. Indeed, the stalk vanishing conditions that such a perverse sheaf $\mathcal{P}$ satisfies would give $\mathbb{H}^i(X;\mathcal{P}[-n]) = H^i(M)$, for $i < n$, while $H^i(IX) = H^i(M, \partial M)$ if $i < n$. (Here M denotes as before the complement of an open cone neighborhood of x in X.) However, due to the high-connectivity of the links, this goal can be achieved in the case when X is a hypersurface with only isolated singularities, this being in fact the main source of examples for conifold transitions. This fact motivates our study of intersection spaces associated to hypersurfaces with only isolated singularities.

Furthermore, by construction, the intersection space complex $\mathcal{IS}_X$ underlies a mixed Hodge module, therefore its hypercohomology groups carry canonical mixed Hodge structures. This result mirrors the corresponding one for the intersection cohomology complex $\mathcal{IC}_X$.

It follows from the above interpretation of intersection space cohomology that the groups $\mathbb{H}^*(X;\mathcal{IS}_X)$ satisfy Poincaré duality globally, which raises the question whether this duality is induced by a more powerful (Verdier-) self-duality isomorphism $\mathcal{D}(\mathcal{IS}_X) \simeq \mathcal{IS}_X$ in the derived category of constructible bounded sheaf complexes on X. In [BBM], we showed the following:

Theorem 3.2. *([BBM]) If the local monodromy T_x at the singular point x is semi-simple in the eigenvalue 1, then the intersection space complex $\mathcal{IS}_X$ is Verdier self-dual. In particular, for any integer i, there is a non-degenerate pairing*

$$\mathbb{H}^{-i}(X;\mathcal{IS}_X) \times \mathbb{H}^i(X;\mathcal{IS}_X) \to \mathbb{Q}.$$

The assumption on the semi-simplicity of local monodromy in the eigenvalue 1 is satisfied by a large class of isolated singularities, e.g., the weighted homogeneous ones.

[1] Let us recall here the definition of a *perverse sheaf* on a singular space X with only one isolated singular point x. Such a space can be given a Whitney stratification $\mathcal{X}$ with only two strata: $\{x\}$ and $X \setminus \{x\}$. Denote by $i : \{x\} \hookrightarrow X$ and $j : X \setminus \{x\} \hookrightarrow X$ the corresponding closed and open embeddings. Then a complex $K \in D^b_c(X)$, which is constructible with respect to $\mathcal{X}$, is perverse on X if $j^*K[-n]$ is cohomologically a local system on $X \setminus \{x\}$ and, moreover, the following two (stalk and, respectively, co-stalk vanishing) conditions hold:

$$H^j(i^*K) = 0, \text{ for any } j > 0,$$

$$H^j(i^!K) = 0, \text{ for any } j < 0.$$

Let us next recall that the Beilinson-Bernstein-Deligne decomposition [BBD] for the pushforward $Rf_*\mathbb{Q}_{\widetilde{X}}[n]$ of the constant sheaf $\mathbb{Q}_{\widetilde{X}}$ under an algebraic resolution map $f : \widetilde{X} \to X$ splits off the intersection sheaf $\mathcal{IC}_X$ of X plus contributions from the singularities of X. Suppose now that X sits as $X = \pi^{-1}(0)$ in a family $\pi : \widetilde{X} \to S$ of projective hypersurfaces over a small disc around $0 \in \mathbb{C}$ such that $\widetilde{X}$ is smooth, and $X_s = \pi^{-1}(s)$ is smooth over nearby $s \in S$, $s \neq 0$. In this situation, the *nearby cycle functor* ψ_π for π can be defined, and we have the following result:

Theorem 3.3. *([BBM]) If the local monodromy T_x at the singular point x is semi-simple in the eigenvalue 1, then the intersection space complex $\mathcal{IS}_X$ is a direct summand of the nearby cycle complex $\psi_\pi\mathbb{Q}_{\widetilde{X}}[n]$.*

The summand complementary to $\mathcal{IS}_X$ has the interpretation as being contributed by the singularity x, since it is supported only over $\{x\}$. We regard this splitting of nearby cycles as mirroring the above Beilinson-Bernstein-Deligne decomposition theorem in the following sense. For s sufficiently close to 0, there is a map $sp : X_s \to X$, the specialization map, which should be viewed as mirroring a resolution map. Moreover, the nearby cycle complex $\psi_\pi\mathbb{Q}_{\widetilde{X}}[n]$ can be computed by the (derived) pushforward $R(sp)_*\mathbb{Q}_{X_s}[n]$ of the constant sheaf on a nearby smoothing of X. Altogether, we have a decomposition

$$R(sp)_*\mathbb{Q}_{X_s}[n] \simeq \mathcal{IS}_X \oplus \mathcal{C}, \tag{7}$$

with $\mathcal{C}$ a perverse sheaf supported on the singular set $\{x\}$.

Finally, in [BBM] we prove the following result which mirrors the existence of the Kähler package on intersection cohomology groups:

Theorem 3.4. *([BBM]) If the local monodromy T_x at x is semi-simple in the eigenvalue 1, and the global monodromy T acting on $H^*(X_s)$ is semi-simple in the eigenvalue 1, then the hypercohomology groups $\mathbb{H}^*(X;\mathcal{IS}_X)$ carry pure Hodge structures satisfying the Hard Lefschetz theorem.*

3.2. **Intersection space complex.** Let us now sketch the construction of the perverse sheaf $\mathcal{IS}_X$, see [BBM] and references therein for complete details. We will try to keep the technical details at a minimum, in order not to obscure the presentation.

3.2.1. *Nearby and vanishing cycles.* Let us consider, as before, a hypersurface $X \subset \mathbb{CP}^{n+1}$ with $Sing(X) = \{x\}$. Let $\pi : \widetilde{X} \to S \subset \mathbb{C}$ be a family of hypersurfaces over a small disc S centered at the origin, with $X = \pi^{-1}(0)$, and so that $\widetilde{X}$ is smooth and $X_s := \pi^{-1}(s)$ for $s \neq 0$ is a smooth hypersurface in $\mathbb{CP}^{n+1}$. Let

$$\psi_\pi, \varphi_\pi : D^b_c(\widetilde{X}) \to D^b_c(X)$$

be the *nearby* and *vanishing cycle functors* for π, with monodromy T and resp. $\widetilde{T}$. Then

$$H^i(X_s;\mathbb{Q}) \cong \mathbb{H}^i(X;\psi_\pi\mathbb{Q}_{\widetilde{X}}), \tag{8}$$

and, for the point inclusion $i_x : \{x\} \hookrightarrow X$, with F_x denoting as before the Milnor fiber of the hypersurface singularity germ (X, x), we have

$$H^i(F_x;\mathbb{Q}) \cong H^i(i_x^*\psi_\pi\mathbb{Q}_{\widetilde{X}}) \quad \text{and} \quad \widetilde{H}^i(F_x;\mathbb{Q}) \cong H^i(i_x^*\varphi_\pi\mathbb{Q}_{\widetilde{X}}), \tag{9}$$

with compatible monodromy actions. Note that $Supp(\varphi_\pi\mathbb{Q}_{\widetilde{X}}) = Sing(X) = \{x\}$.

There are canonical morphisms:

$$can : \psi_\pi \to \varphi_\pi \quad \text{and} \quad var : \varphi_\pi \to \psi_\pi$$

so that $can \circ var = \widetilde{T} - 1$, $var \circ can = T - 1$.

The monodromy automorphisms T and $\widetilde{T}$ have Jordan decompositions

$$T = T_u \circ T_s = T_s \circ T_u,$$

where T_s is semisimple (and locally of finite order) and T_u is unipotent, and similarly for $\widetilde{T}$. For any $\lambda \in \mathbb{Q}$ and $K \in D^b_c(\widetilde{X})$, denote by $\psi_{\pi,\lambda}K$ the generalized λ-eigenspace for T, and similarly for $\phi_{\pi,\lambda}K$. There are decompositions

$$\psi_\pi = \psi_{\pi,1} \oplus \psi_{\pi,\neq 1} \quad \text{and} \quad \varphi_\pi = \varphi_{\pi,1} \oplus \varphi_{\pi,\neq 1}$$

so that $T_s = 1$ on $\psi_{\pi,1}$, $\widetilde{T}_s = 1$ on $\varphi_{\pi,1}$, and T_s and $\widetilde{T}_s$ have no 1-eigenspace on $\psi_{\pi,\neq 1}$ and $\varphi_{\pi,\neq 1}$, respectively. Moreover, $can : \psi_{\pi,\neq 1} \to \varphi_{\pi,\neq 1}$ and $var : \varphi_{\pi,\neq 1} \to \psi_{\pi,\neq 1}$ are isomorphisms.

Let $N := \log(T_u)$, and similarly for $\widetilde{N}$. The morphism $\varphi_\pi K \xrightarrow{Var} \psi_\pi K$ is defined by the cone of the pair $(0, N)$. Then $can \circ Var = \widetilde{N}$ and $Var \circ can = N$.

The functors ${}^p\psi_\pi := \psi_\pi[-1]$ and ${}^p\varphi_\pi := \varphi_\pi[-1]$ from $D^b_c(\widetilde{X})$ to $D^b_c(X)$ commute with the Verdier duality functor $\mathcal{D}$ (up to natural isomorphisms), and send perverse sheaves to perverse sheaves. These functors and their decompositions into unipotent and non-unipotent parts lift to the category of mixed Hodge modules, as do the functors can, N, $\widetilde{N}$ and Var. For an introduction to Saito's theory of mixed Hodge modules, the interested reader is advised to consult [Sa89].

3.2.2. *Intersection space complex: construction.* First note that $\psi_\pi \mathbb{Q}_{\widetilde{X}}[n]$, $\varphi_\pi \mathbb{Q}_{\widetilde{X}}[n]$ are perverse sheaves on X. Consider the perverse sheaf

$$\mathcal{C} := \mathrm{Image}(\widetilde{T} - 1) \subseteq \varphi_\pi \mathbb{Q}_{\widetilde{X}}[n], \tag{10}$$

and denote by

$$\iota_\varphi : \mathcal{C} \hookrightarrow \varphi_\pi \mathbb{Q}_{\widetilde{X}}[n]$$

the corresponding inclusion in the abelian category $Perv(X)$. Then $Supp(\mathcal{C}) = \{x\}$, and we have

$$\mathbb{H}^i(X; \mathcal{C}) = \begin{cases} 0 & , \text{ if } i \neq 0, \\ \mathrm{Image}(T_x - 1) & , \text{ if } i = 0. \end{cases} \tag{11}$$

Let

$$\iota := var \circ \iota_\varphi : \mathcal{C} \longrightarrow \psi_\pi \mathbb{Q}_{\widetilde{X}}[n].$$

In view of (8), (11) and the Betti calculation of Theorem 2.1, it is natural to define the intersection space complex by:

$$\mathcal{IS}_X := \mathrm{Coker}\left(\iota : \mathcal{C} \longrightarrow \psi_\pi \mathbb{Q}_{\widetilde{X}}[n]\right) \in Perv(X). \tag{12}$$

Remark 3.5. *If π is a small deformation of X, i.e., if the local monodromy operator T_x is trivial, then $\mathcal{C} \simeq 0$, so we get an isomorphism of perverse sheaves $\mathcal{IS}_X \simeq \psi_\pi \mathbb{Q}_{\widetilde{X}}[n]$. In view of the Betti identity of Theorem 2.1, this isomorphism can be interpreted as a sheaf-theoretical enhancement of the stability result from* [BM11] *mentioned in the Section 2.*

Remark 3.6. *The above construction can be easily adapted to the situation of hypersurfaces with multiple isolated singular points. It then follows from Theorem 3.1 and* [Ba11a]*[Prop.3.6] that the hypercohomology of $\mathcal{IS}_X$ for conifolds X provides the correct count of massless* 3*-branes in type IIB string theory.*

Some of the results described in Section 3.1 can be obtained as direct consequences of the definition of the intersection complex $\mathcal{IS}_X$. We describe some of these instances below. Others require more intricate proofs based on Saito's theory of (mixed) Hodge modules, or, alternatively, on the theory of zig-zags; see [BBM] for complete details on these proofs.

By using the facts stated in Section 3.2.1, it is not hard to see that the intersection space complex $\mathcal{IS}_X$ underlies a mixed Hodge module. More precisely, we have that:

$$\mathcal{IS}_X = \text{coker}\left(\text{Image}(\widetilde{N}) \xrightarrow{Var} {}^p\psi_{\pi,1}\mathbb{Q}_{\widetilde{X}}[n+1]\right) \tag{13}$$

and, as already mentioned, the functors $\widetilde{N}$, *Var* and ${}^p\psi_{\pi,1}$ admit lifts to the category of mixed Hodge modules.

It can also be seen that if T_x is semi-simple in the eigenvalue 1, then:

$$\mathcal{IS}_X \cong \psi_{\pi,1}\mathbb{Q}_{\widetilde{X}}[n].$$

So in this case $\mathcal{IS}_X$ is self-dual, since $\psi_\pi\mathbb{Q}_{\widetilde{X}}[n]$ is self-dual and $\mathcal{D}$ respects the decomposition $\psi_\pi = \psi_{\pi,1} \oplus \psi_{\pi,\neq 1}$. Moreover, in this case, the weight filtration on $\mathbb{H}^i(X, \mathcal{IS}_X)$ coincides (up to a shift) with the monodromy filtration defined by the nilpotent endomorphism N acting on $\mathbb{H}^i({}^p\psi_{\pi,1}) := \mathbb{H}^i(X; {}^p\psi_{\pi,1}\mathbb{Q}_{\widetilde{X}}[n+1])$. So the mixed Hodge structure on $\mathbb{H}^i(X, \mathcal{IS}_X) \cong \mathbb{H}^i({}^p\psi_{\pi,1})$ is pure if and only if $N = 0$, or equivalently if $T = T_s$ on $\mathbb{H}^i({}^p\psi_{\pi,1})$. In other words, one has purity if the action of T on $H^*(X_s)$ is semi-simple in the eigenvalue 1. Moreover, if this is the case, one can show as in [DMSS][Section 3] that the Hard Lefschetz theorem also holds for the hypercohomology groups $\mathbb{H}^i(X; \mathcal{IS}_X)$.

4. Concluding remarks

A natural problem is to extend the construction and study of intersection spaces of complex hypersurfaces beyond the case of isolated singularities. This problem is motivated by string theoretic considerations since, given the success of the use of intersection homology on the one hand and homology of the intersection space on the other hand in the context of the conifold transition, it is natural to investigate the use of such Poincaré duality homology theories in more singular situations encountered in string theory, e.g., for the fibre singularities in F-theory. This is particularly important as the non-uniqueness of the (small) resolutions of singular elliptic fibrations calls for a more model-independent procedure to determine the homology relevant for the physical theory. Recent progress in this direction has been recently made by Banagl and his students, e.g, see [Ba11b]. On the other hand, the sheaf-theoretic approach presented in this note is valid in more general settings, so it can be used to define an intersection space (co)homology theory directly, without having to construct an intersection space at all.

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SECTIONS, MULTISECTIONS, AND $U(1)$ FIELDS IN F-THEORY

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Abstract. We show that genus-one fibrations lacking a global section fit naturally into the geometric moduli space of Weierstrass models. Elliptic fibrations with multiple sections (nonzero Mordell-Weil rank), which give rise in F-theory to abelian $U(1)$ fields, arise as a subspace of the set of genus-one fibrations with multisections. Higgsing of certain matter multiplets charged under abelian gauge fields in the corresponding supergravity theories break the $U(1)$ gauge symmetry to a discrete gauge symmetry group. We demonstrate these results explicitly in the case of bisections, and describe the general framework for multisections of higher degree. We further show that nearly every $U(1)$ gauge symmetry arising in an F-theory model can be "unHiggsed" to an $SU(2)$ gauge symmetry with adjoint matter, though in certain situations this leads to a model in which a superconformal field theory is coupled to a conventional gauge and gravity theory. The only exceptions are cases in which the attempted unHiggsing leads to a boundary point at an infinite distance from the interior of the moduli space.

1. Introduction

F-theory [1, 2, 3] is a nonperturbative approach to string theory in which the axiodilaton $\tau = \chi + ie^{-\phi}$ of type IIB supergravity is specified by means of an auxiliary complex torus (elliptic curve), and 7-branes serve as sources for the RR scalar, providing an opportunity for $SL(2,\mathbb{Z})$-multivaluedness of the τ field. In most work to date, F-theory is compactified on a base B_n of complex dimension n, where the tori $\mathbb{C}/\langle 1, \tau(\xi_1, \dots, \xi_n)\rangle$ parameterized by coordinates ξ_i on the base are assumed to fit together to form a Calabi-Yau $(n+1)$-fold X_{n+1} that is elliptically fibered with section, $\pi : X_{n+1} \to B_n$, so that (after appropriate blowing down) X_{n+1} can be described by a Weierstrass model

$$y^2 = x^3 + fx + g\,, \tag{1}$$

where f, g are sections of line bundles $\mathcal{O}(-4K), \mathcal{O}(-6K)$ on the base B_n (locally described simply as functions of the base coordinates). The 7-branes are located at the *discriminant locus* $\{4f^3 + 27g^2 = 0\}$, in a manner specified by the Kodaira–Néron classification of singular fibers [4, 5].

Recently, Braun and Morrison [6] considered a more general class of F-theory compactification spaces, where the space X_{n+1} has a genus-one (torus) fibration, but no global section. They identified a large number of examples of such genus-one fibrations over the base $B_2 = \mathbb{P}^2$ in the comprehensive list compiled by Kreuzer and Skarke [7] of Calabi-Yau threefolds that are hypersurfaces in toric varieties. Any such X_{n+1} has a *Jacobian fibration* J_{n+1}, which is an elliptically fibered Calabi–Yau with section[1] whose τ function and discriminant locus are identical to those of X_{n+1}. The set of genus-one fibered Calabi–Yau manifolds with the same Jacobian fibration J_{n+1} is known as the Tate-Shafarevich group of J_{n+1}, denoted $Ш(J_{n+1})$, and is identified with

[1]This statement has been mathematically proven only for $n + 1 \leq 3$ [4, 8, 9], but is likely true in arbitrary dimension.

the discrete part of the gauge group of F-theory following [10, 11][2]. Note that $Ш(J_{n+1})$ represents not only a disjoint set of manifolds, but also includes an abelian group structure [12, 13]. Braun and Morrison identified in the examples they studied an apparent deficit in the number of scalar hypermultiplets required for gravitational anomaly cancellation, when the massless scalars are identified only as the complex structure moduli of the smooth genus one fibrations without global section. They resolved this apparent problem by identifying additional massless hypermultiplets at nodes in the discriminant locus of the Jacobian fibrations (more specifically, in the I_1 part of that locus).

While this analysis supports the proposition that genus-one fibrations without a global section are associated with consistent F-theory backgrounds, it also raises several questions, such as whether these backgrounds are connected to other F-theory geometries or form a disjoint component of the moduli space of the theory, and how the additional massless hypermultiplets should be interpreted. In this note, we show how these genus-one fibrations and their Jacobians fit naturally into the connected moduli space of Weierstrass models, and relate them to models with $U(1)$ gauge fields arising from extra sections of the elliptic fibrations. The structure of $U(1)$ gauge fields in F-theory is rather subtle, as they are determined by global features (the Mordell-Weil group) of an elliptic fibration; F-theory models with one or more $U(1)$ fields have been the subject of significant recent research activity (see for example [14, 15, 16, 18, 19, 20, 17, 22, 21, 23]).

In rough outline, the framework developed in this note is as follows: over any complex n-dimensional base B_n, there is a space $\mathcal{W}$ of Weierstrass models, parameterized by the sections f, g in (1). Any Calabi-Yau $(n+1)$-fold with a genus-one fibration has a multisection of some degree k, and its associated Jacobian fibration has a Weierstrass model which is generally singular when $k > 1$ (even in the absence of nonabelian gauge symmetry). We can map the set $\mathcal{M}_k$ of genus-one fibrations with a k-fold multisection (a "k-section", or when $k = 2$, a "bisection") to a subset $\mathcal{J}^k \subseteq \mathcal{W}$ of the set of Weierstrass models, consisting of the Jacobians of those genus-one fibrations. The set of elliptic fibrations with k independent global sections (rank $r = k - 1$ Mordell-Weil group) can also be viewed through singular Weierstrass models as a subset $\mathcal{S}_k \subseteq \mathcal{W}$ of the full space of elliptic fibrations. For Calabi-Yau threefolds, these results follow for any k from the result of Nakayama [24] and Grassi [25] that any elliptically fibered Calabi-Yau threefold with section has a realization as a Weierstrass model that is also Calabi-Yau; as in the case of the statements mentioned earlier concerning Jacobian fibrations of Calabi-Yau genus-one fibrations, this statement has not been mathematically proven for Calabi-Yau fourfolds, but there are no known examples to the contrary. Furthermore, we have

$$\mathcal{S}_k \subseteq \mathcal{J}^k \subseteq \mathcal{W}\,, \tag{2}$$

meaning that the set of models with k independent sections can be viewed as a subset of the larger set of models with a k-fold multisection.

We give explicit formulae describing these inclusions in the case $k = 2$ in the next section, but the inclusion $\mathcal{S}_k \subseteq \mathcal{J}^k$ clearly holds for any k since having k independent sections is a special case of having a k-fold multisection where the k sections can be given distinct global labels. In particular, we can think of the multisection of an $(n+1)$-fold $X_{n+1} \in \mathcal{J}^k$ as a branched cover of the base; the multisection breaks into k distinct global sections on a subspace of moduli space where the branch points coalesce in such a way as to give trivial monodromy among the branches. In this picture, going from a model in $\mathcal{S}_k$ to a model in $\mathcal{J}^k$ can be interpreted physically as a partial Higgsing, where Higgsing some charged matter fields breaks $U(1)^{k-1}$ to a discrete subgroup, under which the remaining fields parameterizing $\mathcal{J}^k$ carry discrete charges. In the

[2]The discrete part of a gauge group corresponds to the set of connected components of the group; a purely discrete gauge group is a finite group such as $\mathbb{Z}_n$

case $k = 2$, for example, we can have matter fields with various integer-valued $U(1)$ charges; if we Higgs matter fields with charge Q, we break $U(1)$ to $\mathbb{Z}_Q$.

In the case $k = 2$, we can also further analyze any model containing a $U(1)$ by considering the explicit form of a Weierstrass model with nonzero Mordell-Weil rank. From this point of view we can demonstrate that every $U(1)$ is associated geometrically with a nonabelian $SU(2)$ (or larger) symmetry arising from a Kodaira type I_2 singularity along a divisor on the base. Starting with such an $SU(2)$ having both adjoint and fundamental matter, there are several possible Higgsing steps: the first leaves us with a $U(1)$ under which the remnant of the adjoint matter has charge[3] 2 and the remnant of the fundamental matter has charge 1; the second Higgsing (of matter fields of charge 2) leaves us with gauge group $\mathbb{Z}_2$ under which the remnant of the original fundamental matter is charged; a final Higgsing of the fields originally carrying charge 1 breaks the residual discrete gauge group and moves the model out of $\mathcal{J}^k$ and into the moduli space $\mathcal{W}$ of generic Weierstrass models.

In §2 we describe the general framework for this geometrical picture explicitly in the case $k = 2$, for a general base manifold B_n. In §3, we show explicitly in 6D how any $U(1)$ gauge field in an F-theory model can be associated with an $SU(2)$ gauge group that has been Higgsed by an adjoint matter field, and we look at several explicit examples. §4 contains some general remarks about the implications of this picture for 6D and 4D supergravity theories, and some comments on further directions for related research.

2. General framework

2.1. Calabi-Yau manifolds with bisections and with two different sections. In [6], an exercise in Galois theory provides an equation for the Jacobian of a genus-one fibration with a bisection

$$y^2 = x^3 - e_2x^2z^2 + (e_1e_3 - 4e_0e_4)xz^4 - (e_1^2e_4 + e_0e_3^2 - 4e_0e_2e_4)z^6\,, \tag{3}$$

where $e_0, \ldots, e_4$ are sections of various line bundles over the base B_n (to be determined below). Completing the cube, changing variables, and setting $z = 1$ puts this in Weierstrass form

$$y^2 = x^3 + \big(e_1e_3 - \frac{1}{3}e_2^2 - 4e_0e_4\big)x + \big(- e_0e_3^2 + \frac{1}{3}e_1e_2e_3 - \frac{2}{27}e_2^3 + \frac{8}{3}e_0e_2e_4 - e_1^2e_4\big)\,. \tag{4}$$

This parameterizes the set of all Jacobians of genus-one fibrations over B_n with bisections, represented through the Weierstrass models (of the Jacobian fibrations). In particular, this describes how $\mathcal{J}^2 \subseteq \mathcal{W}$ for any base B_n.

This class of Weierstrass models is closely related to the Weierstrass form for elliptically fibered Calabi-Yau $(n+1)$-folds on B_n with two (different) sections. Elliptically fibered Calabi-Yau manifolds with two sections can be described as models with a non-Weierstrass presentation (like the E_7 models of [26, 27]) that are smooth for generic moduli. All such $(n+1)$-folds, however, also have a (possibly singular) description as Weierstrass models. In [16], the general form of such a Weierstrass model was given as[4]

$$y^2 = x^3 + \big(e_1e_3 - \frac{1}{3}e_2^2 - b^2e_0\big)x + \big(- e_0e_3^2 + \frac{1}{3}e_1e_2e_3 - \frac{2}{27}e_2^3 + \frac{2}{3}b^2e_0e_2 - \frac{1}{4}b^2e_1^2\big)\,. \tag{5}$$

[3] A field is said to have "charge n" under a $U(1)$ gauge symmetry if it transforms as $e^{in\theta}$ under a gauge transformation $e^{i\theta} \in U(1)$.

[4] We have modified eq. (5.35) of [16] by using a scaling $(f, g) \mapsto (i^4f, i^6g)$ to change the sign of g, and by changing c_j in that paper to e_j here $(j = 0, 1, 2, 3)$.

Note that this equation is equivalent to (4) under the replacement $b^2 \to 4e_4$. The interpretation of this analysis is that, as stated in the introduction,

$$\mathcal{S}_2 \subseteq \mathcal{J}^2 \subseteq \mathcal{W}\,, \tag{6}$$

The condition $e_4 = b^2/4$ is precisely the condition that the branching loci of the bisection associated with a genus-one fibration in $\mathcal{J}^k$ coalesce in pairs so that the total structure is that of an elliptic fibration with two sections.

In [6, 16], it was shown that for both an elliptic fibration with two sections, and for a genus-one fibration with a bisection, there is a natural model with a quartic equation of the general form

$$w^2 = e_0 u^4 + e_1 u^3 v + e_2 u^2 v^2 + e_3 u v^3 + e_4 v^4\,. \tag{7}$$

If $e_4 = b^2/4$, the equation can be rewritten

$$\big(w + \frac{1}{2} b v^2\big)\big(w - \frac{1}{2} b v^2\big) = u(e_0 u^3 + e_1 u^2 v + e_2 u v^2 + e_3 v^3)\,, \tag{8}$$

which makes the two sections manifest: they are given by $u = w \pm \frac{1}{2} b v^2 = 0$. In general, when there are two sections one might need to make a linear redefinition of the variables u, v before (7) can be rewritten in the form (8), but after such a linear redefinition it can always be done.

From the condition that f, g in (1) are sections of the line bundles associated with $-4K, -6K$, we can characterize the line bundles of which the e_i and b are sections. Focusing on the e_i's, we have

$$-4K = 2[e_2] = [e_1] + [e_3] = [e_0] + [e_4]\,, \tag{9}$$

$$-6K = 2[e_1] + [e_4] = [e_0] + 2[e_3]\,. \tag{10}$$

From $2[e_2] = -4K$, we have $[e_2] = -2K$. We also note that $[e_0] = -6K - 2[e_3]$ must be an even divisor class. Choosing $[e_0] \equiv 2L$, with L the class of an arbitrary line bundle, we have

$$[e_0] = 2L \tag{11}$$

$$[e_1] = -K + L \tag{12}$$

$$[e_2] = -2K \tag{13}$$

$$[e_3] = -3K - L \tag{14}$$

$$[e_4] = -4K - 2L \tag{15}$$

$$[b] = -2K - L\,. \tag{16}$$

For any given base, L can be chosen subject to the conditions that $[e_1], [e_3]$ are effective divisors (if this condition is not satisfied, then the only non-vanishing terms in the Weierstrass model are those proportional to powers of e_2, and the discriminant vanishes identically). This constrains the range of possibilities to a finite set of possible strata in the moduli space. The consequences when $[e_4]$ and/or $[e_0]$ fail to be effective are discussed in §2.4.

This analysis shows that for any Calabi-Yau manifold X_{n+1} that is a genus-one fibration lacking a global section but having a bisection, there is a Jacobian fibration J_{n+1}, which has a description as a Weierstrass model through (4). Taking the limit $e_4 \to b^2/4$ gives a Weierstrass model for an elliptically fibered Calabi-Yau $(n+1)$-fold with two sections, which therefore has a Mordell-Weil group of nonzero rank. In terms of the physical language of F-theory, as we describe in more detail in the following sections, this corresponds to the reverse of a process in which a $U(1)$ gauge symmetry is broken by matter fields of charge 2, leaving a discrete $\mathbb{Z}_2$ symmetry. In §3 we describe several explicit examples of this setup in 6D F-theory constructions.

2.2. **Singular fibers of type I_2 in codimension two.** One of the key features of the quartic models is the presence of singular fibers in codimension two of Kodaira type I_2, observed in [16] in the $U(1)$ case, and in [6] in the bisection case. When there is a $U(1)$, these singular fibers determine matter hypermultiplets that are charged under the $U(1)$, and there can be different charges: [16] focussed on the case when the charges are 1 and 2 only and found distinct geometrical interpretations for each of these. The geometric construction of I_2 fibers of charge 1 under $U(1)$ extends to the case of a bisection (in the deformation from $\mathcal{S}_2$ to $\mathcal{J}^2$), as we will now show explicitly. As explained above, the corresponding matter fields will be charged under the discrete $\mathbb{Z}_2$ gauge symmetry. Both the bisection and $U(1)$ cases have a description in terms of the quartic model (7). We begin by considering the I_2 fibers in the genus one (bisection) case where e_4 is generic, and then consider the limit where $e_4 = b^2$ is a perfect square, corresponding to the $U(1)$ model.

The curves of genus one in the quartic model are double covers of $\mathbb{P}^1$ branched in 4 points, as illustrated in the left half of Figure 1. When the 4 branch points come together in pairs, the resulting double cover splits into two curves of genus zero meeting in those two double branch points, as illustrated in the right half of Figure 1. Such fibers in the family have type I_2 in the Kodaira classification.

Thus, to find such a fiber of type I_2 in the quartic model, we seek points on the base B_n for which the right-hand side of the equation (7) takes the form of a perfect square. As we explain in appendix A, we can assume that e_4 does not vanish at such points on the base (if the model is sufficiently generic) and so we write our condition in the form

$$e_0u^4 + e_1u^3v + e_2u^2v^2 + e_3uv^3 + e_4v^4 = e_4(\alpha u^2 + \beta uv + v^2)^2\,, \tag{17}$$

for some unknown α and β. Multiplying out and equating coefficients, it is easy to solve

$$\beta = e_3/2e_4, \quad \alpha = (4e_2e_4 - e_3^2)/8e_4^2$$

and then determine the remaining conditions, which are:

$$e_3^4 - 8e_2e_3^2e_4 + 16e_2^2e_4^2 - 64e_0e_4^3 = 0 \tag{18}$$

$$e_3^3 - 4e_2e_3e_4 + 8e_1e_4^2 = 0 \tag{19}$$

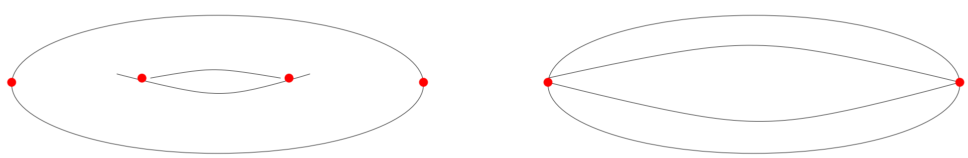

FIGURE 1. Fiber of type I_2 as a degenerate branched cover

To study the solutions of these equations, we introduce an auxiliary variable p and rewrite the equations as

$$p^4 - 8e_2e_4p^2 + 16e_2^2e_4^2 - 64e_0e_4^3 = 0 \tag{20}$$

$$p^3 - 4e_2e_4p + 8e_1e_4^2 = 0 \tag{21}$$

In appendix A we explain how to determine the condition

$$(4e_0e_2 - e_1^2)^2 = 64e_0^3e_4 \tag{22}$$

for these equations to have a common root, and why that root is

$$p = \frac{8e_0e_1e_4}{4e_0e_2 - e_1^2} \tag{23}$$

when all of the coefficient functions $e_0, \ldots, e_4$ are generic. The points we seek can be described as solutions to (22) which also satisfy $e_3 = p$.

We now show how to count the solutions (i.e., the number of I_2 fibers of this type), modifying an argument from [16]. Let us take a limit, replacing e_4 with $\epsilon^2 e_4$ and then taking ϵ very small (with both e_0 and e_1 of order 1). Condition (22) then shows that $4e_0e_2 - e_1^2$ has order ϵ, and (23) shows that p has order $\epsilon^2/\epsilon = \epsilon$. It follows that any simultaneous solution to (22) and $e_3 = p$ can be deformed to a simultaneous solution to (22) and $e_3 = 0$. That is, the isolated I_2 fibers are in one-to-one correspondence with the set

$$\{e_1^4 - 8e_0e_1^2e_2 + 16e_0^2e_2^2 - 64e_0^3e_4 = 0\} \cap \{e_3 = 0\}\,. \tag{24}$$

It follows that the number of I_2 fibers is

$$[4e_1] \cdot [e_3] = 4(-K + L) \cdot (-3K - L)\,, \tag{25}$$

since $e_1^4 - 8e_0e_1^2e_2 + 16e_0^2e_2^2 - 64e_0^3e_4$ is in class $[4e_1]$.

When $e_4 = b^2/4$ so that we have a $U(1)$, the analysis above reproduces the count of I_2 fibers found in [16] which correspond to matter of charge 1 under the $U(1)$ gauge group. It was also observed there (and will be mentioned again below) that when $U(1)$ is further enhanced to $SU(2)$, this matter comes from matter in the fundamental representation of $SU(2)$.

On the other hand, the description of the matter of charge 2 in [16] is a bit different: it occurs where b and e_3 both vanish, and from (8) and (3) we see that both the quartic model and the Jacobian fibration have conifold singularities over each common zero of b and e_3. When we partially Higgs by relaxing the condition $e_4 = b^2/4$, we do a complex structure deformation of that conifold singularity, giving a mass to the gauge field (as is standard in a conifold transition [28]).[5] It would be interesting to find a more geometric interpretation of this massive gauge field, perhaps along the lines of [34, 35].[6]

The Weierstrass model of the Jacobian fibration also has a conifold singularity corresponding to each I_2. For models with two sections, these conifold singularities have a (simultaneous) small resolution, as shown explicitly in [16] by blowing up the second section in the Weierstrass model. However, for Jacobians of models with a bisection, the conifold singularities (i.e., the deformations of those singularities whose corresponding hypermultiplet had charge 1 before Higgsing) have no Calabi–Yau resolution, which led to the question raised in [6] of whether these are genuinely new F-theory models.

2.3. **Generalizations and geometry.** In principle, our explicit analysis of bisections could be extended to the spaces $\mathcal{J}^k$ of Jacobian fibrations associated with genus-one fibered Calabi-Yau manifolds with k-sections and $\mathcal{S}_k$ of elliptically fibered Calabi-Yau manifolds with rank $r = k-1$ Mordell-Weil group in a similar explicit fashion, at least for $k \leq 4$. Explicit formulae for $\mathcal{S}_3, \mathcal{S}_4$, the generic forms of elliptic fibrations with three and four sections respectively, were worked out in [20, 21] and [23], and the analogous formulae for $\mathcal{J}^k$ are known [37] (although unwieldy to manipulate). For $k = 3, 4$, the points in $\mathcal{S}_k$ correspond to singular Weierstrass presentations of Calabi-Yau $(n+1)$-folds with 3, 4 independent sections, which have smooth descriptions similar to the E_6 and D_4 fibrations of [26, 27].

[5] The distinction between conifold singularities which admit a Kähler small resolution and those which do not, and the relation to massive gauge fields, has appeared a number of times in the literature [29, 30, 31, 32, 33].

[6] We thank Volker Braun for emphasizing the crucial role which must be played by massive gauge fields in these models [36].

Even without an explicit description of the general form of a Jacobian fibration with a k-section, it is clear that the framework described in the previous section should generalize. In particular, we expect that any Jacobian fibration J_{n+1} with a multisection will have a discrete gauge group Γ in the corresponding F-theory picture, and that this will match the Tate-Shafarevich group $\Gamma = Ш(J_{n+1})$. There is a simple and natural geometric interpretation of this structure in the M-theory picture. When an F-theory model on J_{n+1} is compactified on a circle S^1, it gives a 5D supergravity theory that can also be described by a compactification of M-theory on a Calabi-Yau $(n+1)$-fold Y_{n+1}. When there is a discrete gauge group Γ in the 6D F-theory model, a nontrivial gauge transformation (Wilson line) around the complex direction gives a set of $|\Gamma|$ distinct 5D vacua associated with J_{n+1}. In the M-theory picture this corresponds precisely to the compactification on the set of distinct genus-one fibered Calabi-Yau manifolds in the Tate-Shafarevich group $Ш(J_{n+1})$.

We can get a clear picture of the meaning of the multiple Calabi-Yau manifolds with the same Jacobian fibration by considering the moduli space for the compactified theory on a circle, which can be analyzed using M-theory. We illustrate this in Figure 2, in which the moduli space $\mathcal{W}$ of Weierstrass models (shown in blue) contains the subset $\mathcal{J}^2$ of Jacobians of models with a bisection, and this in turn contains the subset $\mathcal{S}_2$ of Jacobians of models with two sections. When there are two sections, the second Betti number of the Calabi-Yau increases and there is an additional dimension in the Kähler moduli space, which becomes a modulus in the compactified theory. We have illustrated this extra dimension as a red line emerging from the $\mathcal{S}_2$.

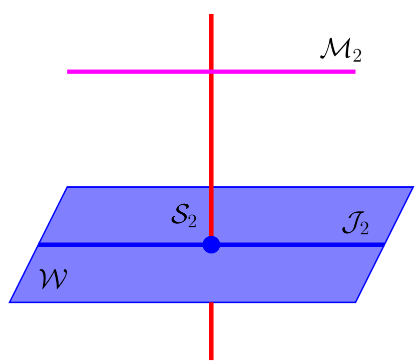

FIGURE 2. Moduli spaces for M-theory compactifications on Calabi-Yau threefolds with different structures of sections (described in text).

What initally seems puzzling is that while the Weierstrass models of Jacobians of genus one fibrations with two sections deform seamlessly to Jacobians of genus one fibrations with bisections (by relaxing the condition that e_4 be a square), and similarly the nonsingular fibrations with two sections deform seamless to genus one fibrations with a bisection, the conifold singularities in the Weierstrass model cannot be resolved in the bisection case. The key to understanding this is to remember that the extra divisor that is present when there are two sections (i.e., a $U(1)$) allows an additional Kähler degree of freedom which in particular allows us to specify the areas of the two components of an I_2 fiber independently. On the other hand, when there is only a bisection, the homology classes of those two components must each be one-half of the homology class of a smooth genus-one fiber; thus, the two components must have the same *area*.

The picture of the M-theory moduli space is thus completed by adding a new component $\mathcal{M}_2$ of smooth genus-one fibrations with a bisection, illustrated in purple in Figure 2. The new component must emerge from the precise value of the additional Kähler classes (red line) at which the two components of an I_2 fiber in the $U(1)$ case have an identical area. (Generally, the red line can be viewed as parameterizing the difference of those areas.) The additional Kähler class in a $U(1)$ thus provides the connection between the Weierstrass models (in which the area of one of the two components is zero, corresponding to the conifold point without a Calabi-Yau resolution) and the bisection models (in which the areas of the two components are equal).

Let us reiterate the crucial point: away from the locus $\mathcal{S}_2$, the complex structures on the Calabi-Yau manifolds represented by the spaces $\mathcal{J}^2$ and $\mathcal{M}_2$ are different (and not even birational to each other), and are only related by the "Jacobian fibration" contruction. However, they determine the same underlying τ function, so the F-theory models are identical. Compactifying on a circle produces two distinct geometries for M-theory models, which is precisely what one expects for a discrete gauge symmetry. Moreover, the "extra" hypermultiplets have different but consistent explanations on the two components of the M-theory moduli space. Along $\mathcal{M}_2$, they are seen as geometric I_2 fibers being wrapped by M2-branes, which were argued to have no continuous gauge charges in [6] (although we now see that they carry $\mathbb{Z}_2$ gauge charges). Along $\mathcal{J}^2$, these same hypermultiplets are seen as complex structure moduli transverse to the $\mathcal{J}^2$ locus (moduli which are absent in $\mathcal{M}_2$).

One of the important lessons that we learn from this picture is that it is important not to discard an F-theory model just because all of the corresponding M-theory models after S^1-compactification are singular. The lack of a nonsingular model means that the M-theory compactification cannot be studied in the supergravity approximation without some additional input to its structure, but such models must be included for a consistent overall picture of the moduli spaces.

2.4. **Enhancement to $SU(2)$.** For any elliptically fibered threefold with nonzero Mordell-Weil rank, we can carry the analysis of §2.1 further, and show that there is a limit in which an extra section in the Mordell-Weil group transforms into a "vertical" divisor class lying over a point in the base B_n. In the F-theory language this corresponds to an enhancement of the $U(1)$ gauge symmetry into a nonabelian gauge group with an $\mathfrak{su}_2$ gauge algebra (or in some special cases, a rank one enhancement of a larger nonabelian gauge group). At least at the level of geometry, this shows that any $U(1)$ gauge group factor in an F-theory construction can be found from the breaking of a nonabelian group containing an $SU(2)$ subgroup by Higgsing a field in the adjoint representation [38]. This fits into a very simple and general story associated with the Weierstrass form (4). Examples of situations where $U(1)$'s can be "unHiggsed" in this fashion were described in [16, 31]. In most situations the unHiggsed model with a vertical divisor is non-singular, though as we show explicitly in the following section, in some cases a singularity is present which can be interpreted either as a coupled superconformal theory, or as an indication that the unHiggsed model is at infinite distance from the interior of moduli.

If the classes associated with the coefficients e_0, e_4 in (4) are both effective, then all coefficients $e_0, \ldots, e_4$ can generically be chosen to be nonzero, and we have a family of Weierstrass models that characterize Jacobian fibrations with a bisection, as discussed in §2.1. Let us consider what happens when e_0 and/or e_4 factorize or vanish either by tuning or because the associated divisors are not effective (in which case these coefficients would automatically vanish since there would be no sections of the associated line bundles).

As described in §2.1, if $e_4 = b^2/4$ is a perfect square, then the bisection becomes a pair of global sections, and the Mordell-Weil rank of the Jacobian fibration rises, which in the F-theory picture corresponds to the appearance of a $U(1)$ gauge factor. The equation is symmetric under $e_i \to e_{4-i}$, however, so we can also take $e_0 = a^2/4$ to produce a global section in another way.

In [16] it was observed that the zeros of b correspond to the intersection points of the two sections, and that the further tuning $b \to 0$, which naïvely would place the two sections on top of each other, in fact leads to a gauge symmetry enhancement to $SU(2)$. We can see this enhancement explicitly by choosing $e_4 = b^2/4 = 0$ so that the structure simplifies further, and the equation of the discriminant factorizes into the form

$$\Delta := 4f^3 + 27g^2 = e_3^2(-18e_0e_1e_2e_3 + 4e_1^3e_3 - e_1^2e_2^2 + 4e_0e_2^3 + 27e_0^2e_3^2)\,. \tag{26}$$

Since neither f nor g are generically divisible by e_3, this corresponds to a family of singular fibers[7] of Kodaira type I_2 along the divisor $\{e_3 = 0\}$, associated with an $\mathfrak{su}_2$ Lie algebra component. (In some special cases the $\mathfrak{su}_2$ can be part of a larger nonabelian algebra, but we focus here on the generic $\mathfrak{su}_2$ case for simplicity.) In the F-theory picture the transition from the model with $b = 0$ to the $U(1)$ model with $b \neq 0$ is described by the Higgsing of an $SU(2)$ gauge group by a matter field in the adjoint representation.

Again, because the equation is symmetric, we can tune $e_0 = a^2/4 = 0$ in a similar fashion, giving a second I_2 singularity on the divisor $\{e_1 = 0\}$. In the F-theory picture this gives a second nonabelian gauge group factor with an $\mathfrak{su}_2$ algebra.

This gives a very generic picture in which, when the divisor classes $-K+L$ and $-3K-L$ are effective, we have a class of models with two A_1 Kodaira singularities on the divisors e_1, e_3. This corresponds in the F-theory picture to a theory with gauge algebra $\mathfrak{su}_2 \oplus \mathfrak{su}_2$. When the divisor classes L and $-4K - 2L$ are effective we can turn on terms $e_0 = a^2/4$ and/or $e_4 = b^2/4$ that turn the "vertical" A_1 Kodaira singularities into global sections (without changing $h^{1,1}(X_{n+1})$); this corresponds in F-theory to Higgsing one or both of the nonabelian gauge groups through an adjoint representation to the $U(1)$ Cartan generator. When $L, -4K - 2L$ are nonzero classes, we can choose e_0 and/or e_4 to be generically nonzero and non-square, which further breaks the $U(1)$'s to a discrete $\mathbb{Z}_2$ symmetry. Because the discrete $\mathbb{Z}_2$ symmetry in the generic bisection model (4) is naturally identified with the center of both $U(1)$ fields, we expect only one $\mathbb{Z}_2$ in the center of the original nonabelian gauge group.

This can be seen geometrically by analyzing the charged matter under each of the $SU(2)$ factors, following [39, 40]. For any F-theory model, the "virtual" or "index" spectrum of massless matter multiplets minus massless vector multiplets can be described in terms of an algebraic cycle of codimension 2 on the base B_n, to each component of which is associated a representation of the gauge group. (For 6D compactifications, one then just counts points in the 2-cycle to determine the multiplicity of the representation, but for 4D compactifications there is an additional quantization which must be performed on each component of the 2-cycle to determine the multiplicity [41, 42], which may depend on the G-flux; what is fixed by the geometry is the set of representations which can appear in the spectrum.) For an I_2 fiber located along a divisor Σ, the virtual adjoint representation in the matter spectrum is associated to the algebraic cycle $\Sigma \cdot \frac{1}{2}(K + \Sigma)$, while the fundamental representation is associated to the cycle[8] $\Sigma \cdot (-8K - 2\Sigma)$. Since we have bifundamental matter at the intersection of $[e_1]$ and $[e_3]$, which counts as $2(-K+L)\cdot(-3K-L)$ fundamentals for each of the $SU(2)$ factors, these bifundamentals account for all of the fundamental matter.[9] Neither adjoints nor bifundamentals tranform nontrivially under the diagonal $\mathbb{Z}_2$ in the combined gauge group. Since in the M-theory picture

[7]Note that when I_2 fibers occur in codimension one on the base, we associate them to $\mathfrak{su}_2$, but when they occur in codimension two on the base, they are responsible for (charged) matter only and no additional gauge symmetry.

[8]More precisely, these cycles are determined by the intersections with $\{f = 0\}$ and $\{g = 0\}$ in the Weierstrass model, as described in [40].

[9]Here we are using the fact that when $\Sigma = -K+L$, we have $-8K-2\Sigma = 2(-3K-L)$ and when $\Sigma' = -3K-L$, we have $-8K - 2\Sigma' = 2(-K + L)$.

the set of divisors must be dual to the set of curves in the Calabi-Yau, the diagonal $\mathbb{Z}_2$ is not part of the gauge group unless some field (associated with a curve in the resolved threefold) transforms under it [43, 44]. Thus, the full gauge group in the model with $e_0 = e_4 = 0$ will be $(SU(2) \times SU(2))/\mathbb{Z}_2$ where the discrete quotient is taken by the diagonal $\mathbb{Z}_2$. Note that if either $[e_0]$ or $[e_4]$ is not effective then the corresponding A_1 (on $[e_1]$ or $[e_3]$) cannot be deformed away in the Weierstrass model while preserving the element of $h^{1,1}(X)$ in the form of a section; in the F-theory picture this corresponds to an $SU(2)$ that does not have massless matter in the adjoint representation.

We can confirm this analysis by exhibiting an explicit element of the Mordell-Weil group of order 2, as predicted by [45] (see also [44]). Namely, when $e_0 = e_4 = 0$ the Weierstrass equation (4) takes the form

$$y^2 = x^3 + \big(-\frac{1}{3}e_2^2 + e_1e_3\big)x + \big(\frac{1}{3}e_1e_2e_3 - \frac{2}{27}e_2^3\big) = \big(x + \frac{1}{3}e_2\big)\big(x^2 - \frac{1}{3}e_2x - \frac{2}{9}e_2^2 + e_1e_3\big)\,. \tag{27}$$

The factorization of the right side of the equation corresponds to a point of order two on each elliptic fiber, and since this factorization is uniform over the base, the locus $\{x = -\frac{1}{3}e_2, y = 0\}$ defines a section which has order two in the Mordell-Weil group. Note that either on the locus $e_1 = 0$ or on the locus $e_3 = 0$, the Weierstrass equation (27) takes the form

$$y^2 = x^3 + \big(-\frac{1}{3}e_2^2 + e_1e_3\big)x + \big(\frac{1}{3}e_1e_2e_3 - \frac{2}{27}e_2^3\big) = \big(x + \frac{1}{3}e_2\big)^2\big(x - \frac{2}{3}e_2\big)\,, \tag{28}$$

showing that the A_1 singularity of the singular fiber is located at $\{x = -\frac{1}{3}e_2, y = 0\}$ in each case, i.e., exactly at the section of order two. This implies that the $\mathbb{Z}_2$ quotient is nontrivial on each $SU(2)$, and thus that the global structure of the group must be $(SU(2) \times SU(2))/\mathbb{Z}_2$ using the diagonal $\mathbb{Z}_2$.

One additional complication that can arise in this picture is when $-4K$ contains a divisor A as an irreducible effective component. In this case, there may be an automatic vanishing of f, g over A giving a nonabelian gauge group, such as in 6D for the non-Higgsable clusters of [46]. In this case, this component must be subtracted out from $-4K$ in computing the complementary divisors on which the $SU(2)$ factors reside, and some of the matter fields may transform under the gauge group living on A as well as one of the $SU(2)$ factors. We describe this mechanism further in the 6D context in the following section.

The upshot of this analysis is that for any elliptically fibered Calabi-Yau manifold with a nonzero rank Mordell-Weil group, a global section can be associated with a divisor class $D = [e_3]$ to which the section can be moved as an A_1 (or higher) Kodaira type singularity. In the language of F-theory geometry (without considering effects such as G-flux relevant in four dimensions, §4.2) this means that any $U(1)$ gauge symmetry can be seen as arising from a broken nonabelian symmetry. Furthermore, there is an intriguing structure in which for each such divisor class D there is a complementary divisor class

$$D' = [e_1] = -4K - D\,, \tag{29}$$

that can (and in some cases must) also be tuned to support an A_1 singularity, which may be associated with a second independent section. In situations where $-4K$ has a base locus over which f, g have enforced vanishing associated with Kodaira singularities giving nontrivial gauge groups, the base locus must also be subtracted out in (29). In the next section we give several explicit examples of how this works in some 6D models.

3. 6D EXAMPLES

The arguments given up to this point have been very general, and in principle apply to elliptically fibered Calabi-Yau manifolds in all dimensions where a suitable Weierstrass model is available. In this section we consider some simple explicit examples of 6D F-theory models to illustrate some of the general points. We begin by describing explicitly the way in which any $U(1)$ in 6D can be seen as arising from an $SU(2)$ factor that has been Higgsed by turning on a vacuum expectation value for an adjoint hypermultiplets. We then describe some general aspects of models with bisections in this context, and conclude by explicitly analyzing various possible ways in which the "unHiggsing" to $SU(2)$ may encounter problems with singularities. While such singularities do not arise in most cases, we identify one situation where such a singularity arises, which can only be removed by blowing up the F-theory base manifold.

Before beginning, let us recall how the various moduli spaces of Weierstrass models are linked together through transitions involving coupling to 6D superconformal field theories [47], sometimes called "tensionless string transitions" [48, 49]. As we tune the coefficients of a Weierstrass model over a fixed base B_2, various singularities are encountered that have explanations in terms of nonabelian gauge symmetry or the massless matter spectrum. However, if a singular point P is encountered at which f has multiplicity at least 4, and g has multiplicity at least 6, the model has a superconformal field theory sector and another branch emerges in which a tensor multiplet is activated [3, 50]. The other branch consists of Weierstrass models over the blowup $\mathrm{Bl}_P(B_2)$ of B_2 at P, and the area of the exceptional curve of the blowup serves as the expectation value of the scalar in the new tensor multiplet. We generally refer to such points as "$(4, 6)$ points."

Even more special is the case in which either (f, g) have multiplicities at least $(8, 12)$ at a point, or have multiplicities of at least $(4, 6)$ along a curve. In this case, the total space of the fibration is not Calabi-Yau, and in fact any resolution of the space in algebraic geometry has no nonzero holomorphic 3-forms. It is known that the points in the moduli space of Weierstrass models at which such singularities occur are boundary points of moduli at infinite distance from the interior of the moduli space [51, 52].

3.1. $U(1)$ **from a Higgsed** $SU(2)$ **in 6D.** In six dimensions, we can demonstrate explicitly that in most situations a $U(1)$ can be enhanced to an $SU(2)$ in a conventional F-theory model on the same base (i.e., one not involving a superconformal theory or at infinite distance from the interior of the moduli space) by considering general classes of acceptable $U(1)$ model in which tuning $b^2 \to 0$ in (5) need not introduce a $(4, 6)$ point. We can also identify some situations in which this tuning does necessarily lead to such a singularity. A forced $(4, 6)$ point can in principle occur in one of two ways: first, if $[e_3]$ contains a curve of negative self intersection over which f, g are required to vanish to high degree, and second if $[e_3]$ has nonzero intersection with another curve C or combination of curves $A, B, \ldots$ over which f, g vanish to high enough degree to force a $(4, 6)$ vanishing at an intersection point. We outline the general structure of the analysis here, and describe some special cases in the later parts of this section.

First, let us consider the case where C is a curve in the class $[e_3]$, C is irreducible, and $[e_1] = -4K - [e_3]$ does not contain as irreducible components any curves of self-intersection below -2. We consider the Weierstrass model of the form (5), and take the limit as $b^2 \to 0$, which produces an $SU(2)$ over C with matter in the adjoint representation. For the enhancement to $SU(2)$ with an adjoint or higher representation to occur on a curve C, the curve must have genus $g > 0$. This follows from the general result [53] that every representation of $SU(N)$ with a Young diagram having more than one column makes a positive contribution to the genus of the curve through the anomaly equations. It was shown in [46] that a curve of positive genus cannot have negative self-intersection without forcing a $(4, 6)$ vanishing all along the curve. So $C \cdot C \geq 0$ and f, g cannot be required to vanish on the irreducible curve for a generic Weierstrass model over the given base. To see where there are enhanced singularities at points on C in the $SU(2)$ model, we can use the 6D anomaly cancellation conditions [54, 55, 56, 57, 58, 59]. For a generic curve C in the class $[e_3]$, where $SU(2)$ matter is only in A hypermultiplets that transform under the adjoint (symmetric) representation and x fundamental hypermultiplets, the anomaly conditions read

$$K \cdot C = \frac{1}{6}\left[4(1-A) - x\right] \tag{30}$$

$$C \cdot C = -\frac{1}{3}\left[8(1-A) - x/2\right] . \tag{31}$$

Solving these equations gives

$$A = g = 1 + \frac{1}{2}(K \cdot C + C \cdot C) \tag{32}$$

and

$$x = 2C \cdot (-4K - C) = 2[e_3] \cdot [e_1] . \tag{33}$$

When $e_0 = a^2/4 = 0$ and there are $SU(2)$ gauge factors supported on both e_1 and e_3, this shows that all matter fields – and hence all enhanced singularities – arise at the intersection points between these two curves. Note that this is the same conclusion about the matter spectrum that we reached in §2.4 in a different way. Note also that the location of the singularities associated with the matter charged under the $SU(2)$ on C is the same whether or not we take the $a \to 0$ limit, and that this matter corresponds to the extra charged matter fields found on the I_2 locus in §2.2.

Additional complications can arise if e_1 or e_3 are reducible, particularly when either or both contain irreducible factors that carry nontrivial Kodaira singularities. In such cases, f, g will vanish on the associated curve A, with an extra nonabelian gauge group factor, according to the classification of non-Higgsable clusters in [46]. To show when a $U(1)$ that arises in a Weierstrass form (5) can be associated with a broken $SU(2)$ in a conventional F-theory model without changing the base, we need to prove that in these cases a $(4, 6)$ point cannot be introduced by taking the $b \to 0$ limit. There can also be more complicated singularities introduced if the curve C is not a generic curve in the class $[e_3]$ and itself has singularities. There is not yet a complete dictionary relating codimension two singularities of this type to matter representations, though there has been some recent progress in this direction [60, 61, 62]. We do not consider such cases here in any detail, though an example is discussed in §4.1; here we assume that the curve C is taken to be generic in the class $[e_3]$, so the statement that a $U(1)$ can be viewed as a Higgsing of an $SU(2)$ model should be understood as involving the Higgsing of an $SU(2)$ model with a generic C given $[e_3]$, with further tuning of C carried out as necessary to achieve the given $U(1)$ model of the form (5).

If $[e_3]$ intersects a curve A in $[e_1]$ that carries a nonabelian gauge group G_A (again, assuming A is a generic curve in its class), some of the matter charged under the $SU(2)$ living on the curve C will also be charged under G_A. This must occur in such a way that setting $b^2 \to 0$ does not increase the degree of vanishing of f, g on A, or the spectrum of fields charged under the $U(1)$ would not match the spectrum of fields charged under the $SU(2)$ in the $b \to 0$ limit determined as above by the anomaly conditions. Indeed, explicit analysis of the possibilities shows that such an intersection can occur only when A is a -3 or -4 curve. In these cases, when $[e_3] \cdot A \neq 0$, the degrees of vanishing of f, g on A are increased above the minimal Kodaira levels, and G_A carries an enhanced gauge group with charged matter that also carries charges under the $U(1)$ or $SU(2)$ on C in a consistent fashion. When A is a -5 (or less) curve, there is a $(4, 6)$ point on A even in the $U(1)$ model (5), so no such conventional $U(1)$ theory can be constructed. We consider some explicit examples of these cases in the subsequent sections and demonstrate the unconventional presence of a superconformal theory explicitly for -5 curves in §3.6.

In a similar fashion, we can analyze the special cases where e_3 contains a curve D of negative self-intersection as an irreducible component. Note that if $d|e_3$ and also $d|b$, then we can move the factor of d from e_3 into e_1 (with two factors of d extracted from b^2 and moved into e_0). Thus, if $[e_3]$ contains $[d]$ as a component, and $b = -2K - L$ is such that $[b] - [d]$ is effective, we can tune $b = db'$ and the analysis becomes that of the previous case. So we need only consider situations where e_3 contains an irreducible component D that is not a component of b. It turns out this is possible for curves of self-intersection $-3, -4, -5$, and -6; in each of these cases there are configurations where e_3 contains such curves as a component but b does not. When the parameter b is tuned to vanish, the enhancement to $SU(2)$ is combined with an enhancement of the gauge group over D in a way that is consistent with anomaly cancellation and does not introduce $(4, 6)$ points. For curves of self-intersection -7 or below, the $U(1)$ model already has $(4, 6)$ singularities, so there are no conventional models. We give some examples of these kinds of configurations in the subsequent parts of this section.

Although a single curve of negative self-intersection contained in e_3 does not lead to a problematic singularity, there are also situations where e_3 contains a more complicated configuration of intersecting negative self-intersection curves. In particular, there exist non-Higgsable clusters identified in [46] that contain intersecting -3 and -2 curves. In such a situation, as we show explicitly below, a $(4, 6)$ point can arise at the intersection between these curves when a $U(1)$ is unHiggsed to $SU(2)$ by taking the $b \to 0$ limit. This is the one situation we have clearly identified in which such a singularity can arise.

This argument shows that a $U(1)$ gauge factor in a 6D F-theory model over any base can be viewed as arising from an $SU(2)$ gauge group supported on a corresponding effective irreducible divisor class $[e_3]$, after Higgsing a matter hypermultiplet in the adjoint representation; in a wide range of situations the unHiggsing results in a conventional F-theory model with reduced Mordell-Weil rank, though in certain special cases the unHiggsing either gives rise to a model which is coupled to a superconformal theory or is at infinite distance from the interior of moduli space. This general framework gives strong restrictions on the ways in which $U(1)$ factors can arise in 6D F-theory models, and illuminates the structure of the Mordell-Weil group for elliptically fibered Calabi-Yau threefolds over general bases.

3.2. **6D theories on $\mathbb{P}^2$ with two sections or a bisection.** As a simple specific example of a class of 6D theories that illustrate the general structure of models with bisection, two sections ($U(1)$) and enhanced $(SU(2) \times SU(2))/\mathbb{Z}_2$ gauge group, we consider the case of 6D F-theory compactifications on the simplest base surface $B_2 = \mathbb{P}^2$. Models of this type with $U(1)$ fields were considered from the point of view of supergravity and anomaly equations in [14], and an explicit F-theory analysis and Calabi-Yau constructions were given in [16]. In this case, $-K = 3H$, where H is the hyperplane (line) divisor with $H \cdot H = 1$. Tuning an I_2 singularity along a degree d curve C in $\mathbb{P}^2$ by adjusting the degrees of vanishing of f, g, Δ along C to be $0, 0, 2$, respectively, gives an F-theory model with gauge group $SU(2)$. A generic curve of degree d has genus $g = (d-1)(d-2)/2$, and the associated $SU(2)$ gauge group has a matter content consisting of g massless hypermultiplets in the adjoint representation and $24d - 2d^2$ multiplets in the fundamental representation (note that for $SU(2)$, unlike $SU(N)$ for $N > 2$, the antisymmetric representation is trivial). By tuning higher order singularities in the curve C, some of the adjoint matter fields can be transformed into higher-dimensional matter fields, with a simple relation between the matter representations and contribution to the arithmetic genus of C, as described in [53, 60].

To describe the class of Calabi-Yau threefolds on $\mathbb{P}^2$ associated with a Jacobian fibration with a bisection, we consider Weierstrass equations of the form (4), where the classes of the e_i are given in (11–15). We parameterize the set of models of interest by $[e_3] = -3K - L = mH$, where

m corresponds to the degree of a curve in the class $[e_3]$. For any m in the range $0 \leq m \leq 12$ there is a class of Weierstrass models of the form (4) that give "good" F-theory models without $(4, 6)$ points (points which, if present, would involve coupling to superconformal field theories or would violate the Calabi-Yau condition). The generic model in each of these classes corresponds to a Jacobian fibration, and in the F-theory picture there is a discrete $\mathbb{Z}_2$ gauge group, with a number of charged matter hypermultiplets. For $3 \leq m \leq 9$, both $[e_0]$ and $[e_4]$ are effective; in this range of models, there is a subset of models with $e_4 = b^2/4$ a perfect square, giving an extra section contributing to the Mordell-Weil rank, which is associated in the F-theory picture with a $U(1)$ gauge factor, and there is also a (partially overlapping) subset of models with $e_0 = a^2/4$ with another $U(1)$ factor. Either or both of these $U(1)$ factors can be further enhanced to an $SU(2)$ by fixing $b^2 = 0$ or $a^2 = 0$. When both factors are enhanced ($e_0 = e_4 = 0$) the total gauge group is $(SU(2) \times SU(2))/\mathbb{Z}_2$. When $m < 3$ or $m > 9$ the story is similar but one of the two $SU(2)$ factors is automatically imposed by the non effectiveness of the divisor $[e_4]$ or $[e_0]$; in these cases there is only one possible $U(1)$ factor.

This class of models can be understood most easily in the F-theory picture starting from the locus $e_0 = e_4 = 0$ where the gauge algebra is $\mathfrak{su}_2 \oplus \mathfrak{su}_2$. In this case, the two $\mathfrak{su}_2$ summands are associated with 7-branes wrapped on divisors D, D' given by curves of degrees m and $12-m$ in the classes $[e_3], [e_1]$. The spectrum of the theory consists of $m(12-m)$ bifundamental hypermultiplets (associated with the intersection points of D, D'), and $(m-1)(m-2)/2, (11-m)(10-m)/2$ fields in the adjoint representation of each $SU(2)$. The limiting cases $m = 0, 12$ correspond to situations with only a single $SU(2)$ factor and no fundamental hypermultiplets. In all cases, an $SU(2)$ on a curve of degree $d \geq 3$ has adjoint hypermultiplets, of which one can be used to Higgs the nonabelian gauge group to a $U(1)$. Under this Higgsing, the remaining adjoints become scalar fields of charge 2 under the resulting $U(1)$, while fundamentals acquire a charge of 1. When $3 \leq m \leq 9$, such Higgsing to abelian factors is possible for both $SU(2)$ factors; for other values only one of the groups can be Higgsed. Once one or both of the nonabelian factors are Higgsed to $U(1)$ fields, a further breaking can be done by making e_0 or e_4 a generic non-square. This corresponds to using the charge 2 fields to Higgs the $U(1)$ to a discrete gauge group $\mathbb{Z}_2$. Under this Higgsing, the charge 1 fields retain a charge under the discrete gauge group. It is straightforward to check that the numbers of fields in each of these models satisfies the gravitational anomaly cancellation condition $H - V = 273 - 29T$, and matches with the results of [14, 16, 6] for the various component theories.

In particular, note that for $m = 3$ the $SU(2)$ gauge group on $D = [e_3]$ only has a single adjoint field, so after breaking to $U(1)$ there are only charge 1 hypermultiplets. Thus, in this case there is no way of breaking to a model with a bisection and residual discrete gauge group. Note also that by tuning a non-generic singularity on the curve C carrying an $SU(2)$ factor, it should be possible to construct higher dimensional representations of $SU(2)$, which will correspond to larger charges $Q \geq 3$ after breaking to $U(1)$, and which can give rise to higher order discrete gauge groups $\mathbb{Z}_Q$. We return to this issue in §4.1. In Table 1, we provide an explicit list of the charges that arise for the $SU(2)$ and $U(1)$ factors in the various relevant components of the Weierstrass moduli space $\mathcal{S}_2, \mathcal{J}^2$

3.3. **6D theories on $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.** A similar structure will hold on any base B_2 that supports an elliptically fibered Calabi-Yau threefold; a classification of such bases was given in [46], and a complete list of toric bases was given in [65]. As another example we consider the Hirzebruch surface $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$.

For $\mathbb{F}_0$, a basis of $h^{1,1}$ is given by S, F with $S \cdot S = F \cdot F = 0, S \cdot F = 1$. A divisor $D = aS + bF$ is effective if $a, b \geq 0$, and the anticanonical class is $-K = 2S + 2F$. The genus of a curve in the

m	$n_{\rm a}$	$n_{\rm f}$	n_2	n_1
1	0	22	–	–
2	0	40	–	–
3	1	54	0	108
4	3	64	4	128
5	6	70	10	140
6	10	72	18	144
7	15	70	28	140
8	21	64	40	128
9	28	54	54	108
10	36	40	70	80
11	45	22	88	44
12	55	0	108	0

TABLE 1. Table of $SU(2)$ charges in adjoint and fundamental, and $U(1)$ charges in associated theory, when e_3 describes a curve of degree m in $\mathbb{P}^2$. Note that $SU(2)$ and $U(1)$ charges match with Higgsing description ($n_2 = 2(n_{\rm a} - 1), n_1 = 2n_{\rm f}$) as well as with charges computed in [14, 16]. Note also that n_1 matches for $m, 12 - m$, in agreement with the general picture that all charged matter lies on intersection points of $[e_1], [e_3]$ for the $(SU(2) \times SU(2))/\mathbb{Z}_2$ theory.

class $C = aS + bF$ can be computed as

$$(K + C) \cdot C = 2g - 2 = 2(ab - a - b)\,. \tag{34}$$

The genus is nonzero iff $2 \leq a, b$.

The range of possible models (4) with a bisection is thus given by $[e_3] = aS + bF$ with $0 \leq a, b \leq 8$. The values of a, b for which the curves $[e_3], [e_1]$ both have nonzero genus and associated $SU(2)$s can be broken is $2 \leq a, b \leq 6$. Within this range we have the full set of possible enhancements of a model of type (4); there is a model with $(SU(2) \times SU(2))/\mathbb{Z}_2$ symmetry, where either or both $SU(2)$'s can be broken to $U(1)$ or further to the discrete $\mathbb{Z}_2$ symmetry. Again, counting charged multiplets confirms that anomaly cancellation in both the nonabelian and abelian theories matches with the Higgsing process. The spectrum of charged matter fields for an $SU(2)$ tuned on a divisor $aS + bF$ consists of $g = ab - a - b + 1$ adjoints and $16(a + b) - 4ab$ fundamentals. As in the $\mathbb{P}^2$ case, the number of fundamental fields is symmetric under $a \leftrightarrow 8 - a, b \leftrightarrow 8 - b$ ($e_1 \leftrightarrow e_3$), corresponding to the fact that all charged matter in the overall $(SU(2) \times SU(2))/\mathbb{Z}_2$ theory is contained in the adjoints and $8(a + b) - 2ab$ bifundamental fields.

3.4. **6D theories on $\mathbb{F}_3$.** Some interesting points are illuminated by examples on the Hirzebruch surface $\mathbb{F}_3$. Here we have a basis of curves S, F with $S \cdot S = -3, S \cdot F = 1, F \cdot F = 0$. The canonical class is $-K = 2S + 5F$, and there is an automatic vanishing of f, g, Δ to degrees $2, 2, 4$ giving an $SU(3)$ gauge group supported on the divisor S in a generic elliptic fibration.

The simplest irreducible curve e_3 that can give rise to a $U(1)$ factor is $C = 2\tilde{S} = 2S + 6F$, since e_4 must be effective; a generic curve in this class C is irreducible and has genus 2. Choosing

$$[e_3] = 2S + 6F, \quad \Rightarrow \quad [e_1] = 6S + 14F\,. \tag{35}$$

We note that $[e_1] \cdot S = -4$, so $[e_1]$ contains S as an irreducible component with multiplicity at least 2. There is an $SU(3)$ over S, but this does not cause any problems since $S \cdot [e_3] = 0$ so there is no matter charged under the $SU(3)$ that interacts with the $SU(2)$ supported on C or the corresponding $U(1)$ when $e_4 = b^2/4 \neq 0$ or discrete group $\mathbb{Z}/2$ when e_4 is non-square. So this case works like the others above, with $[e_1] \cdot [e_3] = 28$ bifundamental matter fields.

The next case of interest is

$$[e_3] = 2S + 7F, \quad \Rightarrow \quad [e_1] = 6S + 13F\,. \tag{36}$$

In this case the curve C defined by the vanishing locus of e_3 is generically a smooth irreducible curve of genus 3. In this case, $C \cdot S = 1$, so there is matter charged under the gauge group lying over S. To analyze this explicitly, we see that $[e_0] = 8S + 16F, [e_1] = 6S + 13F, [e_2] = 4S + 10F$ contain the irreducible component S with multiplicities 3, 2, and 1 respectively. From (5), (26), this shows that f, g, Δ vanish to degrees $2, 3, 6$ at generic points over S, and to degrees $2, 3, 8$ at points of intersection $[e_3] \cdot S$. As discussed in §3.1, in this situation the gauge group over the -3 curve has an algebra that is larger than the minimal $\mathfrak{su}_3$ for a generic model over $\mathbb{F}_3$. The configuration in this case is similar to the $(-3, -2)$ non-Higgsable cluster [46], in which a -3 curve carries a $\mathfrak{g}_2$ algebra, and there are matter fields charged under both this algebra and an $\mathfrak{su}_2$ on a curve that intersects the -3 curve.

We can also analyze the case where $C = -K = 2S + 5F$, where C is reducible and contains S as a component in a similar fashion, which also gives a $(2, 3, 6)$ vanishing on S, with a similar interpretation

3.5. $\mathbb{F}_4$. The analysis in the case of a -4 curve in $\mathbb{F}_4$ is similar to $\mathbb{F}_3$. The curve S has $S \cdot S = -4$. For the minimal irreducible case $[e_3] = 2S + 8F$, there is an $SU(2)$ with adjoint matter that does not intersect S. For $[e_3] = 2S + 9F$, we have

$$[e_2] = 4S + 12F = S + X^{(2)}_{\text{eff}} \tag{37}$$

$$[e_1] = 6S + 16F = 2S + X^{(1)}_{\text{eff}} \tag{38}$$

$$[e_0] = 8S + 20F = 3S + X^{(0)}_{\text{eff}} \tag{39}$$

where $X^{(a)}_{\text{eff}}$ are effective divisors that contain no further components of S. We can read off the order of vanishing of f, g, Δ from (5) and (26) as $(2, 3, 6)$ on S, enhanced to $(2, 3, 8)$ on $[e_3] \cdot S$, so again we have hypermultiplets charged under the gauge group on S as well as the $SU(2)$ on $[e_3]$. For curves such as $[e_3] = -K = 2S + 6F$, where $[e_3]$ contains S as an irreducible component, a similar analysis holds.

3.6. $\mathbb{F}_5$ **and** -5 **curves.** Now let us consider a -5 curve, beginning with the case of $\mathbb{F}_5$. As in the previous cases, for $[e_3] = 2S + 10F$, there is no intersection with S and the $SU(2)$ story is as above. For the next interesting case, however, we have

$$[e_3] = 2S + 11F \tag{40}$$

$$[e_2] = 4S + 14F = 2S + X^{(2)}_{\text{eff}} \tag{41}$$

$$[e_1] = 6S + 17F = 3S + X^{(1)}_{\text{eff}} \tag{42}$$

$$[e_0] = 8S + 21F = 4S + X^{(0)}_{\text{eff}}\,. \tag{43}$$

Now, analyzing (5) and (26) we find vanishing orders of f, g, Δ on S of $(3, 4, 9)$, enhanced to $(4, 6, 12)$ on $S \cdot [e_3]$, even when $b^2 \neq 0$. Thus, there cannot be a $U(1)$ model based on (5) using $e_3 = 2S + 11F$ (unless the intersection point is blown up, giving a model on a different base).

More generally, we can show that a $U(1)$ based on an extra section can never be constructed on any curve $[e_3] = C$ if $C \cdot A > 0$ for some curve A of self-intersection -5 or less. The argument basically follows exactly the same steps as above. In general, as described in [46], from $[e_2] = -2K$ it follows that e_2 vanishes to degree 2 on A just as in the $\mathbb{F}_5$ case. We have $-4K \cdot A = -12$ and $[e_3] \cdot A > 0$, so $[e_1] \cdot A = (-4K - [e_3]) \cdot A < -12$ and e_1 vanishes to degree 3 on A. From $[e_3] \cdot A > 0$, it follows that $L \cdot A \leq -10$, so $[e_0] \cdot A \leq -20$, and e_0 vanishes to order 4 on S. Thus, no $U(1)$ can be built using (5) on any curve e_3 that has positive intersection with a curve A of self-intersection -5. The condition on each term is stronger as the self-intersection decreases further, so the same result holds for any curve of self-intersection < -5.

Now, let us consider the case that e_3 itself has a -5 curve D as a component. For this to happen we must have $[e_3] \cdot D < 0$, but as argued in §3.1 we should also have $[e_4] \cdot D \geq 0$, or we could move the associated factor out of e_3 and into e_1. This can lead to a conventional model when $[e_3] \cdot D = -2$ or -3. In these cases, e_0, e_1, e_2, e_3 vanish to degrees $3, 2, 2, 1$ on D, and f, g vanish to degrees $3, 4$. In the limit $b^2 \to 0$, f, g vanish to degrees $3, 5$ and the symmetry is enhanced to $\mathfrak{e}_7$. Note that when $[e_3] \cdot D = -1$, e_0, e_1 vanish to degrees $4, 3$ on D, giving multiplicities $(4, 5)$ along D that are enhanced to $(4, 6)$ at points of intersection with the remainder of e_3, so such models are not conventional even before unHiggsing.

As an example of a conventional model of this type, consider on $\mathbb{F}_5$ the $U(1)$ model given by (5) with

$$[e_3] = 2S + 8F = S + X_{\text{eff}}^{(3)} \tag{44}$$

$$[e_2] = 4S + 14F = 2S + X_{\text{eff}}^{(2)} \tag{45}$$

$$[e_1] = 6S + 20F = 2S + X_{\text{eff}}^{(1)} \tag{46}$$

$$[e_0] = 8S + 26F = 3S + X_{\text{eff}}^{(0)} \,. \tag{47}$$

As discussed above, this gives $(3, 4)$ vanishing on the -5 curve S in the $U(1)$ model, enhanced to $(3, 5)$ at points of intersection with e_1. When $b \to 0$, the group is enhanced to $(3, 5)$ on the whole curve S, with further enhancement to $(4, 5)$ at points of intersection with e_3.

3.7. -6 **curves.** The situation for -6 curves is very similar to that for -5 curves. There is a coupled superconformal theory if $[e_3]$ has positive intersection with a -6 curve, but $[e_3]$ can contain a -6 curve D as a component if $[e_3] \cdot D = -4$, in which case e_0, e_1, e_2, e_3 vanish to orders $3, 2, 2, 1$ on D and the story is similar to the above. In this case, however, e_1 does not intersect D, so there are no points where this intersection increases the degree of the singularity.

3.8. -7 **curves.** There are no conventional $U(1)$ configurations of the form (5) where e_3 either intersects or contains a curve D of self-intersection -7 or below. The closest to an acceptable configuration is when $[e_3] \cdot D = -5$, in which case e_0, e_1, e_2, e_3 vanish to orders $3, 3, 2, 1$ on D. This leads to a $(3, 5)$ vanishing of (f, g) on D, which is however enhanced to a $(4, 6)$ vanishing at the point where $[e_0] - [D]$ intersects D (of which there is at least one since $[e_0] \cdot D = -20$). Any other combination of intersections leads to a similar singularity. A similar problem arises for curves of self-intersection -8 or below.

3.9. **The** $-3, -2$ **non-Higgsable cluster.** Finally, we consider the case where e_3 contains both a -3 curve A and a -2 curve B that intersects A transversely ($A \cdot B = 1$). In this case we find that, at least for some choice of L, a $(4, 6)$ point is forced at the intersection point between A and B. In particular, we choose $L = -2K$, so that $[e_n] = (n - 4)K$. From the analysis in [46], we know that a section of $-4K$ must vanish on A, B to degrees 2, 1 respectively, so

$$[e_0] = 2A + B + X_{\text{eff}}^{(0)} \,. \tag{48}$$

It follows that each of the e_n must contain both A and B as irreducible components at least once, for $n < 4$,

$$[e_1] = A + B + X_{\text{eff}}^{(1)} \tag{49}$$

$$[e_2] = A + B + X_{\text{eff}}^{(2)} \tag{50}$$

$$[e_3] = A + B + X_{\text{eff}}^{(3)} \tag{51}$$

$$[b] = X_{\text{eff}}^{(4)} \,. \tag{52}$$

Now, consider the degrees of vanishing of the various terms in (5). While for $b \neq 0$, there are terms in f and g that only vanish to degrees (3, 5) at the intersection of A and B (namely those proportional to $e_0 b^2, e_0 e_2 b^2$), when we take $b \to 0$, all the remaining non-vanishing terms are of degrees at least $(4, 6)$ at the intersection point.

This means that in such a situation, while there can exist a 6D F-theory model with a $U(1)$ gauge symmetry associated with a nontrivial Mordell-Weil rank, and the Weierstrass coefficients can be tuned to naively produce a nonabelian $SU(2)$ structure, the resulting model might have an isolated $(4, 6)$ point and hence be coupled to a superconformal theory,[10] or in other situations [64] might have $(4, 6)$ singularities all along one or more curves after unHiggsing[11], which indicates that these models are at infinite distance from the interior of the moduli space. There are many known examples of base surfaces that contain $-3, -2$ non-Higgsable clusters; a variety of such examples were constructed in [65, 71]. It would be interesting to analyze in detail the structure of $U(1)$ symmetries that could be tuned over some of these bases.

4. Implications for 6D and 4D F-theory models

4.1. F-theory and supergravity in six dimensions. Six dimensions provides a rich but tractable context in which to study general aspects of string vacua and quantum supergravity theories. In six dimensions, F-theory seems to provide constructions for essentially all known string vacua, and the space of F-theory vacua matches closely with the set of potentially consistent quantum supergravity theories [66, 67, 68, 69, 59]. The class of 6D F-theory constructions based on Weierstrass models of elliptically fibered Calabi-Yau threefolds with section form a single moduli space of smooth components associated with different bases B_2 that are connected through tensionless string transitions [49, 3]; recent work has made progress in providing a global picture of this connected moduli space [59, 46, 70]. The results of [6] raised a question of whether genus-one fibrations without section might constitute a class of F-theory models that were disconnected from the rest of the F-theory moduli space. The picture outlined in this note makes it clear that in fact the Jacobian fibrations for threefolds without section fit neatly into the connected moduli space of Weierstrass models. Furthermore, this picture sheds light on how $U(1)$ gauge fields in 6D F-theory models may be understood in the context of the full moduli space of models.

In [6, 20, 21, 23], a systematic description was given of the general form for Weierstrass models containing one, two, and three $U(1)$ fields. It is known that 6D models can be constructed with up to eight or more $U(1)$ fields; for example, as described in [60] there are F-theory constructions on $\mathbb{P}^2$ with an $SU(9)$ tuned on a curve of genus one that contain an adjoint representation the breaking of which gives gauge factors $U(1)^8$, and in [71] a class of $\mathbb{C}^*$-bases B_2 were found with varying automatic ranks for the Mordell-Weil group for generic elliptic fibrations; the resulting threefolds are closely related to the Schoen manifold [64]. One such base in particular is a

[10] We would like to thank Jim Halverson for discussions on this point. Analogous curves in 4D F-theory models are identified and classified in [63].

[11] We would like to thank D. Park for discussions on this point.

generalized del Pezzo nine over which the generic elliptic fibration has a rank 8 Mordell-Weil group, corresponding to gauge factors $U(1)^8$. In [14], it was shown from 6D anomaly cancellation arguments that for a pure abelian theory in 6D with no tensor multiplets (corresponding to an F-theory model on $\mathbb{P}^2$) the number of $U(1)$ fields is bounded above by $r \leq 17$. The approach taken in this paper shows that for a single $U(1)$ factor, it is often possible to tune the model so that the $U(1)$ can be seen as arising from an $SU(2)$ or larger nonabelian factor that is Higgsed by VEVs for an adjoint field in a conventional F-theory model. It would be interesting to investigate the possible apparent exceptions to this construction, such as the ones we encountered with base contains a $-3, -2$ cluster, where the F-theory model becomes coupled to a superconformal theory. While in general the construction of higher rank Mordell-Weil models seems very challenging due to the global nature of the sections, it would be very interesting to explore when higher rank abelian models can arise from Higgsed nonabelian gauge symmetries. This would provide a powerful tool for the construction of general models with abelian gauge symmetries, since a systematic analysis of the nonabelian sector is much more straightforward, both in F-theory and 6D supergravity. It would also be interesting to explore in more detail the way in which the basic $SU(2) \to U(1) \to \mathbb{Z}_2$ Higgsing pattern interacts with other nonabelian gauge symmetries which may be present in a given model.

The existence of an underlying $SU(2)$ for many $U(1)$ gauge factors also greatly clarifies the set of possible spectra. The spectrum of $SU(2)$ theories is quite constrained by anomaly cancellation [53], which in turn places strong constraints on the spectrum of possible charges for abelian factors in the 6D supergravity gauge group. When an $SU(2)$ factor is tuned on a curve of genus g over a general base B_2 generically the model will include g symmetric (adjoint) representations and some number of fundamentals. After breaking to a $U(1)$, this gives charges 1 and 2, so these are the only charges expected in generic models. For specially tuned singular curves, however, higher representations of $SU(2)$ are possible.

For example, following the lines of [60], we expect that an $SU(2)$ on a quintic curve on P^2 can carry a 3-symmetric (4-dimensional) representation when the curve is tuned to have a triple point of self-intersection. Group theoretically, this should correspond to an embedding of $\mathfrak{su}_2 \oplus \mathfrak{su}_2 \oplus \mathfrak{su}_2$ in an $\mathfrak{e}_7$ singularity associated with the triple intersection point. After breaking the $SU(2)$ to $U(1)$ by an adjoint VEV, this would give rise to a massless scalar hypermultiplet of charge ± 3 under the $U(1)$. By the mechanism discussed in this paper, such fields could then be used to break the $U(1)$ to a discrete $\mathbb{Z}_3$ gauge symmetry, associated again with a Weierstrass model associated with the Jacobian of an elliptic fibration with a multisection. Exploring the range of possibilities of this type that may be possible for general representations of $SU(2)$ and higher rank nonabelian groups on arbitrary curves on general F-theory bases B_2 promises to provide a rich and interesting range of phenomena. The analysis here shows that there are strong constraints on the charge spectrum for $U(1)$ fields in many 6D F-theory models. These constraints are stronger than those imposed simply by 6D anomaly cancellation. In the spirit of [66], it would be interesting to understand if some of the F-theory constraints on charge structure could be seen as consistency conditions for the low-energy 6D supergravity theories with abelian gauge factors.

4.2. **Four dimensions.** At the level of geometry, the framework developed in this paper should be valid for Calabi-Yau manifolds of any dimension. It has not been shown, however, that all genus-one fibered Calabi-Yau $(n+1)$-folds X_{n+1} that lack a global section have an associated Jacobian fibration J_{n+1} whose total space is Calabi-Yau when $n \geq 3$, so it is possible in principle that the analysis described here can only be applied in a subset of cases where there is a Jacobian fibration available. If so, the application to four-dimensional F-theory constructions would only be relevant in those cases. When a Jacobian fibration is available, however, the analysis of §2 should hold: the Jacobian fibrations of all Calabi-Yau fourfolds with a genus-one fibration but no global section should fit into the moduli space of Weierstrass models over complex threefold bases B_3, with an explicit description of the form (4) when the Jacobian fibration has a bisection. When the section $e_4 = b^2/4$ is a perfect square, the bisection becomes a pair of global sections and the Mordell-Weil rank increases by one. When $b \to 0$, the extra section transforms into a vertical A_1 Kodaira type singularity without changing the total Hodge number $h^{1,1}(X_4)$. In many situations, the physics interpretation of this geometry through F-theory will be the same as in 6 dimensions: the bisection geometry will be associated with a discrete $\mathbb{Z}_2$ gauge symmetry that arises from a broken $U(1)$ gauge field, which in turn can be viewed as coming from an $SU(2)$ gauge group broken by an adjoint VEV. Wrapping the 4D theory on a circle will give distinct vacua, again associated with the Tate-Shafarevich group and in the M-theory picture with a discrete choice of Calabi-Yau fourfold with a genus-one fibration but no section.

We also expect a similar story to hold for higher degree multisections and elliptic fibrations with higher rank Mordell-Weil group. In four dimensions, however, there is additional structure beyond the geometry that can modify this story. In particular, G-flux, associated with 4-form flux of the antisymmetric 3-form potential in the dual M-theory picture, produces a superpotential that gives masses to many of the scalar moduli of the Calabi-Yau geometry. This mechanism can modify the gauge group and matter spectrum of the theory from that described purely by the geometry. At this point a complete understanding of the role of G-flux in F-theory is still lacking, despite some recent progress in this direction [72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 42]. We leave the analysis of how the results in this note are affected by G-flux and the 4D superpotential to further work. The implications of the generic appearance of an $SU(2)$ (or larger) nonabelian enhancement for most $U(1)$ vector fields are, however, a question of obvious phenomenological interest.

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Appendix A. Solving the equations determining I_2 fibers

In this appendix, we will explain how to solve the equations (18), (19) which determine the location of the codimension two I_2 fibers by finding the conditions for the quartic equation to be a square, as in (17). The first observation is that when the coefficient functions $e_0, e_1, \ldots e_4$ are generic, none of them will vanish at any of the solutions to (17).

In the case of e_4, if e_4 vanishes at a solution then $u = 0$ is one of the double roots so e_3 must also vanish. For the remaining root to be double, we also need $e_1^2 = 4e_0e_2$ to vanish, but now we have three conditions on the base and the solutions are in codimension two. The case of e_0 is similar: if it vanishes, then e_1 and $e_3^2 - e_2e_4$ would both also have to vanish.

In the case of e_3 vanishing, β would need to vanish and then the equation would take the form $e_4((e_2/2e_4)u^2 + v^2)^2$. Again we get three condtions: $e_3 = 0$, $e_1 = 0$, and $e_2^2 = 4e_0e_4$ which is of too large a codimension to be generic. The case of e_1 is similar.

Finally, in the case of e_2 vanishing, we have an equation of the form

$$e_4(-(e_3^2/8e_4^2)u^2 + (e_3/2e_4)uv + v^2)^2,$$

and this implies the additional conditions $e_3^3 = -8e_4^2e_1$ and $e_3^4 = 64e_4^3e_0$. Once again we have three conditions and this is not possible.

Now we turn to the solution of (18), (19). As in §2.2, the first step is to introduce an auxiliary variable p, and to express the solutions as the common zeros of two auxiliary polynomials

$$\Phi_1 := p^4 - 8e_2e_4p^2 + 16e_2^2e_4^2 - 64e_0e_4^3 \tag{53}$$

$$\Phi_2 := p^3 - 4e_2e_4p + 8e_1e_4^2 \tag{54}$$

(together with the equation $e_3 = p$). From this, we can form additional polynomials which must vanish on the solution, roughly following the Gröbner basis algorithm (but allowing division by e_1, e_2 or e_4, which are known not to vanish on solutions). This gives the following sequence of polynomials:

$$\Phi_3 := (-\Phi_1 + p\Phi_2)/4e_4 = e_2p^2 + 2e_1e_4p - 4e_2^2e_4 + 16e_0e_4^2 \tag{55}$$

$$\Phi_4 := (-e_2\Phi_2 + p\Phi_3)/2e_4 = e_1p^2 + 8e_0e_4p - 4e_1e_2e_4 \tag{56}$$

$$\Phi_5 := (e_1\Phi_3 - e_2\Phi_4)/2e_4 = (e_1^2 - 4e_0e_2)p + 8e_0e_1e_4 \tag{57}$$

$$\Phi_6 := ((4e_0e_2 - e_1^2)\Phi_4 + pe_1\Phi_5)/4e_2e_4 = 8e_0^2p - e_1(4e_0e_2 - e_1^2) \tag{58}$$

$$\Phi_7 := (8e_0^2\Phi_5 + (4e_0e_2 - e_1^2)\Phi_6)/e_1 = 64e_0^3e_4 - (4e_0e_2 - e_1^2)^2\,. \tag{59}$$

The variable p has been eliminated from Φ_7, so the equation $\Phi_7 = 0$ gives the condition for a p to exist (this is (22)). The equation $\Phi_5 = 0$ can then be solved for p; this gives (23).

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