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ON THE BI-LIPSCHITZ CONTACT EQUIVALENCE OF PLANE COMPLEX FUNCTION-GERMS

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To David Trotman for his sixtieth birthday.

Abstract. In this note, we consider the problem of bi-Lipschitz contact equivalence of complex analytic function-germs of two variables. Basically, it is inquiring about the infinitesimal sizes of such function-germs up to bi-Lipschitz changes of coordinates. We show that this problem is equivalent to right topological classification of such function-germs.

1. Contact equivalence

Two $\mathbb{K}$-analytic function-germs $f, g\colon (\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}, 0)$, at the origin $\mathbf{0}$ of $\mathbb{K}^n$, are ($\mathbb{K}$-analytically) *contact equivalent* if the ideals (in $\mathcal{O}_{\mathbb{K}^n,\mathbf{0}}$) generated by f and, respectively, generated by g are $\mathbb{K}$-analytically isomorphic. As is well known, this classical ($\mathbb{K}$-analytic) contact equivalence admits moduli. For a complete description and answer to Zariski *problème des modules pour les branches planes* in the uni-branch case, see [5], (see also [6] for an answer towards the general case). Over the years several generalizations of the notion of ($\mathbb{K}$-analytic) contact equivalence appeared, and for some rough ones moduli do not exist.

More precisely, we will say that two function-germs $f, g\colon (\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}, 0)$ at the origin $\mathbf{0}$ of $\mathbb{K}^n$ are *bi-Lipschitz contact equivalent* if there exists $H\colon (\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}^n, \mathbf{0})$ a bi-Lipschitz homeomorphism and there exist positive constants A and B, and $\sigma \in \{-1, +1\}$ such that

$$A|f(\mathbf{p})| \leq |g \circ H(\mathbf{p})| \leq B|f(\mathbf{p})| \text{ when } \mathbb{K} = \mathbb{C},$$
$$Af(\mathbf{p}) \leq \sigma \cdot (g \circ H(\mathbf{p})) \leq Bf(\mathbf{p}) \text{ when } \mathbb{K} = \mathbb{R},$$

for any point $\mathbf{p} \in \mathbb{K}^n$ close to $\mathbf{0}$.

When the bi-Lipschitz homeomorphism H is also subanalytic, we will say that the functions f and g are *subanalytically bi-Lipschitz contact equivalent*.

A consequence of the main result of [1] on bi-Lipschitz contact equivalence of Lipschitz function-germs is the following finiteness

Theorem ([1]). *For any given pair n and k of positive integers, the subspace of polynomial function-germs $(\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}, 0)$ of degree smaller than or equal to k has finitely many bi-Lipschitz contact equivalence classes.*

Later on, Ruas and Valette (see [10]) obtained for real mappings a result more general than that of [1], and which again ensures the finiteness of the bi-Lipschitz contact equivalent classes for

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polynomial function-germs $(\mathbb{K}^n, \mathbf{0}) \to (\mathbb{K}, 0)$ with given bounded degree. However, we observe that in the aforementioned papers [1, 10], the proofs of the finiteness theorems for bi-Lipschitz contact equivalence do not say anything about the corresponding recognition problem.

The preprint [2] completely solves the recognition problem of subanalytic contact bi-Lipschitz equivalence for continuous subanalytic function-germs $(\mathbb{R}^2, \mathbf{0}) \to (\mathbb{R}, 0)$ by providing an explicit combinatorial object which completely characterizes the corresponding orbit.

In the present note, we solve the recognition problem for the subanalytic bi-Lipschitz contact equivalence of complex analytic function-germs $(\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$.

Our main result, Theorem 4.2, states that the subanalytic bi-Lipschitz contact equivalence class of a plane complex analytic function-germ $f\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ determines and is determined by purely numerical data, namely: *the Puiseux pairs of each branch of its zero locus, the multiplicities of its irreducible factors and the intersection numbers of pairs of branches of its zero locus.* It is a consequence of Theorem 3.6 which explicits the order of an irreducible function-germ g along real analytic half-branches at $\mathbf{0}$ as an affine function of the contact of the half-branch and the zero locus of g.

Last, combining the main result of [8] and our main result, we eventually get that two complex analytic function germs $f, g\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ are subanalytically bi-Lipschitz contact equivalent if, and only if, they are *right topologically equivalent,* i.e. there exists a homeomorphism $\Phi\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ such that $f = g \circ \Phi$.

2. Preliminaries

We present below some well known material about complex analytic plane curve-germs. It will be used in the description and the proof of our main result.

2.1. Embedded topology of complex plane curves.

Let $f\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ be the germ at $\mathbf{0}$ of an irreducible analytic function. It admits a Puiseux parameterization of the following kind:

$$x \to (x^m, \Psi(x)) \text{ with } \Psi(x) = x^{\beta_1}\varphi_1(x^{e_1}) + \ldots + x^{\beta_s}\varphi_s(x^{e_s}), \tag{1}$$

where each function φ_i is a holomorphic unit at $x = 0$, the integer number m is the multiplicity of the function f at the origin and $(\beta_1, e_1), \ldots, (\beta_s, e_s)$ are the Puiseux pairs of f. Then we can write down,

$$f(x^m, y) = U(x, y)\Pi_{i=1}^m (y - \Psi(\omega^i x)), \tag{2}$$

where ω is a primitive m-th root of unity, the function U is a holomorphic unit at the origin, and Ψ is a function like in Equation (1).

The following relations determines the Puiseux pairs of f. Let us write $\Psi(x) = \sum_{j>m} a_j x^j$ and $e_0 := m$ and $\beta_{s+1} := +\infty$. We recall that

$$\beta_{i+1} = \min\{j : a_j \neq 0 \text{ and } e_i \not| j\} \text{ and } e_{i+1} := \gcd(e_i, \beta_{i+1})$$

for $i = 0, \ldots, s-1$. We deduce that there exists positive integers $m_1, \ldots, m_s$, such that for each $k = 1, \ldots, s$, we find

$$m = e_1 m_1 = e_2 m_2 m_1 = \ldots = e_k (m_k \cdots m_1) \tag{3}$$

We recall that the irreducibility of the function f implies that $e_s = 1$.

Remark 1. *Let $f\colon (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C},0)$ be an irreducible analytic function-germ and let X be its zero locus. The ideal I_X of $\mathbb{C}\{x,y\}$ consisting of all the functions vanishing on X is generated by f. If $g = \lambda f$ is any other generator of I_X, then the functions f and g have the same Puiseux pairs. Thus we will speak of the* Puiseux pairs of the branch *X.*

Let $f_1, f_2\colon (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C},0)$ be irreducible analytic function-germs, and let X_1 and X_2 be the respective zero sets of f_1 and f_2.

The *intersection number at* $\mathbf{0}$ of the branches X_1 and X_2 is defined as:

$$(X_1, X_2)_{\mathbf{0}} = \dim_{\mathbb{C}} \frac{\mathbb{C}\{x,y\}}{(f_1, f_2)}$$

where (f_1, f_2) denotes the ideal generated by f_1 and f_2.

Notation: Let $\Phi : (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C}^2,\mathbf{0})$ be a homeomorphism and let X be a subset germ of $(\mathbb{C}^2,\mathbf{0})$. We will write

$$\Phi : (\mathbb{C}^2, X, \mathbf{0}) \to (\mathbb{C}^2, Y, \mathbf{0})$$

to mean that the subset germ Y is the germ of the image $\Phi(X)$ of X.

The following classical result completely described the classification of embedded complex plane curve germs:

Theorem 2.1 ([3, 11])**.** *Let $f, g\colon (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C},0)$ be reduced analytic function-germs and let X and Y be the respective zero sets of f and g. Let $X = \bigcup_{i=1}^r X_i$ and $Y = \bigcup_{i=1}^s Y_i$ be the irreducible components of X and Y respectively. There exists a homeomorphism $\Phi\colon (\mathbb{C}^2, X, \mathbf{0}) \to (\mathbb{C}^2, Y, \mathbf{0})$ if and only if, up to a re-indexation of the branches of Y, the components X_i and Y_i have the same Puiseux pairs, and each pair of branches X_i and X_j have the same intersection numbers as the pair Y_i and Y_j.*

We end-up this subsection in recalling a recent result of Parusiński [8]. It is as much a generalization of Theorem 2.1 to the non reduced case, as it is an improvement in the sense that it provides a more rigid statement.

Theorem 2.2. *Let $f, g\colon (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C},0)$ be complex analytic function-germs (thus not necessarily reduced). There exists a germ of homeomorphism $\Phi : (\mathbb{C}^2,\mathbf{0}) \to (\mathbb{C}^2,\mathbf{0})$ such that $g \circ \Phi = f$ (the function-germs f and g are then said* topologically right-equivalent*) if, and only if, there exists a one-to-one correspondence between the irreducible factors of f and g which preserves the multiplicities of these factors, their Puiseux pairs and the intersection numbers of any pairs of distinct irreducible components of the respective zero loci of f and g.*

2.2. Lipschitz geometry of complex plane curve singularities.

The Lipschitz geometry of complex plane curve singularities we are interested in is the Lipschitz geometry which comes from being embedded in the plane. It is described in a collection of three articles over 40 years, initiated with the seminal paper [9], followed then by [4] and concluding for now with the recent preprint [7]. Those papers state that the Lipschitz geometry of complex plane curve singularities determines and is determined by the embedded topology of such singularities. The version of this result which we are going to use is the following one:

Theorem 2.3. *Let X and Y be germs of complex analytic plane curves at $\mathbf{0} \in \mathbb{C}^2$. Then, there exists a homeomorphism $\Phi\colon (\mathbb{C}^2, X, \mathbf{0}) \to (\mathbb{C}^2, Y, \mathbf{0})$ if, and only if, there exists a (subanalytic) bi-Lipschitz homeomorphism $H\colon (\mathbb{C}^2, X, \mathbf{0}) \to (\mathbb{C}^2, Y, \mathbf{0})$.*

The version stated above is almost Theorem 1.1 of [7]. The exact statement of Theorem 1.1 of [7] does not require the subanalyticity of the homeomorphism H. However, we observe that the proof presented there actually guarantees the subanalyticity of the mapping H.

3. On the irreducible functions case

This section is devoted to the relation between the order of a given irreducible plane complex function-germ f along any real analytic half-branch germ at the origin $\mathbf{0}$ of $\mathbb{C}^2$, and the contact (at the origin) between the half-branch and the zero locus X of f. (Both notions of order and contact will be recalled below.) Theorem 3.6 is the main result of the section and the key new ingredient to complete the subanalytic bi-Lipschitz contact classification. It states that the contact and the order satisfies an affine relation whose coefficients can be explicitly computed by means of the Puiseux data of X presented in sub-Section 2.1.

We suppose given some local coordinates (w, y) centered at the origin of $\mathbb{C}^2$.

Let Γ be a real-analytic half branch germ at the origin of $\mathbb{C}^2$, that is the image of (the restriction of) a real analytic map-germ $\gamma : (\mathbb{R}_+, 0) \to (\mathbb{C}^2, \mathbf{0})$ defined as $s \to \gamma(s) = (w(s), y(s))$. When Γ is not contained in the y-axis, we can assume that $\gamma(s) = (s^e \mathbf{u}(s), s^{e'} \mathbf{v}(s))$ for positive integers e, e' with $\mathbf{u}(z), \mathbf{v}(z) \in \mathcal{O}_1 := \mathbb{C}\{z\}$ and $\mathbf{u}(0), \mathbf{v}(0) \neq 0$.

When Γ is not contained in the y-axis, we want to find a holomorphic change of coordinates $w \to x(w)$ so that

$$x(z^e \mathbf{u}(z)) = z^e \iff \mathbf{u}(z) \cdot \mathbf{x}(z^e \mathbf{u}(z)) = 1 \tag{4}$$

writing x as $x(w) := w \cdot \mathbf{x}(w)$ for a local holomorphic unit $\mathbf{x}$. Thus Equation (4) admits a holomorphic solution. The mapping $\Theta : (w, y) \to (x(w), y) = (x, y)$ is bi-holomorphic in a neighbourhood of the origin. In the new coordinates (x, y), the mapping γ now writes as $s \to (s^e, s^{e'} \mathbf{v}(s))$.

Vocabulary. A map-germ $\phi : (\mathbb{R}_+, 0) \to (\mathbb{C}^2, \mathbf{0})$ is *ramified analytic* if there exists a function germ $\tilde{\phi} \in \mathcal{O}_1$ and (co-prime) positive integers p, q such that $\phi(t) = \tilde{\phi}(t^{p/q})$. We will further say that ϕ is *a ramified analytic unit* if $\tilde{\phi}$ is a holomorphic unit.

When Γ is not contained in the y-axis, we re-parameterize γ with $s(t) := t^{e/m}$ for $t \in \mathbb{R}_+$, so that $\gamma(t) := \gamma(s(t)) = (t^m, y(t))$ where y is ramified analytic with $y(0) = 0$ and m is the multiplicity of the function f at the origin.

If Γ is contained in the y-axis then we take $s = t$ and Θ is just the identity mapping.

We recall that the Puiseux pairs introduced in sub-Section 2.1 are bi-holomorphic invariant. We denote again $f = f(x, y)$ for $f \circ \Theta^{-1}$ and use the Puiseux decomposition for $f(x^m, y)$ given in Equation (2) to define for each $k = 0, \ldots, s$, the function germ $\Psi_k \in \mathcal{O}_1$ as

$$\begin{aligned} \Psi_0(x) &:= 0, \\ \Psi_k(x) &:= x^{\beta_1} \varphi_1(x^{e_1}) + \ldots + x^{\beta_k} \varphi_k(x^{e_k}) \text{ when } k \geq 1. \end{aligned}$$

Note that $\Psi_k(x) = \theta_k(x^{e_k})$ for some function germ $\theta_k \in \mathcal{O}_1$.

For each $l = 1, \ldots, m$, we can write

$$y(t) = \Psi(\omega^l t) + t^{\lambda_l} u_l(t)$$

where $\lambda_l \in \mathbb{Q}_{>0} \cup \{+\infty\}$ for u_l is a ramified analytic unit, and with the convention that we write the null function 0 as $0 = t^{+\infty} u_l(t)$. Thus the half-branch Γ is contained in X if and only if there exists l such that $\lambda_l = +\infty$.

Notation. *Let* $\lambda := \max_{l=1,\ldots,m} \lambda_l$.

Let $l \in \{1, \ldots, m\}$ so that $\lambda = \lambda_l$. When Γ is not contained in X (equivalently $\lambda < +\infty$) and convening further that $\beta_0 = 0$ and $\beta_{s+1} = +\infty$, there exists a unique integer $k \in \{0, \ldots, s\}$ such that

$$\beta_k \leq \lambda < \beta_{k+1},$$

and consequently we can write
$$y(t) = \Psi_k(\omega^l t) + t^\lambda u(t)$$
for u a ramified analytic unit. (Note that $\Psi = \Psi_k + R_k$ where $R_k(x) = (\Psi - \Psi_k)(x) = O(x^{\beta_{k+1}})$.)

Evaluating the function f along the parameterized arc $t \to \gamma(t)$ using Equation (2) gives
$$f(\gamma(t)) = f(t^m, y(t)) = f(t^m, \Psi_k(t) + t^\lambda u(t)) = U(t)\Pi_{i=1}^m[\Psi_k(\omega^l t) + t^\lambda u(t) - \Psi(\omega^i t)]$$
where $t \to U(t)$ is a ramified analytic unit. Since the function $t \to f(\gamma(t))$ is a ramified analytic function, there exist a ramified analytic unit V and a number $\nu \in \mathbb{Q}_{>0} \cup \{+\infty\}$ such that
$$f(\gamma(t)) = t^\nu V(t). \tag{5}$$
The number ν of Equation (5) is called the *order of the function f along the parameterized curve $t \to \gamma(t)$*.

Lemma 3.1. *1) Assume Γ is contained in the y-axis. The order of the function f along the parameterized curve $t \to \gamma(t) = (0, t^{e'}\mathbf{v}(t))$ is $\nu = m \cdot e'$.*
2) Assume Γ is not contained in the y-axis. The order of ν the function f along the parameterized curve $t \to \gamma(t)$ is given by
$$\nu = e_k\lambda + (e_0 - e_1)\beta_1 + \ldots + (e_{k-1} - e_k)\beta_k \in \mathbb{Q}_{>0} \cup \{+\infty\}.$$

Proof. If Γ is contained in the y-axis, then the order of f along $t \to (0, t^{e'}\mathbf{v}(t))$ is $m \cdot e'$.

We can assume that Γ is parameterized as $\mathbb{R}_+ \ni t \to \gamma(t) = (t^m, \psi_k(t) + t^\lambda u(t))$.

For $i \in \{1, \ldots, m\}$ such that $l-i$ is not a multiple of m_1, the order of $\Psi_k(\omega^l t) + t^\lambda u(t) - \Psi(\omega^i t)$ is β_1. There are $m - 1 - (e_1 - 1) = e_0 - e_1$ such indices i.

For any $0 < j < k$, when $i \in \{1, \ldots, m-1\}$ is such that $l - i$ is a multiple of $m_1 \ldots m_j$ but not a multiple of $m_1 \ldots m_{j+1}$, the order of $\Psi_k(\omega^l t) + t^\lambda u(t) - \Psi(\omega^i t)$ is β_j. There are $e_{j-1} - e_j$ such indices.

When $i \in \{1, \ldots, m\}$ is such that $l - i$ is a multiple of $m_1 \ldots m_k$, the order of
$$\Psi_k(\omega^l t) + t^\lambda u(t) - \Psi(\omega^i t)$$
is λ. There are e_k such indices.

We just add-up all these orders to get the desired number ν, once we have checked that this sum does not depend on the index l such that $\lambda = \lambda_l$. Let $r \in \{1, \ldots, m\}$ be an index such that $\lambda_r = \lambda$. Thus $y(t) = \Psi_k(w^r t) + t^\lambda u_r(t)$. If $l - r$ is not a multiple of $m_1 \cdots m_k$, then we check again that $0 = y(t) - y(t) = t^\lambda(u_l(t) - u_r(t)) + t^{\beta_j}W$ for a ramified analytic unit W and $\beta_j \le \beta_{k-1} < \lambda$, which is impossible. Necessarily $l - r$ is a multiple of $m_1 \cdots m_k$ and thus $\Psi_k(w^r t) = \Psi_k(\omega^l t)$, so that ν is well defined. ☐

Now we can introduce a sort of normalized parameterization of real analytic half-branch germs in order to do bi-Lipschitz geometry. More precisely,

Definition 3.2. *A* (real) analytic arc *(at the origin of $\mathbb{C}^2$) is the germ at $0 \in \mathbb{R}_+$ of a mapping $\alpha : [0, \epsilon[\to \mathbb{C}^2$ defined as $t \to (x(t), y(t))$ such that:*

0) the mapping α is not constant and $\alpha(0) = \mathbf{0}$,

1) there exists a positive integer e such that $t \to \alpha(t^e)$ is (the restriction of) a real analytic mapping,

2) the arc is parameterized by the distance to the origin in the following sense: there exists positive constants $a < b$ such that for $0 \le t \ll 1$ the following inequalities hold,
$$at \le |\alpha(t)| \le bt.$$

We will denote any analytic arc by its defining mapping α. Note that the semi-analyticity of the image of an analytic arc α implies a much better asymptotic than that proposed in the definition, namely we know that $|\alpha(t)| = \alpha_1 t + t\delta(t)$, with $\alpha_1 > 0$ and where δ is ramified analytic such that $\delta(0) = 0$.

Let α be a real analytic arc. The function $t \to f \circ \alpha(t)$ is ramified analytic, thus as already seen in Equation (5) can be written as $f \circ \alpha(t) = t^{\nu_f(\alpha)} V(t)$ for a ramified analytic function V and $\nu_f(\alpha) \in \mathbb{Q}_{>0} \cup \{+\infty\}$. The *order of the function f along the real analytic arc α* is the well defined rational number $\nu_f(\alpha)$.

Let C be a real-analytic half-branch germ at the origin of $\mathbb{C}^2$. Let α and β be two real analytic arcs parameterizing C. We check with an easy computation that $\nu_f(\alpha) = \nu_f(\beta)$. Thus we introduce the following

Definition 3.3. *The* order of the function f along the real analytic half-branch C *is the well defined number $\nu_f(C) := \nu_f(\delta)$ for any analytic arc δ parameterizing C.*

Let us denote $X(r) = \{\mathbf{p} \in X \ : |\mathbf{p}| = r\}$ for r a positive real number.

Let α be any analytic arc. The *contact (at the origin) between the analytic arc α and the complex curve-germ X* is the rational number defined as

$$c(\alpha, X) = \lim_{t \to 0+} \frac{\log(\operatorname{dist}(\alpha(t), X(|\alpha(t)|)))}{\log(t)}.$$

Let C be the image of the analytic arc α above. Given any other analytic arc β parameterizing C, it is a matter of elementary computations to check that $c(\alpha, X) = c(\beta, X)$. Thus we present the following

Definition 3.4. *The* contact between the real-analytic half-branch C and the curve X *is $c(C, X) := c(\delta, X)$ for any analytic arc δ parameterizing C.*

Let Γ be a real analytic half-branch at the origin of $\mathbb{C}^2$. Let γ be a parameterization of Γ of the form $\mathbb{R}_+ \ni t \to (0, y(t))$ when Γ is contained in the y-axis, where y is a ramified analytic function-germ. When Γ is not contained in the y-axis, possibly after a holomorphic change of coordinates at the origin of $\mathbb{C}^2$, we consider a parameterization of Γ of the form $\mathbb{R}_+ \ni t \to (t^m, y(t))$ for y ramified analytic.

When the half-branch Γ is not contained in X (and regardless of its position relatively to the y-axis), as already seen above, we can write $y(t)$ as $y(t) = \Psi_k(\omega^l t) + t^\lambda u(t)$ where $\beta_k \leq \lambda < \beta_{k+1}$ for some integer $k \in \{0, \ldots, s\}$, with u a ramified analytic unit and $l \in \{1, \ldots, m\}$. Let μ be the order of $|\gamma(t)|$ at $t = 0$, that is the positive rational number μ such that $|\gamma(t)| = M t^\mu + o(t^\mu)$ for a positive constant M. Thus we find

Lemma 3.5. *The contact between Γ and X is $c(\Gamma, X) = \frac{\lambda}{\mu}$.*

Proof. Up to a linear change of coordinates we can assume that the tangent cone at the origin of the (irreducible) curve X is just the x-axis. Writing $\gamma(t) = (x(t), y(t))$, the half-branch is tangent to the x-axis if and only if $\lim_{t \to 0} x(t)^{-1} y(t) = 0$. When Γ is transverse to the x-axis, we have $k = 0$ in the writing of $y(t)$ above, so that $\mu = \lambda$ and thus $c(\Gamma, X) = 1$.

When the half-branch Γ is tangent to the x-axis, we deduce $\mu = m$ since the tangency hypothesis implies that $y(t) = o(t^m)$. Thus the mapping $t \to \gamma(t^{\frac{1}{m}}) = (t, y(t^{\frac{1}{m}}))$ is an analytic arc parameterizing Γ. In particular we must have $\lambda > m$.

Notation. Up to the end of this proof we will use the notation *Const* to mean a positive constant we do not want to precise further.

Let $\rho : (\mathbb{R}_+, 0) \to (\mathbb{R}_+, 0)$ be the function defined as $\rho(t) := \operatorname{dist}(\gamma(t^{\frac{1}{m}}), X)$. First, since γ is tangent to X and the function ρ is continuous and subanalytic, there exists a positive rational number c such that

$$\rho(t) = Const \cdot t^c + o(t^c). \tag{6}$$

Second, we obviously have for t positive and small enough $\rho(t) \leq t^{\frac{\lambda}{m}}|u(t)|$ so that we deduce from Equation (6) that $c \geq \frac{\lambda}{m}$.

Let $r(t) := |\gamma(t^{\frac{1}{m}})|$, so that we find $r(t) = t + o(t)$. Let $t \to \phi(t)$ be any analytic arc on X such that $\rho(t) = |\phi(t) - \gamma(t^{\frac{1}{m}})|$. From Equation (6) we get

$$||\phi(t)| - r(t)| \leq Const \cdot t^c. \tag{7}$$

Writing $\phi = (x_\phi, y_\phi)$, we see from Equation (7) that $x_\phi(t) = t + O(t^c)$. Let $\xi : (\mathbb{R}_+, 0) \to (\mathbb{C}, 0)$ be the ramified analytic function of the form $t \to \xi(t) := t^{\frac{1}{m}}[1 + O(t^{c-1})]$ and such that $\xi(t)$ is a m-th root of $x_\phi(t)$. Thus $y_\phi(t) = \Psi(\omega^i \xi(t))$ for some $i \in \{1, \ldots, m\}$ and we observe that $y_\phi(t) = \Psi(\omega^i t^{\frac{1}{m}}) + o(t^{\frac{\lambda}{m}})$. Since $y(t) = \Psi_k(\omega^l t^{\frac{1}{m}}) + t^{\frac{\lambda}{m}} u(t^{\frac{1}{m}})$, with u a ramified analytic function, and $|y_\phi(t) - y(t^{\frac{1}{m}})| \leq Const \cdot t^c$, we deduce that $\Psi_k(\omega^i T) = \Psi_k(\omega^l T)$. But this implies that $c \leq \frac{\lambda}{m}$, and thus $c = \frac{\lambda}{m}$.

From Equation (7) we deduce that

$$\rho(t) \leq \operatorname{dist}(\gamma(t^{\frac{1}{m}}), X(r(t))) \leq Const \cdot t^c. \tag{8}$$

Combining Equation (6) and Equation (8) we get the result. □

The next result will be key for Theorem 4.2, the main result of this note, is indeed the new ingredient to the range of questions we are dealing with here. We recall that the Puiseux data notation convenes that $e_{-1} = \beta_0 = 0$, $e_0 = m$ and $\beta_{s+1} = +\infty$.

Theorem 3.6. *Let Γ be a real analytic half-branch at the origin of $\mathbb{C}^2$ as above. The order of the function f along Γ is given by*

$$\nu_f(\Gamma) = e_k \cdot c(\Gamma, X) + (e_0 - e_1)\frac{\beta_1}{m} + \ldots + (e_{k-1} - e_k)\frac{\beta_k}{m} \tag{9}$$

$$= e_k \left(c(\Gamma, X) - \frac{\beta_k}{m} \right) + \sum_{i=\min(k-1,0)}^{k-1} e_i \left(\frac{\beta_{i+1}}{m} - \frac{\beta_i}{m} \right), \tag{10}$$

where the integer number $k \in \{0, \ldots, s\}$ in Equations (9) and (10) is uniquely determined when $c(\Gamma, X) < +\infty$ by the following condition:

$$\beta_k \leq m \cdot c(\Gamma, X) < \beta_{k+1}.$$

Proof. It is just a rewriting of Lemma 3.1 in term of the size t of any arc parameterizing Γ and uses Lemma 3.5. □

A direct consequence of the above result is the following result about bi-Lipschitz contact equivalence.

Proposition 3.7. *Let $(\mathbb{C}^2, X, \mathbf{0})$ and $(\mathbb{C}^2, Y, \mathbf{0})$ be two germs of irreducible complex plane curves defined by reduced function-germs f and g respectively. If there exists a subanalytic bi-Lipschitz homeomorphism $H \colon (\mathbb{C}^2, X, \mathbf{0}) \to (\mathbb{C}^2, Y, \mathbf{0})$ then there exist positive constants $0 < A < B < +\infty$ such that in a neighbourhood of the origin we find*

$$A|f| \leq |g \circ H| \leq B|f|.$$

Proof. If it is not true, it happens along a real-analytic half-branch C. Necessarily such a half-branch C must be tangent to the curve X. Taking a parameterization of C by an arc α, we can for instance assume that $(f \circ \alpha(t))^{-1}(g \circ H \circ \alpha(t))$ goes to 0 as t goes to 0. Let ν be the order of $f(\alpha(t))$ and ν' the order of $g(H(\alpha(t)))$. Theorem 3.6 provides

$$\begin{aligned}\nu &= (e_0 - e_1)\frac{\beta_1}{m} + \ldots + (e_{k-1} - e_k)\frac{\beta_k}{m} + e_k \cdot c(C, X) \\ \nu' &= (e_0 - e_1)\frac{\beta_1}{m} + \ldots + (e_{k'-1} - e_{k'})\frac{\beta_{k'}}{m} + e_{k'} \cdot c(H^{-1}(C), Y).\end{aligned}$$

From the proofs of Lemma 3.1 and Lemma 3.5 we know that

$$\beta_{k'} \leq m \cdot c(H^{-1}(C), Y) < \beta_{k'+1} \text{ and } \beta_k \leq m \cdot c(C, X) < \beta_{k+1}.$$

Since the contact is a bi-Lipschitz invariant we get $c(C, X) = c(H^{-1}(C), Y)$. Besides $\nu' > \nu$, thus we deduce $k' > k$. This latter inequality implies

$$m \cdot c(H^{-1}(C), Y) \geq \beta_{k'} \geq \beta_{k+1} > m \cdot c(C, X),$$

which is impossible. □

4. Main Result

Let $f\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$ be a germ of analytic function. Let $f = f_1^{\mathbf{m}_1} \cdots f_r^{\mathbf{m}_r}$ be the irreducible decomposition of the function, where $f_1, \ldots, f_r$ are irreducible function-germs and $\mathbf{m}_1, \ldots, \mathbf{m}_r$, the corresponding respective multiplicities, are positive integer numbers.

Let X_i be the zero locus of f_i, let m_i be the multiplicity of f_i at $\mathbf{0}$ and let $(\beta_j^{(i)}, e_j^{(i)})_{j=1}^{s_i}$ be its Puiseux pairs. Let Γ be a real analytic half-branch at the origin. Let $c_i := c(\gamma, X_i)$ be the contact of Γ with X_i and let $\nu_i = \nu_{f_i}(\Gamma)$ be the order of f_i along Γ.

Since we have defined in Section 3 the order of an irreducible function-germ along Γ, the order of f along Γ is defined as the sum of the order of each of its irreducible component weighted by the corresponding multiplicity (as a factor of the irreducible decomposition of f). From Theorem 3.6 we deduce straightforwardly the next

Lemma 4.1. *The order ν of the function f along Γ is*

$$\begin{aligned}\nu &:= \mathbf{m}_1 \cdot \nu_1 + \ldots + \mathbf{m}_r \cdot \nu_r \\ &= \sum_{i=1}^{r} \mathbf{m}_i \left[e_{k_i}^{(i)} \left(c_i - \frac{\beta_{k_i}^{(i)}}{m} \right) + \sum_{j=\min(k_i-1,0)}^{k_i-1} e_j^{(i)} \left(\frac{\beta_{j+1}^{(i)}}{m} - \frac{\beta_j^{(i)}}{m} \right) \right]\end{aligned}$$

where each of the integer $k_i \in \{0, \ldots, s_i\}$ is uniquely determined when $c_1 \cdots c_r < +\infty$ by the condition

$$\beta_{k_i}^{(i)} \leq m_i \cdot c_i < \beta_{k_i+1}^{(i)}.$$

The main result of this note is the following:

Theorem 4.2. *Let f and g be two analytic function-germs $(\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$. Let $f = f_1^{\mathbf{m}_1} \cdots f_r^{\mathbf{m}_r}$ and $g = g_1^{\mathbf{n}_1} \cdots g_s^{\mathbf{n}_s}$ be respectively the irreducible decompositions of the functions f and g. Let X_i be the zero locus of f_i and Y_j be the zero locus of g_j.*

The functions f and g are subanalytically bi-Lipschitz contact equivalent if, and only if, possibly up to a re-indexation of the irreducible factors f_i:

0) $r = s$,

1) the multiplicities of each corresponding factors are equal, that is $\mathbf{m}_i = \mathbf{n}_i$, *2) the Puiseux pairs of f_i and g_i are the same, and*

3) for any pair i, j, the intersection numbers $(X_i, X_j)_{\mathbf{0}}$ and $(Y_i, Y_j)_{\mathbf{0}}$ are equal.

In particular, f and g are subanalytically bi-Lipschitz contact equivalent if, and only if, they are right topologically equivalent.

Proof. First (and possibly after a re-indexation of the irreducible factors f_i) assume that,
- $r = s$,
- the intersection numbers $(X_i, X_j)_{\mathbf{0}}$ and $(Y_i, Y_j)_{\mathbf{0}}$ are equal for any $i \neq j$ and,
- the Puiseux pairs of the functions f_i and g_1 are equal and,
- the multiplicities $\mathbf{m}_i$ and $\mathbf{n}_i$ are equal, for $i = 1, \ldots r$.

From Theorem 2.3 we deduce there exists $H\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ a subanalytic bi-Lipschitz homeomorphism such that $H(X_i) = Y_i$ for any $i = 1, \ldots r$. For each $i = 1, \ldots, r$, Proposition 3.7 implies there exist positive constants $0 < A_i < B_i < +\infty$ such that in a neighbourhood of the origin we find

$$A_i|f_i| \leq |g_i \circ H| \leq B_i|f_i|.$$

Thus the functions f and g are bi-Lipschitz contact equivalent (via h).

Conversely, we assume now that there exists $H\colon (\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0})$ a subanalytic bi-Lipschitz homeomorphism such that there exist positive constants $A < B$ such that in a neighbourhood of the origin the following inequalities hold true:

$$A|f| \leq |g \circ H| \leq B|f|. \tag{11}$$

We immediately find $H(X) = Y$ and $r = s$. Up to re-indexation of the branches Y_i, we also have $H(X_i) = Y_i$ for $i = 1, \ldots, r$. Using Theorem 2.3 again we deduce that the intersection numbers $(X_i, X_j)_{\mathbf{0}}$ and $(Y_i, Y_j)_{\mathbf{0}}$ are equal for any $i \neq j$ (let us denote each such number by $I_{i,j}$), the Puiseux pairs of the function-germs f_i and g_i are equal. It remains to prove that the multiplicities $\mathbf{m}_i$ and $\mathbf{n}_i$ are also equal, for $i = 1, \ldots, r$. In order to prove that $\mathbf{m}_1 = \mathbf{n}_1$, let C be any real-analytic half-branch such that the contact $c = c(C, X_1)$ is sufficiently large (and finite) and also such that the others contacts $c(C, X_i)$, for $i = 2, \ldots, r$, are equal to the intersection number $I_{i,1} := (X_i, X_1)_{\mathbf{0}}$ (see [4] for details). Since H is a subanalytic bi-Lipschitz homeomorphism such that $H(X_i) = Y_i$ for any $i = 1, \ldots, r$, the image $H(C)$ is still a real analytic half-branch. Since bi-Lipschitz homeomorphisms preserve the contact, we deduce that $c = c(H(C), Y_1)$ and each contact $c(H(C), Y_i)$ is equal to the contact $(Y_i, Y_1)_{\mathbf{0}}$, for $i = 2, \ldots, r$. In other words we see

$$\nu_g(H(C)) = c \cdot \mathbf{n}_1 + I_{2,1} \cdot \mathbf{n}_2 + \ldots + I_{r,1} \cdot \mathbf{n}_r \tag{12}$$

and

$$\nu_f(C) = c \cdot \mathbf{m}_1 + I_{2,1} \cdot \mathbf{m}_2 + \ldots + I_{r,1} \cdot \mathbf{m}_r. \tag{13}$$

Combining Equation (11) from the hypothesis, with Equations (12) and (13) we conclude that

$$c\mathbf{n}_1 + I_{2,1}\mathbf{n}_2 + \ldots + I_{r,1}\mathbf{n}_r = c\mathbf{m}_1 + I_{2,1}\mathbf{m}_2 + \cdots + I_{r,1}\mathbf{m}_r.$$

Since the half-branch C can be chosen asymptotically arbitrarily close to X_1, its contact c goes $+\infty$, and thus we find $\mathbf{m}_1 = \mathbf{n}_1$. The same procedure can be applied for each remaining $i = 2, \ldots, r$, substituting i for 1, thus we conclude that

$$\mathbf{m}_i = \mathbf{n}_i \text{ for } i = 1, \ldots, r,$$

thus proving what we wanted. □

The first immediate consequence of our main result is the following:

Corollary 4.3. *Let f and g be two analytic function-germs $(\mathbb{C}^2, \mathbf{0}) \to (\mathbb{C}, 0)$. They are bi-Lipschitz contact equivalent if, and only if, they are subanalytically bi-Lipschitz contact equivalent.*

The second consequence is:

Corollary 4.4. *The subanalytic bi-Lipschitz contact equivalence classification of complex analytic plane function-germs has countably many equivalence classes.*

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